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On the superlinear Steklov problem involving the $p(x)$-Laplacian

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Abstract. This paper is concerned with the existence and multiplicity of solutions for $p(x)$-Laplacian Steklov problem without the well-known Ambrosetti–Rabinowitz type growth conditions. By means of critical point theorems with Cerami condition, under weaker conditions, existence and multiplicity results of the solutions are proved.

Keywords: variational method, nonstandard growth conditions, generalized Sobolev spaces, Cerami condition.

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1 Introduction

The study of differential equations and variational problems with nonstandard $p(x)$-growth conditions has received more and more interest in recent years. The specific attention accorded to such kinds of problems is due to their applications in mathematical physics. More precisely, such equations are used to model phenomena which arise in elastic mechanics or electrorheological fluids, we can refer the reader to [15]. This kind of problems has been the subject of a sizeable literature and many results have been obtained, see for example [1, 6, 8, 17] and references therein.

In this paper we discuss the existence and multiplicity of solutions for the following Steklov problem involving the $p(x)$-Laplacian

$$\begin{aligned} \Delta_{p(x)} u &= |u|^{p(x)-2} u \quad \text{in } \Omega, \\ |\nabla u|^{p(x)-2} \frac{\partial u}{\partial \nu} &= f(x,u) \quad \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary, $\Delta_{p(x)} u := \operatorname{div}(|\nabla u|^{p(x)-2} \nabla u)$ denotes the $p(x)$-Laplace operator, $p \in C_+(\overline{\Omega}) := \left\{ p \in C(\overline{\Omega}) \ : \ p^- := \inf_{x \in \overline{\Omega}} p(x) > 1 \right\}$ and $f \colon \partial\Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function.

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Define the family of functions

$$\mathcal{F} = \Big\{ G_\gamma \mid G_\gamma(x,t) = f(x,t)t - \gamma F(x,t);\ \ \gamma \in [p^-, p^+] \Big\},$$

where $p^+ := \sup_{x\in\overline{\Omega}} p(x)$.

Noticing that when $p(x) = p$ is a constant, $\mathcal{F} = \{f(x,t)t - pF(x,t)\}$ consists of only one element.

We limit ourselves to the subcritical case, i.e. we assume that

($\mathbf{f_1}$) there exist $c > 0$ and $q \in C_+(\partial\Omega)$ with $q(x) < p^\partial(x)$ for each $x \in \partial\Omega$, such that

$$|f(x,t)| \leq c\Big(1 + |t|^{q(x)-1}\Big)$$

for each $(x,t) \in \partial\Omega \times \mathbb{R}$, where

$$p^\partial(x) := \begin{cases} \frac{(N-1)p(x)}{N-p(x)} & \text{if } p(x) < N \\ \infty & \text{if } p(x) \geq N. \end{cases}$$

Problems like (1.1) have been largely considered in the literature in the recent years. In [8], the authors have studied the case $f(x,u) = \lambda|u|^{p(x)-2}u$. They proved the existence of infinitely many eigenvalue sequences and that unlike the p-Laplacian case, there does not exist a principal eigenvalue and the set of all eigenvalues is not closed under some assumptions. Moreover, they presented some sufficient conditions for the infimum of all eigenvalues to be zero and positive, respectively. In [14], the authors have studied the inhomogeneous Steklov problems involving the p-Laplacian. They studied this class of inhomogeneous Steklov problems in the cases of $p(x) = p = 2$ and of $p(x) = p > 1$, respectively. Recently, in [1] the authors obtained results on existence and multiplicity of solutions for problem (1.1) in the case $q^- > p^+$, under ($\mathbf{f_1}$) and the following conditions:

(**AR**) There exists $M > 0$ and $\theta > p^+$ such that

$$0 < \theta F(x,s) \leq f(x,s)s, \quad |s| \geq M, \quad x \in \overline{\Omega},$$

where $F(x,t) = \int_0^t f(x,s)\,ds$ for $x \in \partial\Omega$ and $t \in \mathbb{R}$.

($\mathbf{f_2}$) $f(x,t) = o(|t|^{p^+-1})$ as $t \to 0$ for $x \in \partial\Omega$ uniformly;

and

($\mathbf{f_3}$) $f(x,-t) = -f(x,t)$ for $(x,t) \in \partial\Omega \times \mathbb{R}$.

Generally, to show the existence of solutions for problems which are superlinear, it is essential to assume the superquadraticity condition (**AR**), which is known as Ambrosetti–Rabinowitz's type condition [2]. It is well known that the main aim of using (**AR**) is to ensure the boundedness of the Palais–Smale type sequences of the corresponding functional. But this condition is very restrictive eliminating many nonlinearities. In fact, there are many functions which are superlinear but do not satisfy (**AR**), see the example in Remark 1.1 below.

