On the existence of bounded solutions for nonlinear second order neutral difference equations

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Abstract. Using the techniques connected with the measure of noncompactness we investigate the neutral difference equation of the following form

$$\Delta \left(r_n \left(\Delta \left(x_n + p_n x_{n-k} \right) \right)^{\gamma} \right) + q_n x_n^{\alpha} + a_n f(x_{n+1}) = 0,$$

where $x \colon \mathbb{N}_k \to \mathbb{R}$, $a, p, q \colon \mathbb{N}_0 \to \mathbb{R}$, $r \colon \mathbb{N}_0 \to \mathbb{R} \setminus \{0\}$, $f \colon \mathbb{R} \to \mathbb{R}$ is continuous and k is a given positive integer, $\alpha \geq 1$ is a ratio of positive integers with odd denominator, and $\gamma \leq 1$ is ratio of odd positive integers; $\mathbb{N}_k := \{k, k+1, \dots\}$. Sufficient conditions for the existence of a bounded solution are obtained. Also a special type of stability is studied.

Keywords: difference equation, Emden–Fowler equation, measures of noncompactness, Darbo's fixed point theorem, boundedness, stability.

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1 Introduction

As it is well known, difference equations serve as mathematical models in diverse areas, such as economy, biology, physics, mechanics, computer science, finance, see for example [1, 7]. One of such models is the Emden–Fowler equation which originated in the gaseous dynamics in astrophysics and further was used in the study of fluid mechanics, relativistic mechanics, nuclear physics and in the study of chemically reacting systems, see [28]. For the reader's convenience, we note that the background for difference equations theory can be found in numerous well-known monographs: Agarwal [1], Agarwal, Bohner, Grace and O'Regan [2], Agarwal and Wong [3], Elaydi [7], Kelley and Peterson [12], and Kocić and Ladas [13].

In the present paper we study using techniques connected with the measure of noncompactness the existence of a bounded solution and some type of its asymptotic behavior to a nonlinear second order difference equation, which can be viewed as a generalization of a discrete Emden–Fowler equation or else it can be viewed as a second order difference equation

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with memory. This makes a problem which we consider different from those already inves-
tigated via techniques of measure of noncompactness, see for example [25] since we do not
expect a direct connection between a fixed point of a suitably defined operator investigated
on a non-reflexive space l^∞ a solution to the problem under consideration. Indeed, this is
the case. What we obtain is that starting from some index which we define the solution is
taken from the fixed point while the previous terms have to be iterated. This also makes the
definition of the operator different from this which would be used for the problem without
dependence on previous terms. It seems that the method which we sketch here would prove
applicable for several other problems. We also note that due to the type of space which we
use, namely l^∞ we cannot apply standard fixed point techniques such as Banach theorem or
Schauder theorem and related results. We expect that our method would apply for systems of
difference equations. However, what we cannot obtain here is the asymptotic stability of the
solution. We will use axiomatically defined measures of noncompactness as presented in the
paper [5] by Banaś and Rzepka.

The problem we consider is as follows

$$\Delta \left(r_n \left(\Delta \left(x_n + p_n x_{n-k} \right) \right)^\gamma \right) + q_n x_n^\alpha + a_n f \left(x_{n+1} \right) = 0, \tag{1.1}$$

where $\alpha \geq 1$ is a ratio of positive integers with odd denominator, $\gamma \leq 1$ is ratio of odd positive
integers, $x \colon \mathbb{N}_k \to \mathbb{R}$ $a, p, q \colon \mathbb{N}_0 \to \mathbb{R}$, $r \colon \mathbb{N}_0 \to \mathbb{R} \setminus \{0\}$, and $f \colon \mathbb{R} \to \mathbb{R}$ is a locally Lipschitz
function with no further growth assumptions. Here $\mathbb{N}_0 := \{0, 1, 2, \dots\}$, $\mathbb{N}_k := \{k, k+1, \dots\}$
where k is a given positive integer, and $\mathbb{R}$ is a set of all real numbers. By a solution of
equation (1.1) we mean a sequence $x \colon \mathbb{N}_k \to \mathbb{R}$ which satisfies (1.1) for every $n \in \mathbb{N}_k$.

