Electronic Journal of Qualitative Theory of Differential Equations
2013, No. **72**, 1–19; **http://www.math.u-szeged.hu/ejqtde/**

On nonlinear evolution variational inequalities involving variable exponent

Mingqi Xiang*
Department of Mathematics, Harbin Institute of Technology
Harbin, 150001, P. R. China

Abstract. In this paper, we discuss a class of quasilinear evolution variational inequalities with variable exponent growth conditions in a generalized Sobolev space. We obtain the existence of weak solutions by means of penalty method. Moreover, we study the extinction properties of weak solutions to parabolic inequalities and provide a sufficient condition that makes the weak solutions vanish in a finite time. The existence of global attractors for weak solutions is also obtained via the theories of multi-valued semiflow.
Keywords: quasilinear evolution variational inequality; variable exponent space; penalty method; extinction; global attractor.
2010 AMS Subject Classifications: 35K86, 35R35, 35D30.

1 Introduction

In this paper we study the following quasilinear parabolic inequality with variable exponent growth conditions

$$\frac{\partial u}{\partial t} - \operatorname{div}\big(a(x,t,\nabla u)|\nabla u|^{p(x,t)-2}\nabla u\big) + f(x,t,u) \geq g(x,t). \tag{1.1}$$

Let $\Omega \subset \mathbb{R}^N (N \geq 2)$ be a bounded open domain with smooth boundary, $0 < T < \infty$ be given and $Q_T = \Omega \times (0,T)$. We denote $\mathcal{K} = \{w \in X(Q_T) \cap C(0,T;L^2(\Omega)),\ \frac{\partial w}{\partial t} \in X'(Q_T) : 0 \leq w(x,0) = u_0(x) \in L^2(\Omega),\quad w(x,t) \geq 0 \text{ a.e. } (x,t) \in Q_T\}$, where $X(Q_T)$ is a variable exponent space (see Definition 2.3 below) and $X'(Q_T)$ is the dual space of $X(Q_T)$. A function $u(x,t) \in \mathcal{K}$ is called a weak solution of parabolic inequality (1.1), if for any $0 \leq v \in X(Q_T)$ there holds

$$\int_0^T \int_\Omega \frac{\partial u}{\partial t}(v-u) + a(x,t,\nabla u)|\nabla u|^{p(x,t)-2}\nabla u \nabla(v-u) + f(x,t,u)(v-u)dxdt$$
$$\geq \int_0^T \int_\Omega g(v-u)dxdt.$$

We assume that

*Corresponding author. Email address: xiangmingqi_hit@163.com.

(H1) $p(x,t), q(x,t)$ are two bounded globally log-Hölder continuous functions in Q_T satisfying the following conditions:

$$\frac{2N}{N+2} < p^- = \inf_{\overline{Q_T}} p(x,t) \le p(x,t) \le \sup_{\overline{Q_T}} p(x,t) = p^+ < \infty,$$
$$1 < q^- = \inf_{\overline{Q_T}} q(x,t) \le q(x,t) \le \sup_{\overline{Q_T}} q(x,t) = q^+ < \infty.$$

(H2) $a(x,t,\xi) : \Omega \times \mathbb{R}^+ \times \mathbb{R}^N \to \mathbb{R}$ and $f(x,t,\eta) : \Omega \times \mathbb{R}^+ \times \mathbb{R}$ are two Carathéodory functions, i.e. $a(x,t,\xi)$ is continuous with respect to ξ and measurable for almost every (x,t), $f(x,t,\eta)$ is continuous with respect to η and measurable for almost every (x,t), and satisfy

$$0 < a_0 \le a(x,t,\xi) \le a_1 < \infty,$$
$$|f(x,t,\eta)| \le b_1(x,t)|\eta|^{q(x,t)-1} + h_1(x,t),$$
$$f(x,t,\eta)\eta \ge b_2(x,t)|\eta|^{q(x,t)},$$

where a_0, a_1 are constants, $b_1(x,t), b_2(x,t)$ are two bounded measurable functions on Q_T with $0 < b_1^0 \le b_1(x,t) \le b_1^1 < \infty, 0 < b_2^0 \le b_2(x,t) \le b_2^1 < \infty$ and $h_1(x,t) \in L^{\frac{q(x,t)}{q(x,t)-1}}(Q_T)$.

(H3) $g(x,t) \in L^{q\prime(x,t)}(Q_T)$, where $q\prime(x,t) = \frac{q(x,t)}{q(x,t)-1}$.

In recent years, the research on various mathematical problems with variable exponent growth conditions is an interesting topic. $p(\cdot)$-growth problems can be regarded as a kind of nonstandard growth problems, and these problems possess very complicated nonlinearities, for instance, the $p(x,t)$-Laplacian operator $-\mathrm{div}(|\nabla u|^{p(x,t)-2}\nabla u)$ is inhomogeneous. These problems are interesting in applications and raise many difficult mathematical problems, they appear in nonlinear elastic, electrorheological fluids, imaging processing and other physics phenomena (see [1–6]). Many results have been obtained on this kind of problems, see [7–12]. Especially, in [13–15], the authors studied the existence and uniqueness of weak solutions for anisotropy parabolic equations under the framework of variable exponent Sobolev spaces. Motivated by the works of [13–15], we shall study the existence and long-time behavior of weak solutions to problem (1.1). When the variable exponent depends only on the space variable x, evolution variational inequality without initial conditions has been studied in [16–18]. For the fundamental theory about variable exponent Lebesgue and Sobolev spaces, we refer to [19–20].

Variational inequalities as the development and extension of classic variational problems, are a very useful tool to research PDEs, optimal control and other fields. In the case $p \equiv$ const, many papers are devoted to the solvability of the different kinds of parabolic variational inequalities, see [21–25]. The method is based on a time discretion and the semigroup property of the corresponding differential quotient. Another approach is available via a suitable penalization method. In these works, a crucial assumption on the obstacles is monotonicity or regularity condition. A new method can be found in [26], where the obstacles are only continuous.

The asymptotic behavior of equations without uniqueness received attention in recent years. Several results concerning the existence of global attractors in the case of nonuniqueness have been proved for parabolic equations. However most of them have been devoted to nondegenerate semilinear parabolic equations. To the best of our knowledge, there are no papers devoted to the existence of global attractors for variable exponent parabolic variational inequalities without uniqueness. In the last years, there have been some theories that can be used to deal with multi-valued semiflows, such as Ball's generalized semiflows theory (see [27, 28]), Melnik and Valero's Multi-valued semiflow theory see [29, 30].

This article is organized as follows. In Section 2, we will give some necessary definitions and properties of variable exponent Lebesgue spaces and Sobolev spaces. Moreover, we introduce the space $X(Q_T)$ and prove some necessary properties, which provide a basic framework to solve our problem. In Section 3, using the result of existence of weak solutions for parabolic equation with penalty term, through a priori estimates, we obtain the existence of weak solutions of parabolic inequality (1.1). In Section 4, under some conditions, we obtain that the solutions of parabolic inequality vanish in a finite time. In Section 5, first we recall the basic theories of multi-valued semiflows, then we study the existence of global attractors in $L^2(\Omega)$ to problem (1.1).

2 Preliminaries

In this section, we first recall some important properties of variable exponent Lebesgue spaces and Sobolev spaces, see [11, 19–20] for the details. A measurable function $p : Q_T \to [1, \infty)$ is called a variable exponent and we define $p^- = \operatorname{ess\,inf}_{z\in Q_T} p(z)$ and $p^+ = \operatorname{ess\,sup}_{z\in Q_T} p(z)$. If p^+ is finite, then the exponent p is said to be bounded. The variable exponent Lebesgue space is

$$L^{p(\cdot)}(Q_T) = \{u : Q_T \to \mathbb{R} \text{ is a measurable function; } \rho_{p(\cdot)}(u) = \int_{Q_T} |u(z)|^{p(z)} dz < \infty\}$$

with the norm

$$\|u\|_{L^{p(\cdot)}(Q_T)} = \inf\{\lambda > 0 : \rho_{p(z)}(\lambda^{-1}u) \le 1\},$$

then $L^{p(\cdot)}(Q_T)$ is a Banach space, and when p is bounded, we have the following relations

$$\min\left\{\|u\|^{p^-}_{L^{p(\cdot)}(Q_T)}, \|u\|^{p^+}_{L^{p(\cdot)}(Q_T)}\right\} \le \rho_{p(\cdot)}(u) \le \max\left\{\|u\|^{p^-}_{L^{p(\cdot)}(Q_T)}, \|u\|^{p^+}_{L^{p(\cdot)}(Q_T)}\right\}.$$

That is, if p is bounded, then norm convergence is equivalent to convergence with respect to the modular $\rho_{p(\cdot)}$. For bounded exponent the dual space $(L^{p(\cdot)}(Q_T))'$ can be identified with $L^{p'(\cdot)}(Q_T)$, where $p'(\cdot) = \frac{p(\cdot)}{p(\cdot)-1}$ is the conjugate exponent of $p(\cdot)$. If $1 < p^- \le p^+ < \infty$, then the variable exponent Lebesgue space $L^{p(\cdot)}(Q_T)$ is separable and reflexive.

