

Two maximum principles for a nonlinear fourth order equation from thin plate theory

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Abstract. We develop two maximum principles for a nonlinear equation of fourth order that arises in thin plate theory. As a consequence, we obtain uniqueness results for the corresponding fourth order boundary value problem under the boundary conditions $w = \Delta w = 0$, as well as some bounds of interest.

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1 Introduction


In the pioneering work [9], Payne introduced a technique, which utilizes a maximum principle for a function defined on solutions to an elliptic differential equation, in order to obtain bounds for the gradient of the solution of the relevant differential equation. Several authors have contributed to the growing literature developing this technique (see the references cited here, especially [23], and the references therein).

This paper employs Payne's technique to treat the following equation that arises in the thin plate theory

$$\Delta(D(x)\Delta w) - (1 - \nu)[D, w] + c(x)f(w) = 0 \quad \text{in } \Omega \subset \mathbb{R}^2, \quad (1.1)$$

where Ω is a bounded domain, $D(x) > 0$ is the flexural rigidity of the plate, $[u, v] = u_{xx}v_{yy} - 2u_{xy}v_{xy} + v_{xx}u_{yy}$, and $0 < \nu < \frac{1}{2}$ is the elastic constant (Poisson ratio) and is defined by $\nu = \lambda/2(\lambda + \mu)$ with material depending constants λ and μ , the so-called Lamé constants. Usually λ and $\mu > 0$ and hence $0 < \nu < \frac{1}{2}$. For metals the value ν is about 0.3. Some exotic materials have a negative Poisson ratio. We have denoted partial derivatives by a subscript and will use the summation convention on repeated indices.

In Section 2, we establish two maximum principles for an auxiliary P function containing the terms $w, |\nabla w|^2, (\Delta w)^2$. We note that Mareno [5, 8] was the first to prove a maximum principle for the equation (1.1).

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Finally, in Section 3 we use these results to prove uniqueness results for classical solutions $C^4(\Omega) \cap C^2(\overline{\Omega})$ and some bounds.

2 Maximum principles

The following maximum principle for second order operators will be useful ([2]).

Theorem 2.1. *Let $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ satisfy the inequality $Lu \equiv \Delta u + \gamma(x)u \geq 0$ in Ω , where $\gamma \geq 0$ in Ω .*

Suppose that Ω lies in the strip of width d , $0 < x_i < d$, for some $i \in \{1, \dots, n\}$ and that

$$\sup_{\Omega} \gamma < \frac{\pi^2}{d^2}. \quad (2.1)$$

Then the function u/φ satisfies a generalized maximum principle in Ω , i.e., there exists a constant $k \in \mathbb{R}$ such that $u/\varphi \equiv k$ in Ω or u/φ does not attain a nonnegative maximum in Ω .

Here

$$\varphi(x) = \cos \frac{\pi(2x_i - d)}{2(d + \varepsilon)} \prod_{j=1}^n \cosh(\varepsilon x_j) \in C^\infty(\overline{\Omega}),$$

where $\varepsilon > 0$ is small.

Similarly, if we replace (2.1) by

$$\sup_{\Omega} \gamma < \frac{4}{d^2 e^2}, \quad (2.2)$$

then u/ψ satisfies a generalized maximum principle in Ω .

Here

$$\psi(x) = 1 - \frac{\gamma d^2}{4} e^{\frac{2x_i}{d}}.$$

We define the function

$$P = \frac{1}{2}D(x)(\Delta w)^2 + C|\nabla w^2| + c(x)F(w),$$

where $F(s) = \int_0^s f(t) dt$, $C > 0$ is a constant and prove the following maximum principle.

Theorem 2.2. *Let $w \in C^4(\Omega)$ be solution of (1.1) and let $c, D \in C^2(\Omega)$, $f \in C^1(\mathbb{R})$. Suppose that the following requirements are satisfied*

$$(a_1) \quad c > 0, F \geq 0,$$

$$(a_2) \quad D/\alpha(1 - 2\nu) + \Delta D \geq (1 - \nu)^2/(1 - 2\nu), \text{ where } \alpha \geq 1 \text{ is a constant,}$$

$$(a_3) \quad D_{ij}^2 \leq C/2, \quad \forall i, j = 1, 2,$$

$$(a_4) \quad cf' + C/\alpha - C^2/D > 0, \quad ((\Delta c + c/\alpha)/|\nabla c|^2)F(cf' + C/\alpha - C^2/D) - f^2 \geq 0 \text{ in } \Omega \times \mathbb{R},$$

$$(a_5) \quad \Omega \text{ lies in the strip of width } \sqrt{\alpha}\pi, \quad 0 < x_i < \sqrt{\alpha}\pi, \text{ for some } i = 1, 2.$$

Then the function P/φ satisfies a generalized maximum principle in Ω .