There has been an interest of many authors to study properties of solutions of the second-
order neutral difference equations; see the papers [6, 8–10, 15–18, 21–23, 26, 27] and the refer-
ences therein. The interesting oscillatory results for first order and even order neutral differ-
ence equations can be found in [14], [19] and [20].

2   Preliminaries

Let $(E, \|\cdot\|)$ be an infinite dimensional Banach space. If X is a subset of E, then $\bar{X}$, Conv X
denote the closure and the convex closure of X, respectively. Moreover, we denote by $\mathcal{M}_E$ the
family of all nonempty and bounded subsets of E and by $\mathcal{N}_E$ the subfamily consisting of all
relatively compact sets.

Definition 2.1. A mapping $\mu \colon \mathcal{M}_E \to [0, \infty)$ is called a measure of noncompactness in E if it
satisfies the following conditions:

1^0 $\ker \mu = \{X \in \mathcal{M}_E \colon \mu(X) = 0\} \neq \varnothing$ and $\ker \mu \subset \mathcal{N}_E$,

2^0 $X \subset Y \Rightarrow \mu(X) \leq \mu(Y)$,

3^0 $\mu(\bar{X}) = \mu(X) = \mu(\text{Conv } X)$,

4^0 $\mu(cX + (1-c)Y) \leq c\mu(X) + (1-c)\mu(Y)$ for $0 \leq c \leq 1$,

5^0 If $X_n \in \mathcal{M}_E$, $X_{n+1} \subset X_n$, $X_n = \bar{X}_n$ for $n = 1, 2, 3, \dots$
     and $\lim_{n \to \infty} \mu(X_n) = 0$ then $\cap_{n=1}^\infty X_n \neq \varnothing$.

The following Darbo's fixed point theorem given in [5] is used in the proof of the main result.

Theorem 2.2. *Let M be a nonempty, bounded, convex and closed subset of the space E and let $T\colon M \to M$ be a continuous operator such that $\mu(T(X)) \leq k\mu(X)$ for all nonempty subset X of M, where $k \in [0,1)$ is a constant. Then T has a fixed point in the subset M.*

We consider the Banach space l^∞ of all real bounded sequences $x\colon \mathbb{N}_k \to \mathbb{R}$ equipped with the standard supremum norm, i.e.

$$\|x\| = \sup_{n\in\mathbb{N}_k} |x_n| \quad \text{for } x \in l^\infty.$$

Let X be a nonempty, bounded subset of l^∞, $X_n = \{x_n : x \in X\}$ (it means X_n is a set of n-th terms of any sequence belonging to X), and

$$\operatorname{diam} X_n = \sup\{|x_n - y_n| : x, y \in X\}.$$

We use a following measure of noncompactness in the space l^∞ (see [4])

$$\mu(X) = \limsup_{n\to\infty} \operatorname{diam} X_n.$$

3 Main result

In this section, sufficient conditions for the existence of a bounded solution of equation (1.1) are derived.

Theorem 3.1. *Assume that $a, p, q\colon \mathbb{N}_0 \to \mathbb{R}$, $r\colon \mathbb{N}_0 \to \mathbb{R} \setminus \{0\}$, and $f\colon \mathbb{R} \to \mathbb{R}$. Let*

$$\alpha \geq 1 \text{ be a ratio of positive integers with odd denominator,} \tag{3.1}$$

$$\gamma \in (0,1] \quad \text{be a ratio of odd positive integers,} \tag{3.2}$$

and let k be a fixed positive integer. Assume that

$$f\colon \mathbb{R} \to \mathbb{R} \text{ is a locally Lipschitz function,} \tag{3.3}$$

and that the sequences $r\colon \mathbb{N}_0 \to \mathbb{R} \setminus \{0\}$, $a, q\colon \mathbb{N}_0 \to \mathbb{R}$ satisfy

$$\sum_{n=0}^{\infty} \left| \frac{1}{r_n} \right|^{\frac{1}{\gamma}} \sum_{i=n}^{\infty} |a_i| < +\infty \quad \text{and} \quad \sum_{n=0}^{\infty} \left| \frac{1}{r_n} \right|^{\frac{1}{\gamma}} \sum_{i=n}^{\infty} |q_i| < +\infty. \tag{3.4}$$

Let the sequence $p\colon \mathbb{N}_0 \to \mathbb{R}$ satisfy the following condition

$$-1 < \liminf_{n\to\infty} p_n \leq \limsup_{n\to\infty} p_n < 1. \tag{3.5}$$

Then there exists a bounded solution $x\colon \mathbb{N}_k \to \mathbb{R}$ of equation (1.1).