In the variable exponent Lebesgue space, Hölder's inequality is still valid. For all $u \in L^{p(\cdot)}(Q_T),\ v \in L^{p'(\cdot)}(Q_T)$ with $p(\cdot) \in (1, \infty)$ the following inequality holds

$$\int_{Q_T} |uv| dz \le \left(\frac{1}{p^-} + \frac{1}{(p')^-}\right) \|u\|_{L^{p(\cdot)}(Q_T)} \|v\|_{L^{p'(\cdot)}(Q_T)} \le 2\|u\|_{L^{p(\cdot)}(Q_T)} \|v\|_{L^{p'(\cdot)}(Q_T)}.$$

Definition 2.1. [11, 13] *We say that a bounded exponent $p : Q_T \to \mathbb{R}$ is globally log-Hölder continuous if p satisfies the following two conditions*

(1) there is a constant $c_1 > 0$ such that

$$|p(y) - p(z)| \leq \frac{c_1}{\log(e + 1/|y-z|)},$$

for all points $y, z \in Q_T$;

(2) there exist two constants $c_2 > 0$ and $p_\infty \in \mathbb{R}$, such that

$$|p(y) - p_\infty| \leq \frac{c_2}{\log(e + |y|)}$$

for all $y \in Q_T$.

Definition 2.2. [13, 15] *Given two bounded globally log-Hölder continuous exponent p, q on Q_T, let (H1) be valid. For any fixed $\tau \in (0, T)$, we define*

$$V_\tau(\Omega) = \{u \in W_0^{1,1}(\Omega) : u \in L^{p(\cdot,\tau)}(\Omega), |\nabla u| \in L^{p(\cdot,\tau)}(\Omega)\},$$

and equip $V_\tau(\Omega)$ with the norm

$$\|u\|_{V_\tau(\Omega)} = \|u\|_{L^{q(\cdot,\tau)}(\Omega)} + \|\nabla u\|_{(L^{p(\cdot,\tau)}(\Omega))^N}.$$

Remark 2.1. *From a similar discussion to that in* [13], *for every $\tau \in (0, T)$, the space $V_\tau(\Omega)$ is a separable and reflexive Banach space.*

Definition 2.3. [13, 15] *Let (H1) be valid, we set*

$$X(Q_T) = \left\{u \in L^{q(x,t)}(Q_T) : |\nabla u| \in L^{p(x,t)}(Q_T), u(\cdot, \tau) \in V_\tau(\Omega) \text{ for a.e. } \tau \in (0, T)\right\},$$

with the norm

$$\|u\| = \|u\|_{L^{q(x,t)}(Q_T)} + \|\nabla u\|_{L^{p(x,t)}(Q_T)}.$$

Remark 2.2. *Following the standard proof for Sobolev spaces, we can prove that $X(Q_T)$ is a Banach space, and it's easy to check that $X(Q_T)$ can be continuously embedded into the space $L^r(0, T; W_0^{1,p^-}(\Omega) \cap L^{q^-}(\Omega))$, where $r = \min\{p^-, q^-\}$.*

By using the same method as in [13], the following theorem can be proved.

Theorem 2.1. ([13]) *The space $C_0^\infty(Q_T)$ is dense in $X(Q_T)$.*

Since $C_0^\infty(Q_T) \subset C^\infty(0, T; C_0^\infty(\Omega))$, we have

Lemma 2.1. *The space $C^\infty(0, T; C_0^\infty)$ is dense in $X(Q_T)$.*

Let $X'(Q_T)$ denote the dual space of $X(Q_T)$, then we have

Theorem 2.2. *A function $g \in X'(Q_T)$ if and only if there exist $\bar{g} \in L^{q'(x,t)}(Q_T)$ and $\bar{G} \in (L^{p'(x,t)}(Q_T))^N$ such that*

$$\int_{Q_T} g\varphi dxdt = \int_{Q_T} \bar{g}\varphi dxdt + \int_{Q_T} \bar{G}\nabla\varphi dxdt. \tag{2.1}$$

Proof. We define a mapping $\Gamma : X(Q_T) \to L^{q(x,t)}(Q_T) \times (L^{p(x,t)}(Q_T))^N$ by $\Gamma(u) = (u, \nabla u)$ for all $u \in X(Q_T)$. Clearly, Γ is an isometric isomorphism from $X(Q_T)$ onto the closed subspace $\Gamma(Q_T) \subset L^{q(x,t)}(Q_T) \times (L^{p(x,t)}(Q_T))^N$. So, we can define a continuous linear functional on $\Gamma(X(Q_T))$

$$F : \Gamma(X(Q_T)) \to \mathbb{R}, \qquad F(\Gamma u) = g(u) \quad \text{for all } u \in X(Q_T).$$

For all $u_1, u_2 \in X(Q_T)$ and $\alpha, \beta \in \mathbb{R}$, we have

$$\begin{aligned} F(\alpha\Gamma u_1 + \beta\Gamma u_2) &= F(\Gamma(\alpha u_1 + \beta u_2)) \\ &= g(\alpha u_1 + \beta u_2) \\ &= \alpha g(u_1) + \beta g(u_2) \\ &= \alpha F(\Gamma u_1) + \beta F(\Gamma u_2), \end{aligned}$$

and for all $u \in X(Q_T)$, there holds

$$|F(\Gamma(u))| = |g(u)| \leq \|g\|_{X'(Q_T)}\|u\|_{X(Q_T)} = \|g\|_{X'(Q_T)}\|\Gamma u\|.$$

Thus F is a continuous linear functional on $\Gamma(X(Q_T))$. By the Hahn–Banach theorem, there exists a linear functional $\tilde{F} \in (L^{q(x,t)}(Q_T) \times (L^{p(x,t)}(Q_T))^N)'$ satisfying

$$\tilde{F}(\Gamma u) = F(\Gamma u), \text{ and } \|\tilde{F}\| = \|F\|.$$

According to the fact that $(L^{q(x,t)}(Q_T) \times (L^{p(x,t)}(Q_T))^N)' = L^{q'(x,t)}(Q_T) \times (L^{p'(x,t)}(Q_T))^N$, there exist $f_0 \in L^{q'(x,t)}(Q_T), f_1, \ldots, f_N \in L^{p'(x,t)}(Q_T)$, such that for all $(u_0, u_1, \ldots, u_N) \in L^{q(x,t)}(Q_T) \times (L^{p(x,t)}(Q_T))^N$, there holds

$$\tilde{F}(u_0, u_1, \ldots, u_N) = \int_{Q_T} f_0 u_0 + f_1 u_1 + \cdots + f_N u_N dxdt.$$

Especially, we have

$$\tilde{F}(u_0, u_1, \ldots, u_N) = F(u_0, u_1, \ldots, u_N) = g(u), \ \forall u \in X(Q_T),$$

where $u_0 = u, u_i = \frac{\partial u}{\partial x_i}, i = 1, \ldots, N$. Letting $\bar{g} = f_0$, $\bar{G} = (f_1, f_2, \ldots, f_N)$, we immediately obtain that (2.1) holds. Conversely, if $\langle g, \varphi\rangle = \int_{Q_T} g\varphi dxdt = \int_{Q_T} \bar{g}\varphi dxdt + \int_{Q_T} \bar{G}\nabla\varphi dxdt$ for all $\varphi \in X(Q_T)$, then by the Hölder inequality we have

$$|\langle g, \varphi\rangle| \leq C(\|\bar{g}\|_{L^{q'(x,t)}(Q_T)} + \|\bar{G}\|_{(L^{p'(x,t)})^N(Q_T)})\|\varphi\|_{X(Q_T)}.$$

Thus, we have $g \in X'(Q_T)$. □

Remark 2.3. *It follows from the proof of Theorem 2.2 that $X(Q_T)$ is reflexive and*

$$X'(Q_T) \hookrightarrow L^{s'}(0, T; W^{-1,(p^+)'}(\Omega) + L^{(q^+)'}(\Omega)),$$

where $s = \max\{p^+, q^+\}$. Similar results have been obtained by O. M. Buhrii in the stationary case, see [19].

Similar to [13], we give the following definition.

Definition 2.4. *We define the space* $W(Q_T) = \{u \in X(Q_T) : \frac{\partial u}{\partial t} \in X'(Q_T)\}$ *with the norm*

$$\|u\|_{W(Q_T)} = \|u\|_{X(Q_T)} + \|\frac{\partial u}{\partial t}\|_{X'(Q_T)}$$

where $\frac{\partial u}{\partial t}$ *is the distribution derivative of* u *with respect to the time variable* t *defined by*

$$\int_{Q_T} \frac{\partial u}{\partial t}\varphi dxdt = -\int_{Q_T} u\frac{\partial \varphi}{\partial t}dxdt, \text{ for all } \varphi \in C_0^\infty(Q_T).$$

Lemma 2.2. [13] *The space* $W(Q_T)$ *is a Banach space.*

Similarly, we have

Theorem 2.3. [13] *The space* $C^\infty(0,T;C_0^\infty(\Omega))$ *is dense in* $W(Q_T)$.