As far as we are aware, elliptic problems like (1.1) involving the $p(x)$-Laplacian operator without the (**AR**) type condition, have not yet been studied. That is why, at our best knowledge, the present paper is a first contribution in this direction. In the present paper, we do not use (**AR**) and we know that without (**AR**) it becomes a very difficult task to get the boundedness. So, using a weaker assumption (**g**) below instead of (**AR**) and some variant min-max theorem, which will be reminded in Section 2, we overcome these difficulties.

At first, we will show the existence of a nontrivial weak solution by means of a version of the mountain pass theorem with the Cerami condition [3, 7]. As we will show later, the hypotheses ($\mathbf{f_1}$) and ($\mathbf{f_2}$) imply the mountain pass geometry for the functional corresponding to problem (1.1). To insure the Cerami condition, we introduce some natural growth hypotheses on the nonlinear term in (1.1). More precisely, we assume that the following hold:

($\mathbf{f_4}$) $\liminf_{|t|\to\infty} \frac{f(x,t)t}{|t|^{p^+}} = +\infty$ for $x \in \partial\Omega$ uniformly, i.e., f is p^+-superlinear at infinity

(**g**) There exists a constant $\delta \geq 1$, such that for any $(s,t) \in [0,1] \times \mathbb{R}$, for each $G_\gamma \in \mathcal{F}$ and for all $\eta \in [p^-, p^+]$, the inequality

$$\delta G_\gamma(x,t) \geq G_\eta(x,st) \quad \text{holds for a.e. } x \in \overline{\Omega}.$$

Remark 1.1. Obviously, ($\mathbf{f_4}$) can be derived from (**AR**). However, when $p(x) \equiv 2, \delta = 1$ it is easy to see that function

$$f(x,t) = 2t\log(1+|t|) \tag{1.2}$$

does not satisfy (**AR**), while it satisfies the aforementioned conditions.

Remark 1.2. If $f(x,t)$ is increasing in t, then (**AR**) implies (**g**) when t is large enough. In fact, we can take $\delta = \frac{1}{1-\frac{p^+}{\theta}} > 1$, then

$$\delta G_\gamma(x,t) - G\eta(x,st) \geq f(x,t)t - f(x,st)st \geq 0.$$

But, in general, (**AR**) does not imply (**g**), see [17, Remark 3.4].

Secondly, we will prove under some symmetry condition on the function f that the problem (1.1) possesses infinitely many nontrivial weak solutions. The proof is based on a variant of the fountain theorem [13].

By a weak solution to problem (1.1) we understand a function $u \in X := W^{1,p(x)}(\Omega)$ such that

$$\int_\Omega \left(|\nabla u|^{p(x)-2}\nabla u \nabla v + |u|^{p(x)-2}uv\right) dx - \int_{\partial\Omega} f(x,u)v\, d\sigma = 0, \quad \forall v \in X,$$

where $d\sigma$ is the measure on the boundary.

The energy functional corresponding to problem (1.1) is defined as $I\colon X \to \mathbb{R}$,

$$I(u) = \Phi(u) - \Psi(u),$$

where $\Phi(u) = \int_\Omega \frac{1}{p(x)}(|\nabla u|^{p(x)} + |u|^{p(x)})\, dx$ and $\Psi(u) = \int_{\partial\Omega} F(x,u)\, d\sigma$.

Let us note that under the hypothesis ($\mathbf{f_1}$), the functional I is well defined and of class C^1 and the Fréchet derivative is given by

$$\langle I'(u), v\rangle = \int_\Omega \left(|\nabla u|^{p(x)-2}\nabla u \nabla v + |u|^{p(x)-2}uv\right) dx - \int_{\partial\Omega} f(x,u)v\, d\sigma,$$

for any $u, v \in X$. Moreover, the critical points of I are weak solutions of (1.1).

Our main results are stated as follows.

Theorem 1.3. *Assume that the conditions* $(\mathbf{f_1}), (\mathbf{f_2}), (\mathbf{f_4})$ *and* $(\mathbf{g})$ *are satisfied. If* $q^- > p^+$, *then the problem* (1.1) *has at least one nontrivial solution.*

Theorem 1.4. *Assume that* f *satisfies* $(\mathbf{f_1}), (\mathbf{f_3}), (\mathbf{f_4})$ *and* $(\mathbf{g})$. *If* $q^- > p^+$, *then the problem* (1.1) *possesses infinitely many (pairs) of solutions with unbounded energy.*

The present article is composed of three sections. Section 2 contains some useful results on Sobolev spaces with variable exponents. In particular, we recall a weighted variable exponent Sobolev trace compact embedding theorem and some min-max theorems like mountain pass theorem and fountain theorem with the Cerami condition that will be useful later. The proofs of the main results are given in Section 3.

Throughout the sequel, the letters c, c_i, $i = 1, 2, \dots$, denote positive constants which may vary from line to line but are independent of the terms which will take part in any limit process.