Proof. Condition (3.5) implies that there exist $n_0 \in \mathbb{N}_0$ and a constant $P \in [0,1)$ such that

$$|p_n| \leq P < 1 \quad \text{for } n \geq n_0. \tag{3.6}$$

Condition (3.4) implies that

$$\sum_{i=0}^{\infty} |a_i| < +\infty,$$

$$\sum_{i=0}^{\infty} |q_i| < +\infty. \tag{3.7}$$

Then there exists $n_1 \in \mathbb{N}_0$ such that $\sum_{i=n_1}^{\infty} |a_i| < 1$. Hence, by (3.2),

$$\sum_{n=n_1}^{\infty} \left(\left| \frac{1}{r_n} \right| \sum_{i=n}^{\infty} |a_i| \right)^{\frac{1}{\gamma}} \leq \sum_{n=n_1}^{\infty} \left| \frac{1}{r_n} \right|^{\frac{1}{\gamma}} \sum_{i=n}^{\infty} |a_i|.$$

The above and condition (3.4) imply that

$$\sum_{n=0}^{\infty} \left(\left| \frac{1}{r_n} \right| \sum_{i=n}^{\infty} |a_i| \right)^{\frac{1}{\gamma}} < +\infty. \tag{3.8}$$

Analogously, we get

$$\sum_{n=n_1}^{\infty} \left(\left| \frac{1}{r_n} \right| \sum_{i=n}^{\infty} |q_i| \right)^{\frac{1}{\gamma}} \leq \sum_{n=n_1}^{\infty} \left| \frac{1}{r_n} \right|^{\frac{1}{\gamma}} \sum_{i=n}^{\infty} |q_i|. \tag{3.9}$$

Recalling that remainder of a series is the difference between the n-th partial sum and the sum of a series, we denote by α_n and by β_n the following remainders

$$\alpha_n = \sum_{j=n}^{\infty} \left(\left| \frac{1}{r_j} \right| \sum_{i=j}^{\infty} |a_i| \right)^{\frac{1}{\gamma}} \quad \text{and} \quad \beta_n = \sum_{j=n}^{\infty} \left(\left| \frac{1}{r_j} \right| \sum_{i=j}^{\infty} |q_i| \right)^{\frac{1}{\gamma}} \tag{3.10}$$

We see, by (3.8) and (3.9) that

$$\lim_{n \to \infty} \alpha_n = 0 \quad \text{and} \quad \lim_{n \to \infty} \beta_n = 0. \tag{3.11}$$

Fix any number $d > 0$. From (3.3), there exists a constant $M_d > 0$ such that

$$|f(x)| \leq M_d \quad \text{for all } x \in [-d, d]. \tag{3.12}$$

Choose a constant C such that

$$0 < C \leq \frac{d - Pd}{\left(2^{\frac{1}{\gamma}-1} (M_d)^{\frac{1}{\gamma}} + 2^{\frac{1}{\gamma}-1} (d^{\alpha})^{\frac{1}{\gamma}} \right)}. \tag{3.13}$$

By condition (3.9) there exists a positive integer n_2 such that

$$\alpha_n \leq C \quad \text{and} \quad \beta_n \leq C \quad \text{for } n \in \mathbb{N}_{n_2}. \tag{3.14}$$

Define set B as follows

$$B := \{ (x_n)_{n=0}^{\infty} : |x_n| \leq d, \text{ for } n \in \mathbb{N}_0 \}.$$

Define a mapping $T\colon B \to l^\infty$ as follows

$$(Tx)_n = \begin{cases} -p_n x_{n-k} - \sum\limits_{j=n}^{\infty} \left(\frac{1}{r_j} \sum\limits_{i=j}^{\infty} \left(a_i f(x_{i+1}) + q_i x_i^\alpha \right) \right)^{\frac{1}{\gamma}}, & \text{for any } n \geq n_3, \\ x_n, & \text{for any } 0 \leq n < n_3, \end{cases} \tag{3.15}$$

where $n_3 = \max\{n_1, n_2\} + k$. Observe that B is a nonempty, bounded, convex and closed subset of l^∞.