Theorem 2.4. [21] *Let* $B_0 \subset B \subset B_1$ *be three Banach spaces, where* B_0, B_1 *are reflexive, and the embedding* $B_0 \subset B_1$ *is compact. Denote by* $W = \{v : v \in L^{p_0}(0,T;B_0),\ \frac{\partial v}{\partial t} \in L^{p_1}(0,T;B_1)\}$, *where* T *is a fixed positive number,* $1 < p_i < \infty,\ i = 0, 1,$ *then* W *can be compactly embedded into* $L^{p_0}(0,T;B)$.

As $p^- > \frac{2N}{N+2},\ N \geq 2$, the following theorem can be proved similarly to that in [13], thus we omit its proof.

Theorem 2.5. [13] $W(Q_T)$ *can be continuously embedded into* $C(0,T;L^2(\Omega))$. *Furthermore, for all* $u, v \in W(Q_T)$ *and* $s, t \in [0,T]$ *the following rule for integration by parts is valid*

$$\int_s^t \int_\Omega \frac{\partial u}{\partial t}vdxd\tau = \int_\Omega u(x,t)v(x,t)dx - \int_\Omega u(x,s)v(x,s)dx - \int_s^t \int_\Omega u\frac{\partial v}{\partial t}dxd\tau.$$

The following theorem gives a relation between almost everywhere convergence and weak convergence.

Theorem 2.6. [9] *Let* $p(x,t) : Q_T \longrightarrow \mathbb{R}$ *be a bounded globally log-Hölder continuous function, with* $p^- > 1$. *If* $\{u_n\}_{n=1}^\infty$ *is bounded in* $L^{p(x,t)}(Q_T)$ *and* $u_n \to u$ for a.e. $(x,t) \in Q_T$, *then there exists a subsequence of* $\{u_n\}_{n=1}^\infty$ *(relabeled by* $\{u_n\}_{n=1}^\infty$*) such that* $u_n \to u$ *weakly in* $L^{p(x,t)}(Q_T)$.

Using penalty method, we transform the problem (1.1) into the following problem

$$\begin{aligned}
&\frac{\partial u}{\partial t} - a(x,t,\nabla u)(\mathrm{div}|\nabla u|^{p(x,t)-2}\nabla u) + f(x,t,u) - \left|\frac{u^-}{\varepsilon}\right|^{q(x,t)-2}\frac{u^-}{\varepsilon} = g(x,t),\ (x,t) \in Q_T,\\
&u(x,t) = 0,\quad (x,t) \in \partial\Omega \times (0,T),\\
&u(x,0) = u_0(x),\quad x \in \Omega,
\end{aligned} \tag{2.2}$$

where $u^- = \max\{-u, 0\}$. The weak solutions of equation (2.2) can be constructed as the limit of a sequence of Galerkin's approximation. The proof relies on Theorem 2.4, Theorem 2.5 and the monotonicity of elliptic part of equation (2.2), we refer the reader to [21–23] for the details.

Definition 2.5. [13, 14] *A function $u_\varepsilon \in X(Q_T)$ with $\frac{\partial u_\varepsilon}{\partial t} \in X'(Q_T)$ is called a weak solution of (2.2), if for all $\varphi \in X(Q_T)$, there holds*

$$\int_{Q_T} \frac{\partial u_\varepsilon}{\partial t}\varphi dxdt + \int_{Q_T} a(x,t,\nabla u_\varepsilon)|\nabla u_\varepsilon|^{p(x,t)-2}\nabla u_\varepsilon \nabla\varphi + f(x,t,u_\varepsilon)\varphi$$
$$-\left|\frac{u_\varepsilon^-}{\varepsilon}\right|^{q(x,t)-2}\frac{u_\varepsilon^-}{\varepsilon}\varphi dxdt = \int_{Q_T} g(x,t)\varphi dxdt,$$

Theorem 2.7. [13, 14] *Let (H1)–(H3) hold, then for each $\varepsilon \in (0,1)$, there exists a weak solution of problem (2.2).*

3 Global existence of solutions

In this section, we prove the global existence of weak solutions of problem (1.1).

Theorem 3.1. *Let (H1)–(H3) hold, then there exists a function $u(x,t) \in \mathcal{K}$ such that for all $v \in X(Q_T)$* with $v(x,t) \geq 0$ for a.e. $(x,t) \in Q_T$, *the following inequality holds*

$$\int_{Q_T} \frac{\partial u}{\partial t}(v-u)dxdt + \int_{Q_T} a(x,t,\nabla u)|\nabla u|^{p(x,t)-2}\nabla u\nabla(v-u) + f(x,t,u)(v-u)dxdt$$
$$\geq \int_0^T \int_\Omega g(v-u)dxdt,$$

Proof. By Theorem 2.7, for every $\varepsilon \in (0,1)$, there exists a weak solution of equation (2.2) satisfying Definition 2.5. In Definition 2.5, we take $\varphi = u_\varepsilon \cdot \chi_{(0,t)}$ as a test function, where $\chi_{(0,t)}$ is defined as the characteristic function of $(0,t)$, $t \in (0,T]$, then

$$\int_{Q_t} \frac{\partial u_\varepsilon}{\partial t}u_\varepsilon dxdt + \int_{Q_t} a(x,t,\nabla u_\varepsilon)|\nabla u_\varepsilon|^{p(x,t)} + f(x,t,u_\varepsilon)u_\varepsilon$$
$$-\left|\frac{u^-}{\varepsilon}\right|^{q(x,t)-2}\frac{u^-}{\varepsilon}u_\varepsilon dxdt = \int_{Q_t} g(x,t)u_\varepsilon dxdt,$$

where $Q_t = \Omega \times (0,t)$.

Using integration by parts (see Theorem 2.5) and Young's inequality, we have

$$\begin{aligned}&\frac{1}{2}\int_\Omega |u_\varepsilon(x,t)|^2dx - \frac{1}{2}\int_\Omega |u_\varepsilon(x,0)|^2dx + \int_{Q_t} a(x,t,\nabla u_\varepsilon)|\nabla u_\varepsilon|^{p(x,t)}\\ &\quad + f(x,t,u_\varepsilon)u_\varepsilon + (\frac{1}{\varepsilon})^{p(x,t)-1}|u_\varepsilon^-|^{q(x,t)}dxdt\\ \leq& C\int_{Q_t} |g(x,t)|^{q'(x,t)}dxdt + \frac{b_2^0}{2}\int_{Q_t} |u_\varepsilon|^{q(x,t)}dxdt.\end{aligned}$$

By (H2) and (H3), there holds

$$\int_\Omega |u_\varepsilon(x,t)|^2dx + \int_{Q_t} |\nabla u_\varepsilon|^{p(x,t)} + |u_\varepsilon|^{q(x,t)} + (\frac{1}{\varepsilon})^{q(x,t)-1}|u_\varepsilon^-|^{q(x,t)}dxdt \leq C, \tag{3.1}$$

where C is a constant independent of ε and t. Taking $\varphi = -\frac{u_\varepsilon^-}{\varepsilon}$ as a test function in Definition 2.5, by Young's inequality, we have

$$
\begin{aligned}
&\frac{1}{\varepsilon}\int_{Q_T} \frac{\partial u_\varepsilon}{\partial t}(-u_\varepsilon^-)dxdt + \frac{1}{\varepsilon}\int_{Q_T} a(x,t,\nabla u_\varepsilon)|\nabla u_\varepsilon|^{p(x,t)-2}\nabla u_\varepsilon(-\nabla u_\varepsilon^-) + f(x,t,u_\varepsilon)(-u_\varepsilon^-) \\
&\quad + \left|\frac{u_\varepsilon^-}{\varepsilon}\right|^{q(x,t)} dxdt \\
&= \int_{Q_T} g(x,t)(-\frac{u_\varepsilon^-}{\varepsilon})dxdt \le \frac{1}{2}\int_{Q_T} |g(x,t)|^{q'(x,t)}dxdt + \frac{1}{2}\int_{Q_T} \left|\frac{u_\varepsilon^-}{\varepsilon}\right|^{q(x,t)} dxdt.
\end{aligned}
\tag{3.2}
$$