2 Preliminaries

To discuss problem (1.1), we need some theory of variable exponent Lebesgue–Sobolev spaces. For convenience, we only recall some basic facts which will be used later. For details, we refer to [9, 10, 12].

For $p \in C_+(\overline{\Omega})$, we designate the variable exponent Lebesgue space by

$$L^{p(x)}(\Omega) = \left\{ u : \Omega \to \mathbb{R} \text{ is measurable and } \int_\Omega |u(x)|^{p(x)}\, dx < +\infty \right\}$$

equipped with the so-called Luxemburg norm

$$|u|_{p(x)} = \inf\left\{ \lambda > 0 : \int_\Omega \left| \frac{u(x)}{\lambda} \right|^{p(x)} dx \le 1 \right\}.$$

Proposition 2.1. *If* $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ *is a Carathéodory function and satisfies* $|f(x,t)| \le a(x) + b|t|^{\frac{p_1(x)}{p_2(x)}}$ *for any* $(x,t) \in \overline{\Omega} \times \mathbb{R}$, *where* $p_i \in C_+(\overline{\Omega}$, $i = 1, 2$, $a \in L^{p_2(x)}(\overline{\Omega})$, $a(x) \ge 0$ *and* $b \ge 0$ *is a constant, then the Nemytsky operator from* $L^{p_1(x)}(\overline{\Omega})$ *to* $L^{p_2(x)}(\overline{\Omega})$ *defined by* $N_f(u)(x) = f(x, u(x))$ *is a continuous and bounded operator.*

As in the constant exponent case, the generalized Lebesgue–Sobolev space $W^{1,p(x)}(\Omega)$ is defined as

$$W^{1,p(x)}(\Omega) = \left\{ u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega) \right\},$$

with the norm

$$\|u\| = |u|_{p(x)} + |\nabla u|_{p(x)}.$$

With such norms, $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$ are separable, reflexive and uniformly convex Banach spaces.

Proposition 2.2. *Let* $\rho(u) = \int_\Omega \left(|\nabla u|^{p(x)} + |u|^{p(x)} \right) dx$. *For* $u, u_n \in W^{1,p(x)}(\Omega)$, $n = 1, 2, \dots$ *we have*

1. $\rho\left(u \big/ |u|_{p(x)} \right) = 1.$
2. $\|u\| < 1 (= 1, > 1) \iff \rho(u) < 1 (= 1 > 1).$

3. $\|u\| < 1 \Longrightarrow \|u\|^{p^+} \le \rho(u) \le \|u\|^{p^-}$.
4. $\|u\| > 1 \Longrightarrow \|u\|^{p^-} \le \rho(u) \le \|u\|^{p^+}$.
5. *Then the following statements are equivalent to each other.*
   (a) $\lim_{n\to\infty} \|u_n - u\| = 0$.
   (b) $\lim_{n\to\infty} \rho(u_n - u) = 0$.
   (c) $u_n \to u$ *in measure in* Ω *and* $\lim_{n\to\infty} \rho(u_n) = \Phi(u)$.

Let $a\colon \partial\Omega \to \mathbb{R}$ be measurable. Define the weighted variable exponent Lebesgue space by

$$L^{p(x)}_{a(x)}(\partial\Omega) = \left\{ u : \partial\Omega \to \mathbb{R} \text{ is measurable and } \int_{\partial\Omega} |a(x)||u(x)|^{p(x)} d\sigma < +\infty \right\}$$

with the norm

$$|u|_{p(x),a(x)} = \inf\left\{ \tau > 0 : \int_{\partial\Omega} |a(x)|\Big|\frac{u(x)}{\tau}\Big|^{p(x)} d\sigma \le 1 \right\}.$$

Then, $L^{p(x)}_{a(x)}(\partial\Omega)$ is a Banach space.

In particular, when $a(x) \equiv 1$ on $\partial\Omega$, $L^{p(x)}_{a(x)}(\partial\Omega) = L^{p(x)}(\partial\Omega)$ and $|u|_{p(x),a(x)} = |u|_{p(x),\partial\Omega}$.

For $A \subset \overline{\Omega}$, denote by $p^-(A) = \inf_{x\in A} p(x)$, $p^+(A) = \sup_{x\in A} p(x)$. Define

$$p^{\partial}(x) := \begin{cases} \frac{(N-1)p(x)}{N-p(x)} & \text{if } p(x) < N \\ \infty & \text{if } p(x) \ge N. \end{cases}$$

and

$$p^{\partial}_{r(x)}(x) := \frac{r(x)-1}{r(x)} p^{\partial}(x),$$

where $x \in \partial\Omega$, $r \in C(\partial\Omega)$ with $r^-(\partial\Omega) > 1$.

Recall the following embedding theorem.