We will prove that the mapping T has a fixed point in B. This proof will follow in several subsequent steps.

Step 1. Firstly, we show that $T(B) \subset B$.

We will use classical inequality

$$(a+b)^s \leq 2^{s-1}(a^s + b^s), \quad a, b > 0, \ s \geq 1 \tag{3.16}$$

and the fact $t \to t^{1/\gamma}$ is nondecreasing. If $x \in B$, then for $n < n_3$ $|(Tx)_n| = |x_n| \leq d$ and by (3.15), we get for any $n \geq n_3$

$$|(Tx)_n| \leq |p_n| \, |x_{n-k}| + \left| \sum_{j=n}^{\infty} \left(\frac{1}{r_j} \sum_{i=j}^{\infty} \left(a_i f(x_{i+1}) + q_i x_i^\alpha \right) \right)^{\frac{1}{\gamma}} \right|$$

$$\leq |p_n| \, |x_{n-k}| + \sum_{j=n}^{\infty} \left(\left| \frac{1}{r_j} \sum_{i=j}^{\infty} \left(a_i f(x_{i+1}) + q_i x_i^\alpha \right) \right| \right)^{\frac{1}{\gamma}}$$

$$\leq |p_n| \, |x_{n-k}| + \sum_{j=n}^{\infty} \left(\left| \frac{1}{r_j} \right| \sum_{i=j}^{\infty} \left(|a_i| \, |f(x_{i+1})| + |q_i| \, |x_i|^\alpha \right) \right)^{\frac{1}{\gamma}}.$$

From (3.12), taking into account that $x_{n-k} \in B$, and because of $x_i \in B$ we have $|x_i|^\alpha \leq d^\alpha$. Thus

$$|(Tx)_n| \leq |p_n| \, d + \sum_{j=n}^{\infty} \left(\left| \frac{1}{r_j} \right| \sum_{i=j}^{\infty} \left(|a_i| \, M_d + |q_i| \, d^\alpha \right) \right)^{\frac{1}{\gamma}}$$

$$\leq |p_n| \, d + \sum_{j=n}^{\infty} \left(\left| \frac{1}{r_j} \right| \left(\sum_{i=j}^{\infty} |a_i| \, M_d + \sum_{i=j}^{\infty} |q_i| \, d^\alpha \right) \right)^{\frac{1}{\gamma}}.$$

By inequality (3.16), we have

$$|(Tx)_n| \leq |p_n| \, d + 2^{\frac{1}{\gamma}-1} \sum_{j=n}^{\infty} \left(\left(\left| \frac{1}{r_j} \right| \sum_{i=j}^{\infty} |a_i| \, M_d \right)^{\frac{1}{\gamma}} + \left(\left| \frac{1}{r_j} \right| \sum_{i=j}^{\infty} |q_i| \, d^\alpha \right)^{\frac{1}{\gamma}} \right)$$

$$\leq |p_n| \, d + 2^{\frac{1}{\gamma}-1} (M_d)^{\frac{1}{\gamma}} \sum_{j=n}^{\infty} \left(\left| \frac{1}{r_j} \right| \sum_{i=j}^{\infty} |a_i| \right)^{\frac{1}{\gamma}} + 2^{\frac{1}{\gamma}-1} (d^\alpha)^{\frac{1}{\gamma}} \sum_{j=n}^{\infty} \left(\left| \frac{1}{r_j} \right| \sum_{i=j}^{\infty} |q_i| \right)^{\frac{1}{\gamma}}.$$