Since $\int_{Q_T} \frac{\partial u_\varepsilon}{\partial t}(-u_\varepsilon^-)dxdt = \frac{1}{2}\int_\Omega |u_\varepsilon^-(x,T)|^2dx \ge 0$ and

$$
\begin{aligned}
&\int_{Q_T} a(x,t,\nabla u_\varepsilon)|\nabla u_\varepsilon|^{p(x,t)-2}\nabla u_\varepsilon(-\nabla u_\varepsilon^-) + f(x,t,u_\varepsilon)(-u_\varepsilon^-)dxdt \\
&= \int_{Q_T} a(x,t,\nabla u_\varepsilon)|\nabla u_\varepsilon^-|^{p(x,t)} + f(x,t,-u_\varepsilon^-)(-u_\varepsilon^-)dxdt \ge 0,
\end{aligned}
$$

by (3.2), we obtain $\int_{Q_T} |\frac{u_\varepsilon^-}{\varepsilon}|^{q(x,t)}dxdt \le C$. Thus

$$
\|u_\varepsilon\|_{L^\infty(0,T;L^2(\Omega))} + \|\nabla u_\varepsilon\|_{L^{p(x,t)}(Q_T)} + \|u_\varepsilon\|_{L^{q(x,t)}(Q_T)} + \left\|\frac{u_\varepsilon^-}{\varepsilon}\right\|_{L^{q(x,t)}(Q_T)} \le C. \tag{3.3}
$$

Since

$$
\begin{aligned}
&\int_{Q_T} \left|a(x,t,\nabla u_\varepsilon)|\nabla u_\varepsilon|^{p(x,t)-2}\nabla u_\varepsilon\right|^{p'(x,t)} dxdt \\
\le& C\int_{Q_T} |\nabla u_\varepsilon|^{p(x,t)}dxdt \\
\le& C\max\left\{\|\nabla u_\varepsilon\|_{L^{p(x,t)}(Q_T)}^{p_-}, \|\nabla u_\varepsilon\|_{L^{p(x,t)}(Q_T)}^{p_+}\right\} \le C,
\end{aligned}
$$

we have

$$
\left\| \, |a(x,t,\nabla u_\varepsilon)|\nabla u_\varepsilon|^{p(x,t)-2}\nabla u_\varepsilon| \, \right\|_{L^{p'(x,t)}(Q_T)} \le C. \tag{3.4}
$$

Similarly,

$$
\|f(x,t,u_\varepsilon)\|_{L^{q'(x,t)}(Q_T)} \le C \text{ and } \left\| \left|\frac{u_\varepsilon^-}{\varepsilon}\right|^{q(x,t)-2} \frac{u_\varepsilon^-}{\varepsilon}\right\|_{L^{q'(x,t)}(Q_T)} \le C. \tag{3.5}
$$

For all $\varphi \in X(Q_T)$, from Definition 2.5 we have

$$
\begin{aligned}
\left|\int_{Q_T} u_\varepsilon \varphi dxdt\right| &= \left| -\int_{Q_T} a(x,t,\nabla u_\varepsilon)|\nabla u_\varepsilon|^{p(x,t)-2}\nabla u_\varepsilon \nabla\varphi + f(x,t,u_\varepsilon)\varphi \right. \\
&\qquad \left. - \left|\frac{u_\varepsilon^-}{\varepsilon}\right|^{q(x,t)-2}\frac{u_\varepsilon^-}{\varepsilon}\varphi dxdt + \int_{Q_T} g(x,t)\varphi dxdt\right| \\
&\le C\|\varphi\|_{X(Q_T)}.
\end{aligned}
\tag{3.6}
$$

By (3.6), we get

$$\left\|\frac{\partial u_\varepsilon}{\partial t}\right\|_{X'(Q_T)} = \sup_{\|\varphi\|_{X(Q_T)}\le 1}\left|\int_{Q_T}\frac{\partial u_\varepsilon}{\partial t}\varphi dxdt\right| \le C. \tag{3.7}$$

From (3.3)–(3.5) and (3.7), there exists a subsequence of $\{u_\varepsilon\}_{\varepsilon>0}$, still denoted by $\{u_\varepsilon\}_{\varepsilon>0}$, such that

$$\begin{cases} u_\varepsilon \overset{*}{\rightharpoonup} u \quad \text{weakly * in } L^\infty(0,T;L^2(\Omega)), \\ u_\varepsilon \rightharpoonup u \quad \text{weakly in } X(Q_T), \\ a(x,t,\nabla u_\varepsilon)\left|\nabla u_\varepsilon\right|^{p(x,t)-2}\nabla u_\varepsilon \rightharpoonup \xi \;\text{ weakly in } (L^{p'(x,t)}(Q_T))^N, \\ f(x,t,u_\varepsilon) \rightharpoonup \eta \quad \text{weakly in } L^{q'(x,t)}(Q_T), \\ \frac{\partial u_\varepsilon}{\partial t} \rightharpoonup \beta \quad \text{weakly in } X'(Q_T), \\ u_\varepsilon^- \longrightarrow 0 \quad \text{strongly in } L^{q(x,t)}(Q_T). \end{cases} \tag{3.8}$$

First, we will prove

$$\eta = f(x,t,u),\ \beta = \frac{\partial u}{\partial t} \text{ and } u \ge 0 \text{ for a.e. } (x,t)\in Q_T.$$

For all $\varphi \in C_0^\infty(Q_T)$, there holds $\int_{Q_T}\frac{\partial u_\varepsilon}{\partial t}\varphi dxdt = -\int_{Q_T} u_\varepsilon \frac{\partial \varphi}{\partial t}dxdt$. Due to (3.8), the left-hand side of this equality converges to $\int_{Q_T}\beta\varphi dxdt$, while the right-hand side converges to $-\int_{Q_T} u\frac{\partial\varphi}{\partial t}dxdt$, thus we have $\beta = \frac{\partial u}{\partial t}$.

Since

$$X(Q_T) \hookrightarrow L^r(0,T;W_0^{1,p^-}(\Omega)\cap L^{q^-}(\Omega)), \text{ where } r=\min\{p^-,q^-\},$$
$$X'(Q_T) \hookrightarrow L^{s'}(0,T;W^{-1,(p^+)'}(\Omega)+L^{(q^+)'}(\Omega)), \text{ where } s=\max\{p^+,q^+\}.$$

and

$$W_0^{1,p^-}(\Omega)\cap L^{q^-}(\Omega) \hookrightarrow\hookrightarrow L^2(\Omega) \hookrightarrow W^{-1,\lambda}(\Omega),$$

where $\lambda = \min\{2,(p^+)',(q^+)'\}$, by Theorem 2.4, there exists a subsequence of u_ε (still denoted by u_ε) such that $u_\varepsilon \to u$ in $L^r(0,T;L^2(\Omega))$ and $u_\varepsilon \to u$ for a.e. $(x,t)\in Q_T$. Thus, we obtain that $f(x,t,u_\varepsilon)\to f(x,t,u)$ for a.e. $(x,t)\in Q_T$ and $u_\varepsilon^- \to u^-$ for a.e. $(x,t)\in Q_T$. By Theorem 2.6, we get $\eta = f(x,t,u)$. Moreover, from (3.8) we have $u^- = 0$ for a.e. $(x,t)\in Q_T$, that is $u(x,t)\ge 0$ for a.e. $(x,t)\in Q_T$.

Second, we prove that $u(x,0)=u_0(x)$ and $\xi = a(x,t,\nabla u)\left|\nabla u\right|^{p(x,t)-2}\nabla u$. By (3.1), up to a subsequence of $\{u_\varepsilon\}_{\varepsilon>0}$, we have $u_\varepsilon(x,T)\to\tilde{u}$ weakly in $L^2(\Omega)$. For all $\eta(t)\in C^1[0,T]$ and $\varphi\in C_0^\infty(\Omega)$, there holds

$$\int_0^T\int_\Omega \frac{\partial u_\varepsilon}{\partial t}\eta(t)\varphi(x)dxdt = \int_\Omega u_\varepsilon(x,T)\eta(T)\varphi(x) - u_0(x)\eta(0)\varphi(x)dx - \int_0^T\int_\Omega u_\varepsilon\frac{\partial\eta}{\partial t}\varphi dxdt,$$

Letting $\varepsilon\to 0$ and using integration by parts, we obtain

$$\int_\Omega(\tilde{u}-u(x,T))\eta(T)\varphi(x) - (u(x,0)-u_0(x))\eta(0)\varphi(x)dx = 0.$$

Choosing $\eta(T)=1$ and $\eta(0)=0$, or $\eta(T)=0$ and $\eta(0)=1$, then by the density of $C_0^\infty(\Omega)$ in $L^2(\Omega)$, we obtain that $\tilde{u}=u(x,T)$ and $u(x,0)=u_0(x)$.