Theorem 2.3 ([8, Theorem 2.1])**.** *Assume that the boundary of* Ω *possesses the cone property and* $p \in C(\overline{\Omega})$ *with* $p^- > 1$. *Suppose that* $a \in L^{r(x)}(\partial\Omega)$, $r \in C(\partial\Omega)$ *with* $r(x) > \frac{p^{\partial}(x)}{p^{\partial}(x)1}$ *for all* $x \in \partial\Omega$. *If* $q \in C(\partial\Omega)$ *and*

$$1 \le q(x) < p^{\partial}_{r(x)}(x), \quad \forall x \in \partial\Omega.$$

Then, there exists a compact embedding $W^{1,p(x)}(\partial\Omega) \hookrightarrow L^{q(x)}_{a(x)}(\partial\Omega)$. *In particular, there is a compact embedding* $W^{1,p(x)}(\Omega) \hookrightarrow L^{q_0(x)}(\partial\Omega)$ *where* $1 \le q_0(x) < p^{\partial}(x), \forall x \in \partial\Omega$.

Next we give the definition of the Cerami condition which introduced by G. Cerami [4].

Definition 2.4. Let X be a Banach space and $I \in C^1(E,\mathbb{R})$. Given $c \in \mathbb{R}$, we say that I satisfies the Cerami c condition (we denote condition (C_c)), if

(C_1) any bounded sequence $(u_n) \subset E$ such that $I(u_n) \to c$ and $I'(u_n) \to 0$ has a convergent subsequence;

(C_2) there exist constants $\alpha, r, \beta > 0$ such that

$$\|I'(u)\|\|u\| \ge \beta, \quad \forall u \in I^{-1}([c-\alpha, c+\alpha]) \text{ with } \|u\| \ge r.$$

If $I \in C^1(E,\mathbb{R})$ satisfies condition (C_c) for every $c \in \mathbb{R}$, we say that I satisfies condition (C).

Note that condition (C) is weaker than the (PS) condition. However, it was shown in [3, 5] that from condition (C) it can obtain a deformation lemma, which is fundamental in order to get some min-max theorems. More precisely, let us recall the version of the mountain pass lemma with Cerami condition which will be used in the sequel.

Theorem 2.5 (See [3, 7]). *Let X a Banach space, $I \in C^1(E,\mathbb{R})$, $e \in X$ and $r > 0$ be such that $\|e\| > r$ and*

$$\inf_{\|u\|=r} I(u) > I(0) \geq I(e).$$

If I satisfies the condition (C) with

$$c := \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I(\gamma(t)),$$

where $\Gamma := \{\gamma \in C([0,1],E) : \gamma(0) = 0,\ \gamma(1) = e\}$. Then c is a critical value of I.

We also introduce the fountain theorem with the condition (C) which is a variant of [16, 19]. Let X be a reflexive and separable Banach space, from [18], then there are $\{e_j\} \subset E$ and $\{e_j^*\} \subset E^*$ such that

$$X = \overline{\operatorname{span}\{e_j : j = 1,2,\dots\}}, \quad X^* = \overline{\operatorname{span}\{e_j^* : j = 1,2,\dots\}}$$

and

$$\langle e_i^*, e_j\rangle = \delta_{ij},$$

where $\delta_{i,j}$ denotes the Kronecker symbol. For convenience, we write

$$X_j = \operatorname{span}\{e_j\}, \qquad Y_k = \bigoplus_{j=1}^{k} X_j, \qquad Z_k = \bigoplus_{j=k}^{\infty} X_j,$$

and denote

$$S_\alpha = \{u \in X : \|u\| = \alpha\}.$$

Theorem 2.6 ([13]). *Assume that $I \in C^1(X,\mathbb{R})$ is an even functional and satisfies condition (C). For each $k = 1,2,\dots$, there exist $\rho_k > r_k > 0$ such that*

(i) $b_k := \inf\limits_{u\in Z_k\cap S_{r_k}} I(u) \xrightarrow[k\to+\infty]{} +\infty;$

(ii) $a_k := \max\limits_{u\in Y_k\cap S_{\rho_k}} I(u) \leq 0.$

Then, I has a sequence of critical values tending to $+\infty$.

We need the following lemma.

Lemma 2.7. *For $\alpha \in C_+(\partial\Omega)$, $\alpha(x) < p^\partial(x)$ for any $x \in \partial\Omega$, define*

$$\beta_k = \sup_{u\in Z_k\cap S_1} |u|_{\alpha(x),\partial\Omega}.$$

Then $\lim\limits_{k\to\infty} \beta_k = 0$.

Proof. It is clear that the sequence (β_k) is nonincreasing and positive, so (β_k) converges to $l \geq 0$. Let $u_k \in Z_k \cap S_1$ such that $0 \leq l - |u_k|_{L^{p(x)}(\partial\Omega)} \leq \frac{1}{k}$. Passing if necessary to a subsequence, there exists a subsequence, still noted by (u_k), such that (u_k) converges weakly to u in X.