Here $\varphi(x) = \cos \frac{\pi(2x_i - d)}{2(d + \varepsilon)} \prod_{j=1}^n \cosh(\varepsilon x_j)$.

Similarly, if (a₁)–(a₄) hold with $\alpha > d^2 e^2/4$ and if (a₅) is replaced by

(a₆) Ω lies in the strip of width $d, 0 < x_i < d$, for some $i = 1, 2$,

then the function P/ψ satisfies a generalized maximum principle in Ω . Here $\psi(x) = 1 - \frac{d^2}{4\alpha} e^{\frac{2x_i}{d}}$.

Proof. From equation (1.1) we get

$$\Delta^2 w = -D^{-1}[\Delta D \Delta w + 2D_i \Delta w_i - (1 - \nu)[D, w] + c(x)f(w)]$$

and hence

$$\Delta\left(\frac{1}{2}D(x)(\Delta w)^2\right) = -\frac{1}{2}\Delta D(\Delta w)^2 + (1 - \nu)[D, w]\Delta w + D|\nabla(\Delta w)|^2 - c(x)f(w)\Delta w.$$

Since

$$[D, w] = \Delta D \Delta w - D_{ij}w_{ij},$$

we obtain that

$$\Delta\left(\frac{1}{2}D(x)(\Delta w)^2\right) = \frac{1 - 2\nu}{2}\Delta D(\Delta w)^2 - (1 - \nu)D_{ij}w_{ij}\Delta w + D|\nabla(\Delta w)|^2 - c(x)f(w)\Delta w.$$

A computation shows that

$$\Delta C|\nabla w^2| \geq Cw_{ij}w_{ij} + 2Cw_i(\Delta w)_i,$$

$$\Delta\left(c(x) \int_0^w f(t) dt\right) = \Delta cF(w) + c(x)f'(w)|\nabla w|^2 + 2f(w)c_i w_i + c(x)f(w)\Delta w.$$

Adding and using (a₂) we get

$$\begin{aligned} \Delta P + \frac{P}{\alpha} &\geq \frac{(1 - \nu)^2}{2}(\Delta w)^2 - (1 - \nu)D_{ij}w_{ij}\Delta w + D|\nabla(\Delta w)|^2 + Cw_{ij}w_{ij} \\ &\quad + 2Cw_i(\Delta w)_i + \Delta cF(w) + cf'(w)|\nabla w|^2 + 2f(w)c_i w_i + \frac{C}{\alpha}|\nabla w|^2 + \frac{c}{\alpha}F(w). \end{aligned}$$

We observe that

$$\frac{(1 - \nu)^2}{2}(\Delta w)^2 - (1 - \nu)D_{ij}w_{ij}\Delta w + 2D_{ij}^2 w_{ij}^2 \geq 0.$$

Consequently adding and subtracting $2D_{ij}^2 w_{ij}^2$ in order to complete the square of the first two terms and using the fact that

$$-2D_{ij}^2 w_{ij}^2 \geq -Cw_{ij}w_{ij},$$

we get

$$\begin{aligned} \Delta P + \frac{P}{\alpha} &\geq D|\nabla(\Delta w)|^2 + 2Cw_i(\Delta w)_i + \Delta cF(w) + cf'(w)|\nabla w|^2 + 2f(w)c_i w_i \\ &\quad + \frac{C}{\alpha}|\nabla w|^2 + \frac{c}{\alpha}F(w). \end{aligned}$$

Completing the square of the first two terms see that

$$\Delta P + \frac{P}{\alpha} \geq \Delta cF(w) + \left(cf'(w) + \frac{C}{\alpha} - \frac{C^2}{D}\right)|\nabla w|^2 + 2f(w)c_i w_i + \frac{c}{\alpha}F(w). \quad (2.3)$$

Using the first inequality in (a₄), adding and subtracting $(f^2 c_i c_i)/(c f' + C/\alpha - C^2/D)$ to the previous inequality we are left with

$$\Delta P + \frac{P}{\alpha} \geq \frac{c_i c_i}{c f' + \frac{C}{\alpha} - \frac{C^2}{D}} \left(\frac{\Delta c + c/\alpha}{c_i c_i} F(w) \left(c f'(w) + \frac{C}{\alpha} - \frac{C^2}{D} \right) - f^2 \right) \geq 0 \quad \text{in } \Omega,$$

by the second inequality in (a₄).