Taking $\varphi=u-u_\varepsilon$ as a test function in Definition 2.5 , then using Theorem 2.5 and $u_\varepsilon(x,0)=u_0(x)$, we get

$$\begin{aligned}
&\int_{Q_T}\frac{\partial u}{\partial t}(u-u_\varepsilon)+a(x,t,\nabla u_\varepsilon)|\nabla u_\varepsilon|^{p(x,t)-2}\nabla u_\varepsilon\nabla(u-u_\varepsilon)+f(x,t,u_\varepsilon)(u-u_\varepsilon)\\
&\quad-g(u-u_\varepsilon)dxdt\\
=&\int_{Q_T}\frac{\partial u_\varepsilon}{\partial t}(u-u_\varepsilon)+a(x,t,\nabla u_\varepsilon)|\nabla u_\varepsilon|^{p(x,t)-2}\nabla u_\varepsilon\nabla(u-u_\varepsilon)+f(x,t,u_\varepsilon)(u-u_\varepsilon)\\
&\quad-g(u-u_\varepsilon)dxdt+\int_{Q_T}\frac{\partial(u-u_\varepsilon)}{\partial t}(u-u_\varepsilon)dxdt\\
=&\int_{Q_T}\left|\frac{u_\varepsilon^-}{\varepsilon}\right|^{q(x,t)-2}\frac{u_\varepsilon^-}{\varepsilon}(u-u_\varepsilon)dxdt+\int_{Q_T}\frac{\partial(u-u_\varepsilon)}{\partial t}(u-u_\varepsilon)dxdt\\
\geq&\int_{Q_T}\frac{\partial(u-u_\varepsilon)}{\partial t}(u-u_\varepsilon)dxdt\\
=&\frac{1}{2}\int_\Omega|u(x,T)-u_\varepsilon(x,T)|^2dx\\
\geq&0,
\end{aligned}$$

and further

$$\begin{aligned}
&\int_{Q_T}a(x,t,\nabla u_\varepsilon)|\nabla u_\varepsilon|^{p(x,t)}dxdt\\
\leq&\int_{Q_T}a(x,t,\nabla u_\varepsilon)|\nabla u_\varepsilon|^{p(x,t)-2}\nabla u_\varepsilon\nabla udxdt+\int_{Q_T}\frac{\partial u}{\partial t}(u-u_\varepsilon)-g(u-u_\varepsilon)dxdt\\
&+\int_{Q_T}f(x,t,u_\varepsilon)udxdt-\int_{Q_T}f(x,t,u_\varepsilon)u_\varepsilon dxdt.
\end{aligned}$$

Thus, by Fatou's lemma, we obtain

$$\begin{aligned}
\limsup_{\varepsilon\to0}\int_{Q_T}a(x,t,\nabla u_\varepsilon)|\nabla u_\varepsilon|^{p(x,t)}dxdt\leq&\int_{Q_T}\xi\nabla udxdt\\
=\lim_{\varepsilon\to0}\int_{Q_T}a(x,t,\nabla u_\varepsilon)|\nabla u_\varepsilon|^{p(x,t)-2}\nabla u_\varepsilon\nabla udxdt,&
\end{aligned}$$

that is

$$\limsup_{\varepsilon\to0}\int_{Q_T}a(x,t,\nabla u_\varepsilon)|\nabla u_\varepsilon|^{p(x,t)-2}\nabla u_\varepsilon\nabla(u_\varepsilon-u)dxdt\leq0. \tag{3.9}$$

As $\{a(x,t,\nabla u_\varepsilon)\}$ is uniformly bounded and equi-integrable in $L^1(Q_T)$, there exists a subsequence of $\{u_\varepsilon\}$ (for convenience still relabeled by $\{u_\varepsilon\}$) and a^* such that $a(x,t,\nabla u_\varepsilon)\to a^*$ for almost every $(x,t)\in Q_T$ and $\left|(a(x,t,\nabla u_\varepsilon)-a^*)|\nabla u|^{p(x,t)-2}\nabla u\right|^{p'(x,t)}\leq C|\nabla u|^{p(x,t)}\in L^1(Q_T)$, by Lebesgue's dominated convergence theorem, we get

$$a(x,t,\nabla u_\varepsilon)|\nabla u|^{p(x,t)-2}\nabla u\longrightarrow a^*|\nabla u|^{p(x,t)-2}\nabla u\ \text{ strongly in } L^{p'(x,t)}(Q_T).$$

Since

$$
\begin{aligned}
0 \leq & \int_{Q_T} a(x,t,\nabla u_\varepsilon)(|\nabla u_\varepsilon|^{p(x,t)-2}\nabla u_\varepsilon - |\nabla u|^{p(x,t)-2}\nabla u)(\nabla u_\varepsilon - \nabla u) \\
= & \int_{Q_T} a(x,t,\nabla u_\varepsilon)|\nabla u_\varepsilon|^{p(x,t)-2}\nabla u_\varepsilon \nabla(u_\varepsilon - u) - a(x,t,\nabla u_\varepsilon)|\nabla u|^{p(x,t)-2}\nabla u \nabla(u_\varepsilon - u)dxdt,
\end{aligned}
$$

we have

$$
\liminf_{\varepsilon\to 0} \int_{Q_T} a(x,t,\nabla u_\varepsilon)|\nabla u_\varepsilon|^{p(x,t)-2}\nabla u_\varepsilon \nabla(u_\varepsilon - u)dxdt \geq 0. \tag{3.10}
$$

From (3.9)–(3.10) and $\nabla u_\varepsilon \rightharpoonup \nabla u$ in $(L^{p(x,t)}(Q_T))^N$, there holds

$$
\lim_{\varepsilon\to 0} \int_{Q_T} a(x,t,\nabla u_\varepsilon)(|\nabla u_\varepsilon|^{p(x,t)-2}\nabla u_\varepsilon - |\nabla u|^{p(x,t)-2}\nabla u)\nabla(u_\varepsilon - u)dxdt = 0. \tag{3.11}
$$

Similar to the partition in [32], we set $Q_1 = \{(x,t) \in Q_T : \ p(x,t) \geq 2\}$ and $Q_2 = \{(x,t) \in Q_T : \ \frac{2N}{N+2} < p(x,t) < 2\}$, by (3.11), then as $n \to \infty$,

$$
\begin{aligned}
& \int_{Q_1} |\nabla u_\varepsilon - \nabla u|^{p(x,t)} dxdt \\
\leq & C \int_{Q_1} a(x,t,\nabla u_\varepsilon)(|\nabla u_\varepsilon|^{p(x,t)-2}\nabla u_\varepsilon - |\nabla u|^{p(x,t)-2}\nabla u)(\nabla u_\varepsilon - \nabla u)dxdt \\
& \longrightarrow 0
\end{aligned} \tag{3.12}
$$

and

$$
\begin{aligned}
& \int_{Q_2} |\nabla u_\varepsilon - \nabla u|^{p(x,t)} dxdt \\
\leq & C \left\| \left[a(x,t,\nabla u_\varepsilon)(|\nabla u_\varepsilon|^{p(x,t)-2}\nabla u_\varepsilon - |\nabla u|^{p(x,t)-2}\nabla u)(\nabla u_\varepsilon - \nabla u) \right]^{\frac{p(x,t)}{2}} \right\|_{L^{\frac{2}{p(x,t)}}(Q_T)} \\
& \cdot \left\| \left(|\nabla u_\varepsilon|^{p(x,t)} + |\nabla u|^{p(x,t)} \right)^{\frac{2-p(x,t)}{2}} \right\|_{L^{\frac{2}{2-p(x,t)}}(Q_T)} \\
& \longrightarrow 0.
\end{aligned} \tag{3.13}
$$

Combining (3.12) with (3.13), we have $\nabla u_\varepsilon \to \nabla u$ in $(L^{p(x,t)}(Q_T))^N$. Thus there exists a subsequence of $\{u_\varepsilon\}$, still labeled by $\{u_\varepsilon\}$, such that $\nabla u_\varepsilon \to \nabla u$ a.e. $(x,t) \in Q_T$, furthermore, there holds

$$
a(x,t,\nabla u_\varepsilon)|\nabla u_\varepsilon|^{p(x,t)-2}\nabla u_\varepsilon \to a(x,t,\nabla u)|\nabla u|^{p(x,t)-2}\nabla u,
$$

for a.e. $(x,t) \in Q_T$. By Theorem 2.6, we obtain that $\xi = a(x,t,\nabla u)|\nabla u|^{p(x,t)-2}\nabla u$.