By using (3.6), (3.14) and (3.13), we estimate

$$|(Tx)_n| \leq Pd + 2^{\frac{1}{\gamma}-1} (M_d)^{\frac{1}{\gamma}} C + 2^{\frac{1}{\gamma}-1} (d^\alpha)^{\frac{1}{\gamma}} C$$

$$\leq Pd + \left(2^{\frac{1}{\gamma}-1} (M_d)^{\frac{1}{\gamma}} + 2^{\frac{1}{\gamma}-1} (d^\alpha)^{\frac{1}{\gamma}} \right) \frac{d - Pd}{\left(2^{\frac{1}{\gamma}-1} (M_d)^{\frac{1}{\gamma}} + 2^{\frac{1}{\gamma}-1} (d^\alpha)^{\frac{1}{\gamma}} \right)} = d. \tag{3.17}$$

From the above, we have estimation

$$|(Tx)_n| \leq d, \quad \text{for } n \in \mathbb{N}_{n_3}. \tag{3.18}$$

Step 2. *T* is continuous

By assumption (3.3), (3.7), and by definition of B, there exists a constant $c^* > 0$ such that

$$\sum_{i=j}^{\infty} |a_i f(x_{i+1}) + q_i x_i^\alpha| \leq c^*$$

for all $x \in B$. From (3.2), $t \to t^{1/\gamma}$ is locally Lipschitz then it is Lipschitz on closed and bounded intervals. Hence, there exists a constant L_γ such that

$$\left| t^{1/\gamma} - s^{1/\gamma} \right| \leq L_\gamma |t - s| \quad \text{for all } t, s \in [-c^*, c^*]. \tag{3.19}$$

From (3.3), function f is Lipschitz on $[-d, d]$. So, there is a constant $L_d > 0$ such that

$$|f(x) - f(y)| \leq L_d |x - y| \tag{3.20}$$

for all $x, y \in [-d, d]$. From (3.1), $x \to x^\alpha$ is also Lipschitz on $[-d, d]$. Then there is a constant L_α such that

$$|x^\alpha - y^\alpha| \leq L_\alpha |x - y| \quad \text{for all } x, y \in [-d, d]. \tag{3.21}$$

Let $(y^{(p)})$ be a sequence in B such that $\|y^{(p)} - x\| \to 0$ as $p \to \infty$. Since B is closed, $x \in B$. By (3.15) and (3.17), we get for all $n \geq n_3$

$$\left| (Tx)_n - (Ty^{(p)})_n \right| \leq |p_n| \left| x_{n-k} - y_{n-k}^{(p)} \right|$$
$$+ \sum_{j=n}^{\infty} \left| \frac{1}{r_j} \right|^{\frac{1}{\gamma}} \left| \left(\sum_{i=j}^{\infty} \left(a_i f(x_{i+1}) + q_i (x_i)^\alpha \right) \right)^{\frac{1}{\gamma}} - \left(\sum_{i=j}^{\infty} \left(a_i f(y_{i+1}^{(p)}) + q_i \left(y_i^{(p)} \right)^\alpha \right) \right)^{\frac{1}{\gamma}} \right|.$$

From (3.19), we have for all $n \geq n_3$

$$\left| (Tx)_n - (Ty^{(p)})_n \right| \leq |p_n| \left| x_{n-k} - y_{n-k}^{(p)} \right|$$
$$+ \sum_{j=n}^{\infty} \left| \frac{1}{r_j} \right|^{\frac{1}{\gamma}} L_\gamma \left| \sum_{i=j}^{\infty} a_i f(x_{i+1}) + \sum_{i=j}^{\infty} q_i (x_i)^\alpha - \sum_{i=j}^{\infty} a_i f(y_{i+1}^{(p)}) - \sum_{i=j}^{\infty} q_i \left(y_i^{(p)} \right)^\alpha \right|$$
$$\leq |p_n| \left| x_{n-k} - y_{n-k}^{(p)} \right| + L_\gamma \sum_{j=n}^{\infty} \left| \frac{1}{r_j} \right|^{\frac{1}{\gamma}} \sum_{i=j}^{\infty} |a_i| \left| f(x_{i+1}) - f(y_{i+1}^{(p)}) \right|$$
$$+ L_\gamma \sum_{j=n}^{\infty} \left| \frac{1}{r_j} \right|^{\frac{1}{\gamma}} \sum_{i=j}^{\infty} |q_i| \left| (x_i)^\alpha - \left(y_i^{(p)} \right)^\alpha \right|.$$