On the other hand, for every $j \in \mathbb{N}$,

$$\langle e_j^*, u\rangle = \lim_k \langle e_j^*, u_k\rangle = 0.$$

Thus, $u = 0$. According to Theorem 2.3, there is a compact embedding of X into $L^{\alpha(x)}(\partial\Omega)$, which assures that (u_k) converges strongly to 0 in $L^{\alpha(x)}(\partial\Omega)$ and finally that $l = 0$. □

3 Proofs of main results

First of all, we start with the following compactness result which plays the most important role.

Lemma 3.1. *Under the assumptions* $(\mathbf{f_1})$, $(\mathbf{f_4})$ *and* $(\mathbf{g})$, *I satisfies the condition* (C).

Proof. First, we show that I satisfies the condition (C_1). Let $(u_n) \subset E$ be bounded such that $I(u_n) \to c$, $c \in \mathbb{R}$ and $I'(u_n) \to 0$. Hence, (u_n) has a weakly convergent subsequence in X. Passing to a subsequence if necessary, still denoted by (u_n), we may assume that $u_n \rightharpoonup u$ in X. In view of $(\mathbf{f_1})$, $\Psi' : E \to E'$ is completely continuous, then $\Psi'(u_n) \to \Psi'(u)$. As $I'(u_n) = \Phi'(u_n) - \Psi'(u_n) \to 0$, we deduce $\Phi'(u_n) \to \Psi'(u)$. By the fact that Φ' is a homeomorphism in view of Proposition 2.5, we obtain $u_n \to u$ in X.

Now check that I satisfies the condition (C_2) too. Arguing by contradiction, let us suppose that there exist $c \in \mathbb{R}$ and $(u_n) \subset E$ satisfying

$$I(u_n) \to c, \quad \|u_n\| \to +\infty \quad \text{and} \quad \|I'(u_n)\|\|u_n\| \to 0. \tag{3.1}$$

Let

$$\overline{p}_n = \frac{\int_\Omega \left(|\nabla u_n|^{p(x)} + |u_n|^{p(x)}\right) dx}{\int_\Omega \frac{1}{p(x)} \left(|\nabla u_n|^{p(x)} + |u_n|^{p(x)}\right) dx}.$$

Choosing $\|u_n\| > 1$, for $n \in \mathbb{N}$, thus

$$c = \lim_n \left(I(u_n) - \frac{1}{\overline{p}_n}\langle I'(u_n), u_n\rangle\right) = \lim_n \left(\frac{1}{\overline{p}_n}\int_{\partial\Omega} f(x, u_n)u_n \, dx - \int_{\partial\Omega} F(x, u_n)\, dx\right). \tag{3.2}$$

Put $v_n = \frac{u_n}{\|u_n\|}$, then $\|v_n\| = 1$. Up to subsequences, for some $v \in E$, we have

$$\begin{aligned} &v_n \rightharpoonup v \quad \text{in } E, \\ &v_n \to v \quad \text{in } L^{p^+}(\Omega), \\ &v_n(x) \to v(x) \quad \text{a.e. in } \Omega. \end{aligned} \tag{3.3}$$

If $v = 0$, as in [11], we can define a sequence $(t_n) \subset \mathbb{R}$, such that

$$I(t_n u_n) = \max_{t\in[0,1]} I(tz_n). \tag{3.4}$$

Fix any $d > 0$, let $w_n = (2p^+ d)^{\frac{1}{p^-}} v_n$. Since $v_n \rightharpoonup v \equiv 0$ and Ψ is weakly continuous, then

$$\lim_n \Psi(w_n) = \lim_n \int_{\partial\Omega} F\left(x, (2p^+ d)^{\frac{1}{p^-}} v_n\right) = 0. \tag{3.5}$$

Then, for n large enough, we have

$$\begin{aligned}
I(t_n u_n) \geq I(w_n) &= \Phi\Big((2p^+ d)^{\frac{1}{p^-}} v_n\Big) - \Psi(w_n) \\
&= \int_\Omega \frac{1}{p(x)}\Big(|\Delta(2p^+ d)^{\frac{1}{p^-}} v_n|^{p(x)} + |(2p^+ d)^{\frac{1}{p^-}} v_n|^{p(x)}\Big)\,dx - \Psi(w_n) \\
&\geq \int_\Omega \frac{1}{p^+}(2p^+ d)\Big(|\Delta v_n|^{p(x)} + |v_n|^{p(x)}\Big)\,dx - \Psi(w_n) \\
&= 2d - \Psi(w_n) \geq d,
\end{aligned}$$

that is,

$$\lim_n I(t_n u_n) = +\infty. \tag{3.6}$$

As $I(0) = 0$ and $I(u_n) \to c$, it follows that $0 < t_n < 1$, when n is large enough. We have,

$$\big\langle \Phi'(t_n u_n) - \Psi'(t_n u_n), t_n u_n \big\rangle = \big\langle I'_{\lambda_n}(t_n u_n), t_n u_n \big\rangle = t_n \frac{d}{dt}\Big|_{t=t_n} I(t u_n) = 0.$$

