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Qualitative Properties of Nonlinear Volterra Integral Equations

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Abstract

In this article, the contraction mapping principle and Liapunov's method are used to study qualitative properties of nonlinear Volterra equations of the form

$$x(t) = a(t) - \int_0^t C(t,s)g(s,x(s))\, ds, t \geq 0.$$

In particular, the existence of bounded solutions and solutions with various L^p properties are studied under suitable conditions on the functions involved with this equation.

1 Introduction.

Two interesting papers that motivated us to write this article are [5, 6] of Burton. In these two papers the author considered the scalar linear integral equation

$$x(t) = a(t) - \int_0^t C(t,s)x(s)\, ds, t \geq 0, \tag{1.1}$$

and studied the boundedness and various L^p properties of its solutions.

To study qualitative behavior of solutions of equation (1.1), researchers generally assume the forcing function $a(t)$ to be bounded. The most remarkable aspect of Burton's work in these papers is that the function $a(t)$ can be unbounded.

In the present article, we study boundedness and L^p properties of the scalar nonlinear integral equation

$$x(t) = a(t) - \int_0^t C(t,s)g(s,x(s))\, ds, t \geq 0, \tag{1.2}$$

under suitable conditions on the functions a, c, and g. We show the existence of bounded solutions of (1.2) in Section 2 employing the contraction mapping principle and the resolvent function. In our work, we assume that the resolvent function is integrable, which is an important property by itself. The literature on the resolvent and the contraction mapping principle is massive. Becker [1], Burton [2, 3], Corduneanu [7, 8], Eloe and Islam [9], Eloe *et al* [10], Gripenberg [11], Gripenberg *et al* [12], Hino and Murakami [14], Miller [17], and Zhang [20] contain many interesting studies on resolvents including the integrability property. Burton [2, 3, 4], Grossman and Miller [13], Islam and Raffoul [15], and Raffoul [18] contain various studies involving the application of the contraction mapping principle on integral and integrodifferential equations.

In Section 3, we study L^p properties of solutions of (1.2) using a technique we call 'Liapunov's method for integral equation,' which is outlined in [6]. The literature on the Liapunov method is also huge. Classical theory and many examples of Liapunov's method are found, for example, in Burton [2, 3].

2 Bounded Solutions, Contraction Principle.

In this section, we show the existence of bounded solutions of (1.2) using the contraction mapping principle. In Theorems 2.1 and 2.2, we assume $a(t)$ is bounded. Then in Theorems 2.3 and 2.4, we assume $a'(t)$ is bounded where $a(t)$ can be unbounded. In Theorems 2.2 and 2.4, we use the resolvent function in the analysis, and in all theorems in this section except Theorem 2.1, we assume there exists a function h such that $g(t,x) = x + h(t,x)$ where h satisfies the following property.

(H1) $h(t,0) = 0$, and there is a $k > 0$ such that for each $(t,x,y) \in R^+ \times R \times R$, we have

$$|h(t,x) - h(t,y)| \le k|x-y|.$$

Throughout this section, we assume the functions a, g, and C are continuous with respect to their arguments.

The result of our first theorem, Theorem 2.1., exists in the literature in various forms. We start with this theorem because the basic method involving the contraction mapping that is used in this theorem is carried out in all other theorems throughout this section.

Theorem 2.1. Suppose g satisfies the following properties. $g(t,0) = 0$, and there is a $k > 0$ such that for each $(t,x,y) \in R^+ \times R \times R$, $|g(t,x) - g(t,y)| \le k|x-y|$. Assume $a(t)$ is bounded and

$$\sup_{t\ge 0} k \int_0^t |C(t,s)|\, ds \le \alpha < 1.$$

Then there exists a unique bounded continuous solution of (1.2).

Proof. Let M be the Banach space of bounded continuous functions on $[0,\infty)$ with the supremum norm, $||\cdot||$, where $||x|| = \sup_{t\geq 0}|x(t)|$. For each $\phi \in M$, define

$$(T\phi)(t) = a(t) - \int_0^t C(t,s)g(s,\phi(s))\,ds, t \geq 0.$$

We shall show that $T : M \to M$ is a contraction map. Therefore a fixed point of T is a solution of (1.2). It follows from the continuity assumptions on a, g, and C that $(T\phi)(t)$ is continuous in t.

Now

$$\begin{aligned} |(T\phi)(t)| &\leq |a(t)| + \int_0^t |C(t,s)||g(s,\phi(s))|\,ds \\ &\leq |a(t)| + k\alpha||\phi|| \\ &< \infty. \end{aligned}$$

Therefore, $(T\phi)$ is bounded and $T : M \to M$.

For $\phi, \psi \in M$,

$$\begin{aligned} |(T\phi)(t) - (T\psi)(t)| &\leq \int_0^t |C(t,s)||g(s,\phi(s)) - g(s,\psi(s))|\,ds \\ &\leq k\int_0^t |C(t,s)|\,ds||\phi - \psi|| \\ &\leq \alpha||\phi - \psi||. \end{aligned}$$

Since $\alpha < 1$, T is a contraction mapping, which proves (1.2) has a unique bounded continuous solution.

Now we consider a special case, where $g(t,x) = x + h(t,x)$. So (1.2) becomes

$$x(t) = a(t) - \int_0^t C(t,s)[x(s) + h(s,x(s))]\,ds, t \geq 0. \tag{2.1}$$

Suppose $R(t,s)$ satisfies the resolvent equation

$$R(t,s) = C(t,s) - \int_s^t R(t,u)C(u,s)\,du. \tag{2.2}$$

Then by the variation of parameters formula, which can be found in [17, p. 191-192], solution $x(t)$ of equation (2.1) is given by

$$x(t) = a(t) - \int_0^t R(t,s)[a(s) + h(s,x(s))]\,ds, t \geq 0,$$

where a, R, and h are all continuous functions.