By Fatou's Lemma, we have

$$
\begin{aligned}
& \liminf_{\varepsilon\to 0} \int_0^T \int_\Omega a(x,t,\nabla u_\varepsilon)|\nabla u_\varepsilon|^{p(x,t)} + f(x,t,u_\varepsilon)u_\varepsilon dxdt \\
\geq & \int_0^T \int_\Omega a(x,t,\nabla u)|\nabla u|^{p(x,t)} + f(x,t,u)udxdt.
\end{aligned}
$$

Since $u_\varepsilon(x,T) \rightharpoonup u(x,T)$ weakly in $L^2(\Omega)$, we obtain

$$\liminf_{\varepsilon\to 0}\int_\Omega |u_\varepsilon(x,T)|^2dx \geq \int_\Omega |u(x,T)|^2dx.$$

For all $v \in X(Q_T)$ with $v \geq 0$ for a.e. $(x,t) \in Q_T$, we take $\varphi = v - u_\varepsilon$ as a test function in Definition 2.5, then

$$\begin{aligned}
&\int_0^T\int_\Omega \frac{\partial u_\varepsilon}{\partial t}v + a(x,t,\nabla u_\varepsilon)\left|\nabla u_\varepsilon\right|^{p(x,t)-2}\nabla u_\varepsilon\nabla(v-u_\varepsilon) + f(x,t,u_\varepsilon)(v-u_\varepsilon)\\
&- g(v-u_\varepsilon)dxdt\\
=&\int_0^T\int_\Omega \frac{\partial u_\varepsilon}{\partial t}u_\varepsilon dxdt + \int_0^T\int_\Omega \left|\frac{u_\varepsilon^-}{\varepsilon}\right|^{q(x,t)-2}\frac{u_\varepsilon^-}{\varepsilon}(v-u_\varepsilon)dxdt\\
\geq&\frac{1}{2}\int_\Omega \left|u_\varepsilon(x,T)\right|^2dx - \frac{1}{2}\int_\Omega \left|u_\varepsilon(x,0)\right|^2dx,
\end{aligned}$$

and moreover

$$\begin{aligned}
&\liminf_{\varepsilon\to 0}\int_0^T\int_\Omega \frac{\partial u_\varepsilon}{\partial t}v + a(x,t,\nabla u_\varepsilon)|\nabla u_\varepsilon|^{p(x,t)-2}\nabla u_\varepsilon\nabla v + f(x,t,u_\varepsilon)v - g(v-u_\varepsilon)dxdt\\
\geq&\liminf_{\varepsilon\to 0}\int_0^T\int_\Omega a(x,t,\nabla u_\varepsilon)|\nabla u_\varepsilon|^{p(x,t)} + f(x,t,u_\varepsilon)u_\varepsilon dxdt\\
&+\frac{1}{2}\int_\Omega |u(x,T)|^2dx - \frac{1}{2}\int_\Omega |u_0(x)|^2dx,
\end{aligned}$$

Thus, we obtain

$$\begin{aligned}
&\int_0^T\int_\Omega \frac{\partial u}{\partial t}(v-u)dxdt\\
&+\int_0^T\int_\Omega a(x,t,\nabla u)|\nabla u|^{p(x,t)-2}\nabla u\nabla(v-u) + f(x,t,u)(v-u)dxdt\\
\geq&\int_0^T\int_\Omega g(v-u)dxdt,
\end{aligned}$$

Since $u \in X(Q_T)$ and $\frac{\partial u}{\partial t} \in X'(Q_T)$, by Theorem 2.5, we know that $u \in C(0,T;L^2(\Omega))$. Thus $u \in \mathcal{K}$. $\square$

4 Extinction property of weak solutions

In this section, we study the extinction properties of weak solutions of the parabolic inequality (1.1). We say that the weak solutions of problem (1.1) vanish in a finite time, if there exists a time $T_0 > 0$ such that $\|u(x,t)\|_{L^2(\Omega)} = 0$ as $t \geq T_0$.

Lemma 4.1. *We assume that $g = 0$. Then the weak solutions of the parabolic inequality (1.1) satisfy the following equality*

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u^2(x,t)dx + \int_\Omega a(x,t,\nabla u)|\nabla u|^{p(x,t)} + f(x,t,u)udx = 0, \text{ for a.e. } t \in (0,T).$$

Proof. For every fixed $t \in (0,T)$, Δt small enough such that $t+\Delta t \in (0,T)$, we take $v = u(x,t) \pm u(x,t)\chi_{(0,t)}$ and $v = u(x,t) \pm u(x,t)\chi_{(0,t+\Delta t)}$ as the test functions in Theorem 3.1, respectively, then there holds

$$\int_t^{t+\Delta t} \int_\Omega \frac{\partial u}{\partial \tau} u + a(x,\tau,u)|\nabla u|^{p(x,\tau)} + f(x,\tau,u)u dx d\tau = 0, \tag{4.1}$$

dividing (4.1) by $|\Delta t|$, then

$$\frac{1}{|\Delta t|} \int_t^{t+\Delta t} \int_\Omega \frac{\partial u}{\partial \tau} u + a(x,\tau,u)|\nabla u|^{p(x,\tau)} + f(x,\tau,u)u dx d\tau = 0. \tag{4.2}$$

Since

$$\int_0^T \int_\Omega a(x,t,u)|\nabla u|^{p(x,t)} dx dt < \infty, \quad \int_0^T \int_\Omega f(x,t,u)u dx dt < \infty, \quad \int_0^T \int_\Omega \frac{\partial u}{\partial t} u dx dt < \infty,$$

we have

$$\int_\Omega a(x,t,u)|\nabla u|^{p(x,t)} dx, \int_\Omega f(x,t,u)u dx, \int_\Omega \frac{\partial u}{\partial t} u dx \in L^1(0,T),$$

then for a.e. $t \in (0,T)$ the left-hand side of (4.2) has a limit as $\Delta t \to 0$. Thus, by (4.2), as $\Delta t \to 0$ we have

$$\int_\Omega \frac{\partial u}{\partial t} u + a(x,t,u)|\nabla u|^{p(x,t)} + f(x,t,u)u dx = 0. \tag{4.3}$$

By Theorem 2.5, we get $\int_\Omega u^2(x,t)dx = 2\int_o^t \int_\Omega \frac{\partial u}{\partial \tau} u dx d\tau + \int_\Omega u_0^2(x)dx$. Thus $\int_\Omega u^2(x,t)dx$ is a differentiable function with respect to the time variable t for almost every $t \in (0,T)$ and $\frac{d}{dt}\int_\Omega u^2(x,t)dx = 2\int_\Omega \frac{\partial u}{\partial t} u dx$. From (4.3), this lemma is proved. ☐

Theorem 4.1. *We assume that $p(x,t) \equiv p(x), q(x,t) \equiv q(x), \int_\Omega u_0^2(x)dx > 0$ and $p(x) : \overline{\Omega} \to \mathbb{R}$ is Lipschitz continuous, $q(x) : \overline{\Omega} \to \mathbb{R}$ is global log-Hölder continuous. Let $1 \le q'(x) \le p^*(x) = \frac{Np(x)}{N-p(x)}$ and $\frac{1}{p^+} + \frac{1}{q^+} > 1$ hold, then the weak solutions of evolution variational inequality (1.1) vanish in a finite time.*

Proof. Since $1 \le q'(x) \le p^*(x) = \frac{Np(x)}{N-p(x)}$, there exists a continuous embedding $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$ (see [20]). For each fixed $t \in (0,T)$, by Hölder's inequality, we have

$$\begin{aligned}
\int_\Omega u^2(x,t)dx &\le 2\|u\|_{L^{q'(x)}(\Omega)} \|u\|_{L^{q(x)}(\Omega)} \le C\|\nabla u\|_{L^{p(x)}(\Omega)} \|u\|_{L^{q(x)}(\Omega)} \\
&\le C\max\left\{ \left(\int_\Omega |\nabla u|^{p(x)} + |u|^{q(x)} dx \right)^{\frac{1}{p^-}+\frac{1}{q^-}}, \left(\int_\Omega |\nabla u|^{p(x)} + |u|^{q(x)} dx \right)^{\frac{1}{p^+}+\frac{1}{q^+}} \right\}.
\end{aligned}$$

It follows that

$$\int_\Omega |\nabla u|^{p(x)} + |u|^{q(x)} dx \ge \min\left\{ \left(\frac{1}{C}\int_\Omega u^2(x,t)dx \right)^{\mu_-}, \left(\frac{1}{C}\int_\Omega u^2(x,t)dx \right)^{\mu_+} \right\},$$

where $\mu_- = (\frac{1}{p^-}+\frac{1}{q^-})^{-1}$, $\mu_+ = (\frac{1}{p^+}+\frac{1}{q^+})^{-1}$. By Lemma 4.1, we obtain

$$\frac{d}{dt}\int_\Omega u^2(x,t)dx + 2\min\{a_0,b_0\}\min\left\{\left(\frac{1}{C}\int_\Omega u^2(x,t)dx\right)^{\mu_-}, \left(\frac{1}{C}\int_\Omega u^2(x,t)dx\right)^{\mu_+}\right\} \\ \leq 0,\ \text{a.e. } t\in(0,T), \tag{4.4}$$

If $0<\int_\Omega u_0^2(x)dx \leq C$, then $\int_\Omega u^2(x,t)dx \leq C$. From (4.4), we get

$$\frac{d}{dt}\int_\Omega u^2(x,t)dx + 2\min\{a_0,b_0\}\left(\frac{1}{C}\int_\Omega u^2(x,t)dx\right)^{\mu_+} \leq 0,\ \text{for a.e. } t\in(0,T).$$

Denote $C_1 = 2\min\{a_0,b_0\}(\frac{1}{C})^{\mu_+}$ and $T^* = \sup\{t : 0 < \int_\Omega |u|^2dx \leq C\}$. Integrating the above inequality, we obtain

$$\int_\Omega u^2(x,t)dx \leq \left[\left(\int_\Omega |u_0|^2dx\right)^{1-\mu_+} - C_1(1-\mu_+)t\right]^{\frac{1}{1-\mu_+}},\quad \text{for } t\in(0,T^*).$$

Thus we get that $T^* = \frac{1}{C_1(1-\mu_+)}(\int_\Omega |u_0|^2dx)^{1-\mu_+}$ and $\int_\Omega u^2(x,t)dx \equiv 0$ as $t\geq T^*$.