Thus, from (3.6), we obtain that

$$\begin{aligned}
\left(\frac{1}{\overline{p}_{t_n}}\Psi'(t_n u_n) - \Psi(t_n u_n)\right) &= \frac{1}{\overline{p}_{t_n}}\Phi'(t_n u_n) - \Psi(t_n u_n) \\
&= I(t_n u_n) \to \infty \quad \text{ as } n \to \infty,
\end{aligned} \tag{3.7}$$

where $\overline{p}_{t_n} = \frac{\Phi'(t_n u_n)}{\Phi(t_n u_n)}$.

Let $\gamma_{t_n u_n} = \overline{p}_{t_n}$ and $\gamma_{u_n} = \overline{p}_n$, then $\gamma_{t_n u_n}, \gamma_{u_n} \in [p^-, p^+]$. Hence, $G_{\gamma_{t_n u_n}}, G_{\gamma_{u_n}} \in \mathcal{F}$. Using (**g**), (3.7) and the fact that $\inf_n \frac{\overline{p}_{t_n}}{\overline{p}_n \delta} > 0$, we get

$$\begin{aligned}
\frac{1}{\overline{p}_n}\int_{\partial\Omega}\Big(f(x,u_n)u_n - F(x,u_n)\Big)\,dx &= \frac{1}{\overline{p}_n}\int_{\partial\Omega} G_{\gamma_{u_n}}(x,u_n)\,dx \\
&\geq \frac{1}{\overline{p}_n\delta}\int_{\partial\Omega} G_{\gamma_{t_n u_n}}(x,t_n u_n)\,dx \\
&\geq \frac{\overline{p}_{t_n}}{\overline{p}_n\delta}\left(\frac{1}{\overline{p}_{t_n}}\Psi'(t_n u_n) - \Psi(t_n u_n)\right) \to +\infty,
\end{aligned}$$

which contradicts (3.2).

If $v \neq 0$, from (3.1) and Proposition 2.2, we write

$$\int_\Omega \Big(|\nabla u_n|^{p(x)} + |u_n|^{p(x)}\Big)\,dx - \int_{\partial\Omega} f(x,u_n)u_n\,dx = \langle I'(u_n), u_n\rangle = o(1)\|u_n\|, \tag{3.8}$$

that is,

$$\begin{aligned}
1 - o(1) &= \int_{\partial\Omega} \frac{f(x,u_n)u_n}{\rho(u_n)}\,dx \\
&\geq \int_{\partial\Omega} \frac{f(x,u_n)u_n}{\|u_n\|^{p^+}}\,dx \\
&= \int_{\partial\Omega} \frac{f(x,u_n)u_n}{|u_n|^{p^+}}|v_n|^{p^+}\,dx.
\end{aligned} \tag{3.9}$$

Define the set $\omega_0 = \{x \in \partial\Omega : v(x) = 0\}$. Then, for $x \in \omega \setminus \omega_0 = \{x \in \partial\Omega : v(x) \neq 0\}$, we have $u_n(x) \to +\infty$ as $n \to +\infty$. Hence, by $(\mathbf{f_4})$, we obtain

$$\frac{f(x,u_n)u_n}{|u_n|^{p^+}}|v_n|^{p^+} \to +\infty \quad \text{as } n \to \infty.$$

In view of $|\omega \setminus \omega_0| > 0$, by using Fatou's lemma, we get

$$\int_{\omega\setminus\omega_0} \frac{f(x,u_n)u_n}{|u_n|^{p^+}}|v_n|^{p^+}\,dx \xrightarrow[n\to+\infty]{} +\infty. \tag{3.10}$$

On the other hand, from $(\mathbf{f_1})$ and $(\mathbf{f_4})$, there exists $c > -\infty$ such that $\frac{f(x,t)t}{|t|^{p^+}} \geq c$ for $t \in \mathbb{R}$ and a.e. $x \in \partial\Omega$. Moreover, we have $\int_{\omega_0} |v_n|^{p^+}\,dx \xrightarrow[n\to+\infty]{} 0$. Thus, there exists $m > -\infty$ such that

$$\int_{\omega_0} \frac{f(x,u_n)u_n}{|u_n|^{p^+}}|v_n|^{p^+}\,dx \geq c\int_{\omega_0} |v_n|^{p^+}\,dx \geq m > -\infty. \tag{3.11}$$

Combining (3.9), (3.10) and (3.11), there is a contradiction. This completes the proof of lemma 3.1. □

Proof of Theorem 1.3.

By Lemma 3.1, I satisfies conditions (C) in X. To apply Theorem 2.5, we will show that I possesses the mountain pass geometry.