Hence, by (3.20) and (3.21), we obtain for all $n \geq n_3$

$$\left| (Tx)_n - (Ty^{(p)})_n \right| \leq |p_n| \left| x_{n-k} - y_{n-k}^{(p)} \right|$$
$$+ L_\gamma L_d \sum_{j=n}^{\infty} \left| \frac{1}{r_j} \right|^{\frac{1}{\gamma}} \sum_{i=j}^{\infty} |a_i| \left| x_{i+1} - y_{i+1}^{(p)} \right| + L_\gamma L_\alpha \sum_{j=n}^{\infty} \left| \frac{1}{r_j} \right|^{\frac{1}{\gamma}} \sum_{i=j}^{\infty} |q_i| \left| x_i - y_i^{(p)} \right|$$
$$\leq \sup_{i \in \mathbb{N}_0} \left| y_i^{(p)} - x_i \right| \left(|p_n| + L_\gamma L_d \sum_{j=n}^{\infty} \left| \frac{1}{r_j} \right|^{\frac{1}{\gamma}} \sum_{i=j}^{\infty} |a_i| + L_\gamma L_\alpha \sum_{j=n}^{\infty} \left| \frac{1}{r_j} \right|^{\frac{1}{\gamma}} \sum_{i=j}^{\infty} |q_i| \right).$$

Moreover,

$$\forall\, 0 \le n < n_3 \quad \left| (Tx)_n - (Ty^{(p)})_n \right| \le \left\| y^{(p)} - x \right\|$$

Thus, by (3.4) and (3.5), we have

$$\lim_{p \to \infty} \left\| Ty^{(p)} - Tx \right\| = 0 \quad \text{as} \quad \lim_{p \to \infty} \left\| y^{(p)} - x \right\| = 0.$$

This means that T is continuous.

Step 3. Comparison of the measure of noncompactness

Now, we need to compare a measure of noncompactness of any subset X of B and $T(X)$. Let us fix any nonempty set $X \subset B$. Take any sequences $x, y \in X$. Following the same calculations which led to the continuity of the operator T we see that

$$\forall\, n \ge n_3 \quad |(Tx)_n - (Ty)_n| \le |p_n|\,|x_{n-k} - y_{n-k}| + L_\gamma L_\alpha \beta_n\,|x_n - y_n| + L_\gamma L_d \alpha_n\,|x_{n+1} - y_{n+1}|.$$

Taking sufficiently large n, by (3.10) and (3.11), we get

$$L_\gamma L_d \alpha_n \le c_1 < \frac{1-P}{4} \quad \text{and} \quad L_\gamma L_\alpha \beta_n \le c_2 < \frac{1-P}{4}$$

Here c_1, c_2 are some real constants. From (3.6), we have

$$P + c_1 + c_2 < \frac{1+P}{2}.$$

We see that exists n_5 such that

$$\forall\, n \ge n_5 \quad \operatorname{diam}(T(X))_n \le P \operatorname{diam} X_{n-k} + c_1 \operatorname{diam} X_n + c_2 \operatorname{diam} X_{n+1}.$$

This yields by the properties of the upper limit that

$$\limsup_{n \to \infty} \operatorname{diam}(T(X))_n \le P \limsup_{n \to \infty} \operatorname{diam} X_{n-k} + c_1 \limsup_{n \to \infty} \operatorname{diam} X_n + c_2 \limsup_{n \to \infty} \operatorname{diam} X_{n+1}.$$

From the above, for any $X \subset B$, we have $\mu(T(X)) \le (c_1 + c_2 + P)\,\mu(X)$.