If $\int_\Omega u_0^2(x)dx > C$, denote $T_1 = \sup\{t : \int_\Omega u^2(x,t)dx > C\}$, then by (4.4), we have

$$\frac{d}{dt}\int_\Omega u^2(x,t)dx + 2\min\{a_0,b_0\}\left(\frac{1}{C}\int_\Omega u^2(x,t)dx\right)^{\mu^-} \leq 0,\quad \text{for a.e. } t\in(0,T_1).$$

We set $C_2 = 2\min\{a_0,b_0\}(\frac{1}{C})^{\mu_-}$. By the above differential inequality, we get

$$\int_\Omega u^2(x,t)dx \leq \left[\left(\int_\Omega u_0^2dx\right)^{1-\mu_-} - C_2(1-\mu_-)t\right]^{\frac{1}{1-\mu_-}},\quad \text{for } t\in(0,T_1).$$

Thus $T_1<\infty$ and $\int_\Omega u^2(x,T_1)\leq C$. From (4.4), we obtain

$$\frac{d}{dt}\int_\Omega u^2(x,t)dx + 2\min\{a_0,b_0\}\left(\frac{1}{C}\int_\Omega u^2(x,t)dx\right)^{\mu_+} \leq 0,\quad \text{for a.e. } t\in[T_1,T).$$

Similar to the case $\int_\Omega u_0^2(x)dx \leq C$, there exists a $T_2\geq T_1$ such that $\int_\Omega u^2(x,t)dx = 0$ for $t\geq T_2$ □

5 Existence of global attractors

In this section, we prove the existence of global attractors for the multi-valued semiflow. For the convenience of the reader, we recall some basic concepts and results related to the theory of global attractors for multi-valued semiflow, see [25, 29] for details.

Definition 5.1. [27, 31] *Let X be a Banach space, the mapping $\Phi : [0,\infty) \to 2^X$ is called an m-semiflow if the following conditions are satisfied*

(1) $\Phi(0,\omega) = \omega$ for arbitrary $\omega \in X$;

(2) $\Phi(t_1+t_2,\omega)\subset\Phi(t_1,\Phi(t_2,\omega))$ *for all* $\omega\in X, t_1,t_2\geq 0$.

It is called a strict semiflow if $\Phi(t_1+t_2,\omega)=\Phi(t_1,\Phi(t_2,\omega))$, *for all* $\omega\in X, t_1,t_2\in\mathbb{R}^+$.

Definition 5.2. [27, 31] *The set* $\mathcal{A}$ *is said to be a global attractor of the* m*-semiflow* Φ *if the following conditions hold*

(1) $\mathcal{A}$ *is negatively semi-invariant, i.e.* $\mathcal{A}\subset\phi(t,\mathcal{A})$ *for arbitrary* $t\geq 0$*;*

(2) $\mathcal{A}$ *is an absorbing set of* Φ*, i.e.* $\operatorname{dist}(\Phi(t,B),\mathcal{A})\to 0$ *as* $t\to\infty$ *for all bounded subset* $B\subset X$*, where* $\operatorname{dist}(\cdot,\cdot)$ *is defined by*

$$\operatorname{dist}(A,B)=\sup_{a\in A}\inf_{b\in B} d_X(a,b) \text{ for } A,B\subset X;$$

(3) If B *is an absorbing set of* Φ*, then* $\mathcal{A}\subset B$*.*

The following theorem gives a sufficient condition for the existence of a global attractor for the m-semiflow Φ.

Theorem 5.1. [27, 31] *Suppose that the* m*-semiflow* Φ *has the following properties*

(1) Φ *is pointwise dissipative, i.e. there exists* $K>0$ *such that for* $u_0\in X$, $u(t)\in\Phi(t,u_0)$ *there holds* $\|u(t)\|_X\leq K$*, for all* $t\geq t_0(\|u_0\|_X)$*;*

(2) $\Phi(t,\cdot)$ *is a closed map for any* $t\geq 0$*, i.e. if* $\xi_n\in\Phi(t,\eta_n)$, $\xi_n\to\xi$, $\eta_n\to\eta$ *then* $\xi\in\Phi(t,\eta)$*;*

(3) Φ *is asymptotically upper semicompact, i.e. if* B *is a bounded set in* X *such that for some* $T(B)$, $\gamma^+_{T(B)}$ *is bounded, for any sequence* $\xi_n\in\Phi(t_n,B)$ *with* $t_n\to\infty$ *is precompact in* X*. Here* $\gamma^+_{T(B)}$ *is the orbits after the time* $T(B)$*.*

Then Φ *has a compact global attractor* $\mathcal{A}$ *in* X*. Moreover, if* Φ *is a strict* m*-semiflow then* $\mathcal{A}$ *is invariant, i.e.* $\Phi(t,\mathcal{A})=\mathcal{A}$ *for any* $t\geq 0$*.*

By Theorem 3.1, we construct the multi-valued mapping as follows

$$\Phi(t,u_0)=\{u(t): u(\cdot) \text{ is the solution of (1.1) corresponding to } \mathrm{u(0)=u_0}\}.$$

By using the same method in [31], we can check that Φ is a strict m-semiflow in the sense of Definition 5.1.

Lemma 5.1. *Suppose that for each* $T>0$, $g(x,t)\in L^{q'(x,t)}(Q_T)$ *with*

$$\sup_{t\geq 0}\int_t^{t+1}\int_\Omega |g(x,t)|^{q'(x,t)}dxdt\leq\mu,$$

where μ *is a positive constant and* $p^-\geq 2, q^-\geq 2$*, then the* m*-semiflow generated by parabolic inequality (1.1) is pointwise dissipative.*

Proof. Similar to Lemma 4.1, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega u^2(x,t)dx+\int_\Omega a(x,t,\nabla u)|\nabla u|^{p(x,t)}+f(x,t,u)udx\\
\le& C\int_\Omega |g(x,t)|^{q'(x,t)}dx+\frac{b_2^0}{2}\int_\Omega |u(x,t)|^{q(x,t)}dx,\ \text{a.e. } t>0.
\end{aligned}$$

Since $q^- \ge 2$, we have

$$\frac{d}{dt}\int_\Omega u^2(x,t)dx+b_2^0\int_\Omega u^2dx\le C+C\int_\Omega |g(x,t)|^{q'(x,t)}dx,\ \text{a.e. } t>0.$$

By Gronwall's inequality, for t sufficiently large, we obtain

$$\begin{aligned}
\int_\Omega u^2(x,t)dx\le& e^{-b_0t}\int_\Omega u_0^2(x)dx+C(1-e^{-b_0t})+C\int_0^t\int_\Omega |g(x,\tau)|^{q'(x,\tau)}e^{b_0\tau}dxd\tau e^{-b_0t}\\
\le& e^{-b_0t}\int_\Omega u_0^2(x)dx+C(1-e^{-b_0t})+C\Bigg(\int_{t-1}^t\int_\Omega |g(x,\tau)|^{q'(x,\tau)}dxd\tau\\
&+\int_{t-2}^{t-1}\int_\Omega |g(x,\tau)|^{q'(x,\tau)}e^{b_0\tau}dxd\tau+\dots\Bigg)e^{-b_0t}\\
\le& e^{-b_0t}\int_\Omega u_0^2(x)dx+C(1-e^{-b_0t})\\
&+C(1+e^{-b_0}+e^{-2b_0}+\dots)\sup_{t\ge 0}\int_t^{t+1}\int_\Omega |g(x,\tau)|^{q'(x,\tau)}dxd\tau.
\end{aligned}$$

Thus there exist constants $K>0$ and $T_0=T_0(\|u_0\|_{L^2(\Omega)})$ such that $\|u(t)\|_{L^2(\Omega)}\le K$ for all $t\ge T_0$. □

Lemma 5.2. *Suppose that $p^-\ge 2, q^-\ge 2$, then $\Phi(\bar t,\cdot): L^2(\Omega)\to L^2(\Omega)$ is a compact mapping for each $\bar t\in(0,T]$.*

Proof. Assume that B is a bounded set in $L^2(\Omega)$ and $\xi_n\in\Phi(\bar t,B)$, $\bar t\in(0,T]$. By the definition of Φ, there exists a sequence $\{u_n\}$ such that u_n is the solution of (1.1) with the initial data belongs to B and $u_n(\bar t)=\xi_n$. Since $u_n\in X(Q_T)$, $\frac{\partial u_n}{\partial t}\in X'(Q_T)$ and $p^-\ge 2, q^-\ge 2$, similarly to Section 2, there exist a subsequence of $\{u_n\}$ still labeled by $\{u_n\}$ and a function u such that $u_n\to u$ strongly in $L^2(Q_T)$. Since $u_n,\ u\in C(0,T;L^2(\Omega))$, we have $u_n(\bar t)\to u(\bar t)$ in $L^2(\Omega)$. □

Theorem 5.2. *Suppose that for each $T>0$, $g(x,t)\in L^{q'(x,t)}(Q_T)$ with*

$$\sup_{t\ge 0}\int_t^{t+1}\int_\Omega |g(x,t)|^{q'(x,t)}dxdt\le\mu,$$

where μ is a positive constant. Let $p(x,t),q(x,t):\Omega\times(0,\infty)\to\mathbb{R}$ be two bounded globally log-Hölder continuous functions satisfying $2\le p^-=\inf_{\Omega\times(0,\infty)}p(x,t)\le p(x,t)\le p^+=\sup_{\Omega\times(0,\infty)}p(x,t)<\infty$, $2\le q^-=\inf_{\Omega\times(0,\infty)}q(x,t)\le q(x,t)\le q^+=\sup_{\Omega\times(0,\infty)}q(x,t)<\infty$, $a(x,t,\xi), f(x,t,\eta)$ are two Carathéodory functions in assumption (H2), then the m-semiflow Φ generated by parabolic inequality (1.1) has an invariant global attractor in $L^2(\Omega)$.