The desired proof follows from the generalized maximum principle (Theorem 2.1). \square

Now we assume that $C \leq D/\alpha$ and state a similar result.

Theorem 2.3. *Let $w \in C^4(\Omega)$ be solution of (1.1) and let $c, D \in C^2(\Omega)$, $f \in C^1(\mathbb{R})$. Suppose that the following requirements are satisfied*

$$(b_1) \quad c > 0, \Delta(1/c) \leq 0,$$

$$(b_2) \quad D/\alpha(1 - 2\nu) + \Delta D \geq (1 - \nu)^2/(1 - 2\nu), \text{ where } \alpha \geq 1 \text{ is a constant,}$$

$$(b_3) \quad D_{ij}^2 \leq C/2, C \leq D/\alpha, \forall i, j = 1, 2,$$

$$(b_4) \quad f' > 0, F \geq 0, 2FF'' - (F')^2 \geq 0,$$

$$(b_5) \quad \Omega \text{ lies in the strip of width } \sqrt{\alpha}\pi, 0 < x_i < \sqrt{\alpha}\pi, \text{ for some } i = 1, 2.$$

Then the function P/φ satisfies a generalized maximum principle in Ω , where

$$\varphi(x) = \cos \frac{\pi(2x_i - d)}{2(d + \varepsilon)} \prod_{j=1}^n \cosh(\varepsilon x_j).$$

Similarly, if (b₁)–(b₄) hold with $\alpha > d^2 e^2/4$ and if (b₅) is replaced by

$$(b_6) \quad \Omega \text{ lies in the strip of width } d, 0 < x_i < d, \text{ for some } i = 1, 2,$$

then the function P/ψ satisfies a generalized maximum principle in Ω . Here $\psi(x) = 1 - \frac{d^2}{4\alpha} e^{\frac{2x_i}{d}}$.

Proof. Since $C \leq D/\alpha$ inequality (2.3) reduces to

$$\Delta P + \frac{P}{\alpha} \geq \Delta c F(w) + c f' |\nabla w|^2 + 2f(w) c_i w_i + \frac{c}{\alpha} F(w).$$

Adding and subtracting $(f^2 c_i c_i)/(c f')$ to the previous inequality we get

$$\Delta P + \frac{P}{\alpha} \geq \frac{c_i c_i}{c f'} \left(\frac{c \Delta c}{c_i c_i} F(w) f'(w) - f^2 \right) \geq 0.$$

By (b₁) we get $c \Delta c / c_i c_i \geq 2$ and hence

$$\Delta P + \frac{P}{\alpha} \geq \frac{c_i c_i}{c f'} \left(2FF'' - (F')^2 \right) \geq 0 \quad \text{in } \Omega,$$

and the proof follows. \square

Remarks.

1. The function $D = D_0(1 + 1/(x_2 + 1))^3$ constructed in [5] fulfills the requirements of Theorem 2.3 if $\alpha = 1$. Moreover the requirement (b_4) is satisfied by $F(s) = s^4/4 + s^2/2$.
2. Mareno [5, Theorem 2.2] proved that under the hypotheses:

$$(c_1) \quad c > 0, \Delta(1/c) \leq 0,$$

$$(c_2) \quad \Delta D \geq (1 - \nu)^2/2(1 - 2\nu), \Delta D - 4D^{-1}|\nabla D|^2 \geq 0,$$

$$(c_3) \quad f' > 0, F > 0, FF'' - (F')^2 \geq 0,$$

$$(c_4) \quad f'c > \beta (\beta > 0), \beta \geq D \geq D_{ij}D_{ij},$$

the function

$$R = \frac{1}{2}D(x)(\Delta w)^2 + D(x)|\nabla w|^2 + c(x) \int_0^w f(t) dt$$

takes its maximum value on the boundary of Ω .

Here (Theorem 2.3, case $\alpha = 1$) we imposed a geometric restriction on Ω that allowed us to drop the restriction (c_4) imposed by Mareno [5]. Moreover, Theorem 2.2 works without any sign restriction for f' and Δc .

3 Uniqueness results and bounds

With the aid of the above theorem we can establish the uniqueness results.

Theorem 3.1. *Suppose that we are under the above mentioned hypotheses (c_1) – (c_4) [5, Theorem 2.2]. We also assume that $\partial\Omega \in C^{2+\varepsilon}$, $D \in C^2(\bar{\Omega})$ and*

$$\frac{\partial D}{\partial n} - 2kD < 0 \quad \text{on } \partial\Omega, \quad (3.1)$$

where k is the curvature of $\partial\Omega$.