Step 4. Relation between fixed points and solutions

By Theorem 2.2 we conclude that T has a fixed point in the set B. It means that there exists $x \in B$ such that

$$x_n = (Tx)_n.$$

Thus

$$x_n = -p_n x_{n-k} - \sum_{j=n}^{\infty} \left(\frac{1}{r_j} \sum_{i=j}^{\infty} (a_i f(x_{i+1}) + q_i x_i^\alpha) \right)^{\frac{1}{\gamma}}, \quad \text{for } n \in \mathbb{N}_{n_3} \tag{3.22}$$

To show that there exists a correspondence between fixed points of T and solutions to (1.1) we apply operator Δ to both sides of the following equation

$$x_n + p_n x_{n-k} = -\sum_{j=n}^{\infty} \left(\frac{1}{r_j} \sum_{i=j}^{\infty} (a_i f(x_{i+1}) + q_i x_i^\alpha) \right)^{\frac{1}{\gamma}},$$

which is obtained from (3.22). We find that

$$\Delta(x_n + p_n x_{n-k}) = \left(\frac{1}{r_n} \sum_{i=n}^{\infty} (a_i f(x_{i+1}) + q_i x_i^\alpha) \right)^{\frac{1}{\gamma}}, \quad n \in \mathbb{N}_{n_3}.$$

and next

$$\left(\Delta\left(x_n + p_n x_{n-k}\right)\right)^\gamma = \frac{1}{r_n} \sum_{i=n}^{\infty}\left(a_i f(x_{i+1}) + q_i x_i^\alpha\right), \quad n \in \mathbb{N}_{n_3}.$$

Taking operator Δ again to both sides of the above equation we obtain

$$\Delta\left(r_n\left(\Delta\left(x_n + p_n x_{n-k}\right)\right)^\gamma\right) = -a_n f(x_{n+1}) - q_n x_n^\alpha, \quad n \in \mathbb{N}_{n_3}.$$

So, we get equation (1.1) for $n \in \mathbb{N}_{n_3}$. Sequence x, which is a fixed point of mapping T, is a bounded sequence which fulfills equation (1.1) for $n \geq n_3$. If $n_3 \geq k$ the proof is ended. We find previous $n_3 - k + 1$ terms of sequence x by formula

$$x_{n-k+l} = \frac{1}{p_{n+l}}\left(-x_{n+l} + \sum_{j=n+l}^{\infty}\left(\frac{1}{r_j}\sum_{i=j}^{\infty}\left(a_i f(x_{i+1}) + q_i x_i^\alpha\right)\right)^{\frac{1}{\gamma}}\right),$$

where $l \in \{0, 1, 2, \ldots, k-1\}$, which results leads directly from (1.1). It means that equation (1.1) has at least one bounded solution $x \colon \mathbb{N}_k \to \mathbb{R}$.

This completes the proof. $\square$

Now, we give an example of equation which can be considered by our method.

Example 3.2. Take $k = 3$, an arbitrary C^1 function $f \colon \mathbb{R} \to \mathbb{R}$ and consider the following problem

$$\Delta\left((-1)^n \Delta\left(x_n + \frac{1}{2}x_{n-3}\right)^{1/3}\right) + \frac{1}{2^n}\left((x_n)^5 + f(x_{n+1})\right) = 0. \tag{3.23}$$

Taking $\gamma = \frac{1}{3}$, $\alpha = 5$, $r_n = (-1)^n$, $p_n = \frac{1}{2}$, $a_n = q_n = \frac{1}{2^n}$ with $f(x) = x^{\frac{5}{3}}$ we see that $x_n = (-1)^n$ is a bounded solution to (3.23).

Remark 3.3. We note that the previous terms of the solution sequence are not obtained through a fixed point method, but through backward iteration. It is common that one has a 1–1 correspondence between fixed points to a suitably chosen operator and solutions to the problem under consideration. Here we get as a fixed point solution some sequence which, starting from some index, is a solution to the given problem and in which the first terms must be iterated. This procedure must be applied since we see that in equation (1.1) we have to know also earlier terms in order to start iteration; this is the so called iteration with memory. We recall that in recent works concerning application of the measure of noncompactness to discrete equations, only problems without memory have been considered. That is why we had to alter to established procedure to overcome the difficulty arising in this problem. We believe our method would be applicable for several other problems.

4   A special type stability

The type of stability investigated in this paper is contained in the following theorem.