Proof. By Theorem 5.1, we only need to check the hypotheses (2) and (3) in Theorem 5.1. Suppose that $\xi_n \in \Phi(t, \eta_n)$, $\xi_n \to \xi$, $\eta_n \to \eta$, then there exists a sequence $\{u_n\}$ satisfying $u_n(t) = \xi_n, u_n(0) = \eta_n$. Using equation (2.2), we can construct a approximation sequence of u_n, it follows from the same proof as Theorem 3.1 that

$$\begin{cases} u_n(t) \rightharpoonup u(t) \quad \text{weakly in } L^2(\Omega), \text{ for arbitrary } t \in [0,T], \\ u(0) = \eta, \\ \frac{\partial u_n}{\partial t} \rightharpoonup \frac{\partial u}{\partial t} \text{ weakly in } X'(Q_T), \\ \nabla u_n \to \nabla u, \; u_n \to u \quad \text{a.e. } (x,t) \in Q_T, \\ a(x,t,\nabla u_n)|\nabla u_n|^{p(x,t)-2}\nabla u_n \rightharpoonup a(x,t,\nabla u)|\nabla u|^{p(x,t)-2}\nabla u \text{ weakly in } L^{p'(x,t)}(Q_T), \\ f(x,t,u_n) \rightharpoonup f(x,t,u) \text{ weakly in } L^{q'(x,t)}(Q_T). \end{cases}$$

Similar to Section 2, we obtain that $u(t)$ is a solution of the parabolic inequality (1.1) with the initial data $u(0) = \eta$. Thus $\xi \in \Phi(t, \eta)$, that is, $\Phi(t, \cdot)$ is a closed map for any $t \geq 0$.

Suppose that $t_n \to \infty$ and $t_n > t$ for some $t > 0$. Since $\Phi(t_n, B) = \Phi(t + t_n - t, B) \subset \Phi(t, \Phi(t_n - t, B)) \subset \Phi(t, B)$, where B is a bounded set in $L^2(\Omega)$, for $\xi_n \in \Phi(t_n, B)$, we have $\{\xi_n\} \in \Phi(t, B)$. By Lemma 5.2, $\{\xi_n\}$ is precompact in $L^2(\Omega)$. $\square$

Acknowledgment
The author would like to thank the anonymous referee for making valuable suggestions and comments on the original manuscript. This work was supported by the National Natural Science Foundation of China 11371110.

References

[1] V. V. Zhikov, Averaging of functionals of the calculus of variations and elasticity theory, Math. USSR-Izv., 29 (1987), 675–710.

[2] K. R. Rajagopal, M. Růžička, On the modeling of electrorheological materials, Mech. Res. Commun., 23 (1996), 401–407.

[3] Y. M. Chen, S. Levine and M. Rao, Variable exponent linear growth functionals in image restoration, SIAM J. Appl. Math., 66 (2006), 1383–1406.

[4] S. N. Antontsev, S. I. Shmarev, A model porous medium equation with variable exponent of nonlinearity: existence, uniqueness and localization properties of solutions, Nolinear Anal., 60 (2005), 515–545.

[5] V. V. Zhikov, On Lavrentiev's phenomenon, Russ. J. Math. Phys., 3 (1995), 249–269.

[6] V. V. Zhikov, Solvability of the three-dimensional thermistor problem, Proc. Stekolov Inst. Math., 261 (2008), 101–114.

[7] F. Colasuonno, P. Pucci, Multiplicity of solutions for $p(x)$-polyharmonic elliptic Kirchhoff equations, Nonlinear Anal., 74 (2011), 5962–5974.

[8] G. Autuori, F. Colasuonno, P. Pucci, Lifespan estimates for solutions of polyharmonic Kirchhoff systems, Math. Models Methods. Appl. Sci., 22 (2012) 1150009, 36 pages.

[9] Y. Q. Fu, N. Pan, Existence of solutions for nonlinear parabolic problems with $p(x)$-growth, J. Math. Anal. Appl., 362 (2010), 313–326.

[10] Y. Q. Fu, N. Pan, Local Boundedness of Weak Solutions for Nonlinear Parabolic Problem with $p(x)$-Growth, J. Ineq. Appl., (2010) Article ID 163269, 16 pages.

[11] L. Diening, P. Harjulehto, P. Hästö, M. Růžička, Lebesgue and Sobolev spaces with variable exponents, Springer, Berlin, 2011.

[12] Y. Q. Fu, M. Q. Xiang, N. Pan, Regularity of Weak Solutions for Nonlinear Parabolic Problem with $p(x)$-Growth, EJQTDE, 4 (2012), 1–26.

[13] L. Diening, P. Nägele, M. Růžička, Monotone operator theory for unsteady problems in variable exponent spaces, CVEE, (2012), 1209-1231.

[14] S. Antontsev, S. Shmarev, Parabolic equations with anisotropic nonstandard growth conditions, Free Bound. Probl., 60 (2007), 33–44.

[15] S. Antontsev, S. Shmarev, Anistropic parabolic equations with variable nonlinearity, Publ. Mat., 53 (2009), 355-399.

[16] O. M. Buhrii, S. P. Lavrenyuk, On a parabolic variational inequality that generalizes the equation of polytropic filtration, Ukr. Math., 53 (2001), 1027–1042.

[17] O. M. Buhrii, R. A. Mashiyev, Uniqueness of solutions of parabolic variational inequality with variable exponent of nonlinearity, Nonlinear Anal., 70 (2009), 2325–2331.

[18] R. A. Mashiyev, O. M. Buhrii, Existence of solutions of the parabolic variational inequality with variable exponent of nonlinearity, J. Math. Anal. Appl., 377 (2011), 450–463.

[19] O. Kováčik, J. Rákosník, On spaces $L^{p(x)}$ and $W^{k,p(x)}$, Czechoslovak Math. J., 41 (1991), 592–618.

[20] X. L. Fan, D. Zhao, On the spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$, J. Math. Anal. Appl., 263 (2001), 424–446.

[21] J. L. Lions, Queleues Méthodes de Résolution des Problèmes aux Limites Nonlinéaires, Dunod, Paris, 1969.

[22] D. Kinderlehrer, G. Stampacchia, An Introduction to Variational Inequalities, Acad. Press, New York, 1980.

[23] A. Friedman, Variational Principles and Free Boundary Value Problems, Wiley Interscience, New York, 1983.

[24] M. Rudd, K. Schmitt, Variational Inequalities of Elliptic and Parabolic Type, Taiwanese J. Math., 6 (2002), 287–322.

[25] H. Shahgholian, Analysis of the free boundary for the p-parabolic variational problem ($p \geq 2$), Rev. Mat. Iberoamericana, 19 (2003), 797–812.

[26] R. Korte, T. Kuusi, J. Siljander, Obstacle Problem for Nonlinear Parabolic Equations, J. Differential Equations, 246 (2009), 3668-3680.

[27] J. M. Ball, On the asymptotic behavior of generalized processes with applications to nonlinear evolution equations, J. Differential Equations, 27 (1978), 224–265.

[28] J. M. Ball, Continuity properties and global attractor of generalized semiflows and the Navier–Stokes equations, J. Nonlinear Sci., 7 (1997), 475–502.

[29] V. S. Melnik, J. Valero, On attractors of multi-valued semiflows and differential inclusions, Set Valued Anal., 6 (1998), 83–111.

[30] J. W. Cholewa, T. Dlotko, Global Attractors in Abstract Parabolic Problems, Cambridge University Press, Cambridge, 2000.

[31] T. A. Cung, D. K. Tran, On quasilinear parabolic equations involving weighted p-Laplacian operators, Nonlinear Differential Equations Anal., 17 (2010), 195–212.

[32] J. Chabrowski, Y. Q. Fu, Existence of solutions for $p(x)$-Laplacian problems on a bounded domain, J. Math. Anal. Appl., 306 (2005), 604-618. Erratum in: J. Math. Anal. Appl., 323 (2006), 1483.

(Received July 16, 2013)