First, we claim that there exist $\mu, \nu > 0$ such that

$$I(u) \geq \nu, \quad \text{for } u \in X \text{ with } \|u\| = \mu. \tag{3.12}$$

Indeed, since $p^+ < q^- \leq q(x) < p^*(x)$ for all $x \in \partial\Omega$, we have from Theorem 2.3 that $X \hookrightarrow L^{p^+}(\partial\Omega)$ and $X \hookrightarrow L^{q(x)}(\partial\Omega)$ with a continuous and compact embeddings. So, there exist $c_i > 0,\ i = 1,2$ such that

$$|u|_{p^+,\partial\Omega} \leq c_1\|u\| \quad \text{and} \quad |u|_{q(x),\partial\Omega} \leq c_2\|u\|, \quad \forall u \in X. \tag{3.13}$$

Let $\varepsilon > 0$ such that $\varepsilon c_1^{p^+} < \frac{1}{2p^+}$. Using $(\mathbf{f_1})$ and $(\mathbf{f_2})$, it follows that

$$F(x,t) \leq \varepsilon|t|^{p^+} + C(\varepsilon)|t|^{q(x)}, \quad \forall (x,t) \in \partial\Omega \times \mathbb{R}.$$

Therefore, in view (3.13), for $\|u\|$ sufficiently small we get

$$\begin{aligned} I(u) &\geq \frac{1}{p^+}\|u\|^{p^+} - \int_{\partial\Omega} \varepsilon|u|^{p^+}\,d\sigma - \int_{\partial\Omega} C(\varepsilon)|u|^{q(x)}\,d\sigma \\ &\geq \frac{1}{p^+}\|u\|^{p^+} - \varepsilon c_1^{p^+}\|u\|^{p^+} - C(\varepsilon)c_2^{q^-}\|u\|^{q^-} \\ &\geq \|u\|^{p^+}\Big(\frac{1}{2p^+} - C(\varepsilon)c_2^{q^-}\|u\|^{q^- - p^+}\Big). \end{aligned}$$

As $q^- > p^+$, by the standard argument, our claim follows. Next, we affirm that there exists $e \in X \setminus \overline{B_\mu(0)}$ such that

$$I(e) < 0. \tag{3.14}$$

In fact, from $(\mathbf{f_4})$ it follows that for all $M > 0$, there exists a constant $T_M > 0$ depending on M, such that

$$f(x,t) > Mt^{p^+-1} \quad \text{a.e. } x \in \partial\Omega, \quad \forall |t| > T_M.$$

Thus

$$\int_{T_M}^{s} f(x,t)\,dt > \int_{T_M}^{s} Mt^{p^+-1}\,dt = \frac{M}{p^+}(s^{p^+} - T_M^{p^+}), \quad \text{a.e. } x \in \partial\Omega, \quad \forall s > T_M,$$

that is,

$$F(x,s) > \frac{M}{p^+}(s^{p^+} - T_M^{p^+}) + F(x,T_M), \quad \text{a.e. } x \in \partial\Omega, \quad \forall s > T_M.$$

Since $F(x,s)$ is continuous on $\partial\Omega \times [-T_M, T_M]$, there exists a positive constant C_1 such that

$$|F(x,s)| \le C_1 \quad \text{for all } (x,s) \in \partial\Omega \times [-T_M, T_M].$$

Then,

$$F(x,s) \ge \frac{M}{p^+}(s^{p^+} - T_M{}^{p^+}) - C_1 \quad \text{a.e. } x \in \partial\Omega, \quad \forall s \in \mathbb{R}.$$

Hence, for $u_0 \in E$ such that $\|u_0\| = 1$ and $t > 1$ large enough, we obtain

$$I(tu_0) \le \frac{t^{p^+}}{p^-} - \frac{M}{p^+}\int_{\partial\Omega}(t^{p^+}u_0^{p^+} - T_M^{p^+} - C_1)\,d\sigma = \Big(\frac{1}{p^-} - \frac{M}{p^+}|u_0|_{p^+,\partial\Omega}^{p^+}\Big)t^{p^+} + c.$$

As

$$\frac{1}{p^-} - \frac{M}{p^+}|u_0|_{p^+,\partial\Omega}^{p^+} < 0,$$

for $M > 0$ large enough, we deduce

$$I(tu_0) \to -\infty \quad \text{as } t \to +\infty.$$

Thus, there exists $t_0 > 1$ and $e = t_0 u_0 \in X \setminus \overline{B_\mu(0)}$ such that $I(e) < 0$. The statement (3.14) is true.

Finally, in view (3.12), (3.14) and the fact that $I(0) = 0$, I satisfies the mountain pass theorem 2.5. Therefore, I has at least one nontrivial critical point, i.e., (1.1) has a nontrivial weak solution. We are done. □

Proof of Theorem 1.4.