Then $w \equiv 0$ is the only solution of the boundary value problem

$$\begin{cases} \Delta(D(x)\Delta w) - (1 - \nu)[D, w] + c(x)f(w) = 0 & \text{in } \Omega, \\ w = \Delta w = 0 & \text{on } \partial\Omega. \end{cases} \quad (3.2)$$

Proof. According to Theorem 2.2, [5] the function R attains its maximum value on $\partial\Omega$, at a point x_0 . From Hopf's lemma it follows that $\frac{\partial R}{\partial n} > 0$ at x_0 .

A computation shows that

$$\frac{\partial R}{\partial n} = \frac{1}{2} \frac{\partial D}{\partial n} (\Delta w)^2 + D \Delta w \frac{\partial \Delta w}{\partial n} + \frac{\partial D}{\partial n} |\nabla w|^2 + 2D \frac{\partial w}{\partial n} \frac{\partial^2 w}{\partial n^2} + c f(w) \frac{\partial w}{\partial n} + \frac{\partial c}{\partial n} \int_0^w f(t) dt. \quad (3.3)$$

By introducing normal coordinates in the neighborhood of the boundary, we can write (see [23, p. 46, relation 4.3])

$$\Delta w = \frac{\partial^2 w}{\partial n^2} + \frac{\partial^2 w}{\partial s^2} + k \frac{\partial w}{\partial n} \quad \text{on } \partial\Omega, \quad (3.4)$$

where $\frac{\partial w}{\partial s}$ denotes the tangential derivative of w .

Since $w = \Delta w = 0$ on $\partial\Omega$, relation (3.4) becomes

$$\frac{\partial^2 w}{\partial n^2} = -k \frac{\partial w}{\partial n}. \quad (3.5)$$

We note that from $\int_0^x f(t)dt \geq 0$ and $f' > 0$ it follows that $f(0) = 0$.

Hence, using the boundary conditions, relation (3.5) and the fact that $f(0) = 0$, it follows that

$$\frac{\partial R}{\partial n} = \left(\frac{\partial w}{\partial n} \right)^2 \left(\frac{\partial D}{\partial n} - 2kD \right) \leq 0 \quad \text{on } \partial\Omega.$$

This contradicts Hopf's lemma at the point $x_0 \in \partial\Omega$, where R ($R \neq \text{constant}$) assumes its maximum value. Hence R is constant in $\bar{\Omega}$. Thus $\frac{\partial R}{\partial n} = 0$ on $\partial\Omega$ and consequently $\frac{\partial w}{\partial n} = 0$ on $\partial\Omega$. By the boundary conditions it follows that $R \equiv 0$ in $\bar{\Omega}$. Hence $w \equiv 0$ in $\bar{\Omega}$. \square

Theorem 3.2. *Suppose that we are under the hypotheses of Theorem 2.3. We also assume that $\partial\Omega \in C^{2+\varepsilon}$, $D \in C^2(\bar{\Omega})$, $k \geq 0$ and*

$$2k\varphi + \frac{\partial\varphi}{\partial n} > 0 \quad \text{on } \partial\Omega, \quad (3.6)$$

$$\frac{\partial\varphi}{\partial n} > 0 \quad \text{for some } x_0^* \text{ on } \partial\Omega. \quad (3.7)$$

Then $w \equiv 0$ is the only solution of the boundary value problem (3.2).

A similar uniqueness result holds if we replace φ by ψ in (3.6) and (3.7).

Proof. From Theorem 2.3 it follows that the nonconstant function P/φ attains its maximum value at a point $x_0 \in \partial\Omega$.

The generalized maximum principle, [17, Theorem 10, p. 73] tells us that

$$\frac{\partial(P/\varphi)}{\partial n} > 0 \quad \text{at } x_0. \quad (3.8)$$

A calculation shows that

$$\frac{\partial P}{\partial n} = -2kC \left(\frac{\partial w}{\partial n} \right)^2 \leq 0 \quad \text{on } \partial\Omega, \quad (3.9)$$

since the curvature is supposed to be nonnegative.

Hence

$$\frac{\partial(P/\varphi)}{\partial n} = -\frac{C}{\varphi^2} \left(\frac{\partial w}{\partial n} \right)^2 \left(2k\varphi + \frac{\partial\varphi}{\partial n} \right) \leq 0 \quad \text{on } \partial\Omega,$$

which contradicts (3.8).