Theorem 4.1. *Assume that*

$$q_n \equiv 0, \tag{4.1}$$

and conditions (3.2), (3.4) *and* (3.5) *hold. Assume further that there exists a positive constant D such that*

$$|f(u) - f(v)| \leq D\,|u - v|$$

for any $u, v \in \mathbb{R}$. Then equation (1.1) has at least one solution $x \colon \mathbb{N}_k \to \mathbb{R}$ with the following stability property: given any other solution $y \colon \mathbb{N}_k \to \mathbb{R}$ and $\varepsilon > 0$ there exists $n_4 > n_3$ such that for every $n \geq n_4$ the following inequality holds

$$|x_n - y_n| \leq \varepsilon.$$

Proof. From Theorem 3.1, equation (1.1) has at least one bounded solution $x \colon \mathbb{N}_0 \to \mathbb{R}$ which can be rewritten in the form

$$x_n = (Tx)_n,$$

where mapping T is defined by (3.15) for $n \geq n_3$. From the above and condition (4.1), we see that

$$|x_n - y_n| = |(Tx)_n - (Ty)_n|$$

$$\leq |p_n|\,|x_{n-k} - y_{n-k}| + L_\gamma D \sum_{j=n}^{\infty} \left|\frac{1}{r_j}\right|^{\frac{1}{\gamma}} \sum_{i=j}^{\infty} |a_i|\,|x_{i+1} - y_{i+1}|.$$

Note that for n large enough, say $n \geq n_4 \geq n_3$, we have

$$\vartheta := |p_n| + L_\gamma D \sum_{j=n}^{\infty} \left|\frac{1}{r_j}\right|^{\frac{1}{\gamma}} \sum_{i=j}^{\infty} |a_i| < 1.$$

Let us denote

$$\limsup_{n \to \infty} |x_n - y_n| = l,$$

and observe that

$$\limsup_{n \to \infty} |x_n - y_n| = \limsup_{n \to \infty} |x_{n-k} - y_{n-k}| = \limsup_{n \to \infty} |x_{n+1} - y_{n+1}|.$$

Thus, from the above, we have

$$l \leq \vartheta \cdot l.$$

This means that $\limsup_{n \to \infty} |x_n - y_n| = 0$. This completes the proof since for $\varepsilon > 0$ there exists $n_4 \in \mathbb{N}_0$ such that for every $n \geq n_4 \geq n_3$ the following inequality holds

$$|x_n - y_n| \leq \varepsilon.$$

$\square$

Now, we give another example.

Example 4.2. Take $k = 3$, an arbitrary C^1 function $f \colon \mathbb{R} \to \mathbb{R}$ and consider the following problem

$$\Delta\left((-1)^n \Delta\left(x_n + \frac{1}{2}x_{n-3}\right)^{1/3}\right) + \frac{1}{2^n}f(x_{n+1}) = 0. \tag{4.2}$$

Taking $\gamma = \frac{1}{3}$, $r_n = (-1)^n$, $p_n = \frac{1}{2}$, $a_n = \frac{1}{2^n}$, $q_n = 0$ with $f(x) = -x + \sin(\frac{\pi}{2}x)$ we see that $x_n = (-1)^n$ is a bounded solution to (4.2). By Theorem 4.1, this solution has the stability property.

5 Comments

In [25], the authors consider a special type of problem (1.1), namely they investigate the existence of a solution and Lyapunov type stability to the following equation

$$\Delta \left(r_n \Delta x_n \right) = a_n f(x_{n+1}). \tag{5.1}$$

Their main assumption is the linear growth assumption on nonlinear term f. More precisely, they assume that there exists a positive constant M such that $|f(x_n)| \leq M |x_n|$ for all $x \in N_0$. Using ideas developed in this paper we get the following result.

Corollary 5.1. *Assume that* $f \colon \mathbb{R} \to \mathbb{R}$ *satisfies the condition* (3.3) *and the sequences* $r \colon \mathbb{N}_0 \to \mathbb{R} \setminus \{0\}$, $a \colon \mathbb{N}_0 \to \mathbb{R}$ *are such that*

$$\sum_{n=0}^{\infty} \left| \frac{1}{r_j} \right| \sum_{i=n}^{\infty} |a_i| < +\infty.$$

Then, there exists a bounded solution $x \colon \mathbb{N}_0 \to \mathbb{R}$ *of equation* (5.1).

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