The proof is based on the fountain theorem 2.6. According to Lemma 3.1 and $(\mathbf{f_3})$, I is an even functional and satisfies condition (C). We will prove that if k is large enough, then there exist $\rho_k > r_k > 0$ such that

(i) $b_k := \inf\limits_{u \in Z_k \cap S_{r_k}} I(u) \xrightarrow[k\to+\infty]{} +\infty;$

(ii) $a_k := \max\limits_{u \in Y_k \cap S_{\rho_k}} I(u) \le 0.$

In what follows, we will use the mean value theorem in the following form: for every $\beta \in C_+(\partial\Omega)$ and $u \in L^{\beta(x)}(\partial\Omega)$, there is $\zeta \in \partial\Omega$ such that

$$\int_{\partial\Omega} |u|^{\beta(x)}\,d\sigma = |u|_{\beta(x),\partial\Omega}^{\beta(\zeta)}. \tag{3.15}$$

Indeed, it is well known that there is $\zeta \in \partial\Omega$ such that

$$1 = \int_{\partial\Omega} \left(|u| \Big/ |u|_{\beta(x),\partial\Omega} \right)^{\beta(x)} d\sigma = \int_{\partial\Omega} |u|^{\beta(x)} \, d\sigma \Big/ |u|^{\beta(\zeta)}_{\beta(x),\partial\Omega}.$$

Then, (3.15) holds.

(i) Let $u \in Z_k$ such that $\|u\| = r_k \geq 1$ (r_k will be specified below). Using $(\mathbf{f_1})$ and (3.15), we deduce

$$\begin{aligned}
I(u) &\geq \frac{1}{p^+}\|u\|^{p^-} - c\int_{\partial\Omega} |u|^{q(x)} d\sigma - c_1, \\
&\geq \frac{1}{p^+}\|u\|^{p^-} - c|u|^{q(\zeta)}_{q(x),\partial\Omega} - c_2, \quad \text{ where } \zeta \in \partial\Omega, \\
&\geq \begin{cases} \frac{1}{p^+}\|u\|^{p^-} - c - c_2, & \text{if } |u|_{q(x,\partial\Omega)} < 1; \\ \frac{1}{p^+}\|u\|^{p^-} - c\Big(\beta_k\|u\|\Big)^{q^+} - c_2, & \text{if } |u|_{q(x,\partial\Omega)} > 1; \end{cases} \\
&\geq \frac{1}{p^+}\|u\|^{p^-} - c\beta_k^{q^+}\|u\|^{q^+} - c_3, \\
&= r_k^{p^-}\left(\frac{1}{p^+} - c\beta_k^{q^+} r_k^{q^+-p^-} \right) - c_3.
\end{aligned}$$

We fix r_k as follows

$$r_k := \left(cq^+\beta_k^{q^+} \right)^{1/p^- - q^+}.$$

then,

$$I(u) \geq r_k^{p^-}\left(\frac{1}{p^+} - \frac{1}{q^+} \right).$$

Using Lemma 2.7 and the fact $p^+ < q^+$, it follows $r_k \to +\infty$, as $k \to \infty$. Consequently, $I(u) \to +\infty$ as $\|u\| \to \infty$ with $u \in Z_k$. The assertion (i) is valid.

(ii) Since $\dim Y_k < \infty$ and all norms are equivalent in the finite-dimensional space, there exists $C_k > 0$, for all $u \in Y_k$ with $\|u\| \geq 1$, we have

$$\Phi(u) \leq \frac{1}{p^-}\int_\Omega \left(|\Delta u|^{p(x)} + |u|^{p(x)} \right) dx \leq \frac{1}{p^-}\|u\|^{p^+} \leq C_k|u|^{p^+}_{p^+}. \tag{3.16}$$

Next, from $(\mathbf{f_2})$, there exist $R_k > 0$ such that for $|s| \geq R_k$, we have $F(x,s) \geq 2C_k|s|^{p^+}$. Then, for all $(x,s) \in \partial\Omega \times \mathbb{R}$ we get

$$F(x,s) \geq 2C_k|s|^{p^+} - M_k, \quad \text{where} \quad M_k = \max_{|s|\leq R_k} F(x,s). \tag{3.17}$$

Combining (3.16) and (3.17), for $u \in Y_k$ such that $\|u\| = \rho_k > r_k$, we infer that

$$\begin{aligned}
I(u) &= \Phi(u) - \int_{\partial\Omega} F(x,u)\, d\sigma \\
&\leq -C_k|u|^{p^+}_{p^+} + M_k|\partial\Omega| \\
&\leq -\frac{1}{p^-}\|u\|^{p^+} + M_k|\partial\Omega|.
\end{aligned}$$

Therefore, for ρ_k large enough ($\rho_k > r_k$), we get from the above that

$$a_k := \max_{u\in Y_k\cap S_{\rho_k}} I(u) \leq 0.$$

The assertion (ii) holds. Applying the fountain theorem, we achieve the proof of Theorem 1.4.

□

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