It follows from Theorem 2.3 that there exists a constant $\gamma \geq 0$ such that

$$P = \gamma\varphi \quad \text{in } \bar{\Omega}.$$

The case $\gamma > 0$ and (3.7) would imply

$$\frac{\partial P}{\partial n} > 0 \quad \text{at } x_0^*,$$

which contradicts (3.9).

Hence $\gamma = 0$, i.e., $P \equiv 0$ in $\bar{\Omega}$ and the proof follows. \square

Applications.

(a) From Theorem 3.1 we obtain a uniqueness result for convex domains ($k \geq 0$) under the hypothesis $\partial D/\partial n \leq 0$ on $\partial\Omega$.

(b) Suppose that the plate has the shape of the ellipse

$$\partial\Omega : \frac{x_1^2}{\sigma^2} + \frac{(x_2 - d/2)^2}{d^2/4} = 1.$$

We see that relation (3.7) is fulfilled.

In order to get a uniqueness result, it remains to check the validity of (3.6), i.e.,

$$2k\psi + \frac{\partial\psi}{\partial n} > 0 \quad \text{on } \partial\Omega.$$

It suffices to show that

$$2k_{\min}\psi + \frac{\partial\psi}{\partial n} > 0 \quad \text{on } \partial\Omega, \quad (3.10)$$

where $k \geq \min_{\partial\Omega} k = k_{\min} = d/2\sigma^2$.

A computation shows that

$$2k_{\min}\psi + \frac{\partial\psi}{\partial n} = \frac{d}{\sigma^2} + \frac{d^2}{4\alpha} \left(1 - \frac{d}{\sigma^2}\right) e^{2x_2/d} - \frac{dx_2}{2\alpha} e^{2x_2/d} > 0$$

if $\alpha > \sigma^2 e^2 d/2$, where $\sigma^2 \geq d/2$.

Hence, if the conditions (b_1) – (b_4) and (b_6) of Theorem 2.3 hold with

$$\alpha > \sigma^2 e^2 d/2, \quad \text{where } \sigma^2 \geq d/2,$$

then a uniqueness result is valid.

Similarly, if $\partial\Omega$ is the circle of radius $\sigma = d/2$ then the uniqueness result holds if the conditions (b_1) – (b_4) and (b_6) of Theorem 2.3 are satisfied with

$$\alpha > d^3 e^2/8, \quad \text{where } d \geq 4.$$

(c) We note that a uniqueness result holds under a weaker hypothesis on α , namely if $\alpha \geq 1$.

Suppose that $\partial\Omega$ is the ellipse

$$\frac{x_1^2}{\sigma^2} + \frac{(x_2 - \sqrt{\alpha}\pi/2)^2}{\alpha\pi^2/4} = 1.$$

A computation shows that $k_{\min} = \sqrt{\alpha}\pi/2\sigma^2$.

Since the relation (3.7) is fulfilled, we check the validity of (3.6).

Since ε can be chosen small enough, it suffices to show that

$$\frac{2\sqrt{\alpha}\pi}{\sigma^2} \cos t - \frac{2}{\pi} (\sqrt{\alpha}\pi + \varepsilon) t \sin t > 0 \quad \text{on } \left[-\frac{\pi}{2} + \delta, \frac{\pi}{2} - \delta\right], \quad (3.11)$$

where $\delta = \delta(\varepsilon) > 0$ is a small constant.

Inequality (3.11) is valid only if the ellipse is thin, i.e., if σ is small enough.

We note that if the plate has circular shape (i.e., it is a disk of radius $\sqrt{\alpha}\pi/2$) then relation (3.11) does not hold if $\alpha \geq 1$, that is, the uniqueness result fails to be valid. For small values of α the inequality is valid but this result is not of interest.

Theorem 2.3 allows to derive apriori bounds.

If w is a solution of (1.1) in Ω and $\Delta w = 0$ on $\partial\Omega$, then it follows that

$$|\nabla w|^2 \leq \text{const}(\Omega) \max_{\partial\Omega} \left(|\nabla w|^2 + \frac{c(x)}{C} F(w) \right) \quad \text{in } \Omega,$$

where $\text{const}(\Omega)$ is a constant depending only on Ω .

Finally, suppose that w satisfies (1.1) and $w = 0$ on $\partial\Omega$.

Then

$$(\Delta w)^2 \leq \text{const}(\Omega) \max_{\partial\Omega} \left((\Delta w)^2 + \frac{2C}{D(x)} |\nabla w|^2 \right) \quad \text{in } \Omega.$$

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