Theorem 2.2. Suppose h satisfies (H1). Assume $a(t)$ is bounded and

$$\sup_{t\geq 0} k\int_0^t |R(t,s)|\, ds \leq \alpha < 1.$$

Then there exists a unique bounded continuous solution of (2.1).

Proof. Let M be the Banach space of bounded continuous functions on $[0,\infty)$ with the supremum norm. For each $\phi \in M$, define

$$(T\phi)(t) = a(t) - \int_0^t R(t,s)[a(s) + h(s,\phi(s))]\, ds, t \geq 0.$$

It follows from the continuity assumptions on a, h, and R that $(T\phi)(t)$ is continuous in t. Also, one can easily verify from the given assumptions that $|(T\phi)(t)| < \infty$, and $|(T\phi)(t) - (T\psi)(t)| \leq \alpha||\phi - \psi||$ for all $\phi, \psi \in M$. This shows $T : M \to M$ and T is a contraction. Therefore (2.1) has a unique bounded continuous solution.

We now consider an example showing the integrability of the resolvent $R(t,s)$ of (2.2). The method of proof in this example is similar to the proof of Proposition 4 of [19]. We remark that the integrability of $R(t,s)$ is itself an important property which is often assumed in the study of qualitative behaviors of integral equations.

Example. Suppose

$$\sup_{t\geq 0} \int_0^t |C(t,s)|\, ds \leq L < 1.$$

Then

$$\sup_{t\geq 0} \int_0^t |R(t,s)|\, ds \leq l < \infty.$$

Proof. From the resolvent equation (2.2), we get

$$\begin{aligned}
\int_0^t |R(t,s)|\, ds &\leq \int_0^t |C(t,s)|\, ds + \int_0^t \int_s^t |R(t,u)||C(u,s)|\, du\, ds \\
&= \int_0^t |C(t,s)|\, ds + \int_0^t |R(t,u)| \int_0^u |C(u,s)|\, ds\, du \\
&\leq L + \int_0^t |R(t,u)|L\, du.
\end{aligned}$$

Therefore

$$(1-L)\int_0^t |R(t,s)|\ ds \le L.$$

So

$$\sup_{t\ge 0}\int_0^t |R(t,s)|\ ds \le \frac{L}{1-L} := l.$$

A number of researchers have studied the integrability of $R(t,s)$ for various special cases of the kernel $C(t,s)$. For example, in [16, Theorem 6], the authors considered the case where $C(t,s) = A(t-s)B(s)$. They proved that

$$\sup_{t\ge 0}\int_0^t |R(t,s)|\ ds \le J < \infty,$$

provided A and B satisfy certain conditions.

For more on the integrability of $R(t,s)$, we refer to [11] and the references therein.

Now we assume $a'(t)$ and C_t exist and are continuous functions. Differentiating (2.1), we get

$$x'(t) = -C(t,t)x(t) - \int_0^t C_t(t,s)x(s)\ ds + [a'(t) - C(t,t)h(t,x(t)) - \int_0^t C_t(t,s)h(s,x(s))\ ds]. \tag{2.3}$$

So

$$\begin{aligned} x(t) &= x(0)e^{-\int_0^t C(s,s)\ ds} + \int_0^t e^{-\int_u^t C(s,s)\ ds} a'(u)\ du \qquad (2.4)\\ &\quad - \int_0^t e^{-\int_u^t C(s,s)\ ds} \int_0^u C_u(u,s)x(s)\ ds\ du\\ &\quad - \int_0^t e^{-\int_u^t C(s,s)\ ds} C(u,u)h(u,x(u))\ du\\ &\quad - \int_0^t e^{-\int_u^t C(s,s)\ ds} \int_0^u C_u(u,s)h(s,x(s))\ ds\ du\\ &= a(0)e^{-\int_0^t C(s,s)\ ds} + \int_0^t e^{-\int_u^t C(s,s)\ ds} a'(u)\ du\\ &\quad - \int_0^t e^{-\int_u^t C(s,s)\ ds} \int_0^u C_u(u,s)[x(s) + h(s,x(s))]\ ds\ du\\ &\quad - \int_0^t e^{-\int_u^t C(s,s)\ ds} C(u,u)h(u,x(u))\ du \end{aligned}$$

where a, C, C_t, and h are continuous functions. In subsequent results, we shall write $a(0) = a_0$.

Theorem 2.3. Suppose h satisfies (H1). Assume $a'(t)$ is bounded and continuous, $\int_0^t C(s,s)\,ds \to \infty$ as $t \to \infty$, $\int_0^t e^{-\int_u^t C(s,s)\,ds}\,du$ is bounded, and

$$\sup_{t\geq 0}(k+1)\int_0^t e^{-\int_u^t C(s,s)\,ds}\int_0^u |C_u(u,s)|\,ds\,du + k\int_0^t e^{-\int_u^t C(s,s)\,ds}|C(u,u)|du \leq \alpha < 1.$$

Then there exists a unique bounded, continuous solution of (2.1).

Proof. Let M be the Banach space of bounded continuous functions on $[0,\infty)$. For each $\phi \in M$, define

$$\begin{aligned}
(T\phi)(t) &= a_0 e^{-\int_0^t C(s,s)\,ds} + \int_0^t e^{-\int_u^t C(s,s)\,ds} a'(u)\,du \\
&\quad - \int_0^t e^{-\int_u^t C(s,s)\,ds}\int_0^u C_u(u,s)[\phi(s) + h(s,\phi(s))]\,ds\,du \\
&\quad - \int_0^t e^{-\int_u^t C(s,s)\,ds} C(u,u)h(u,\phi(u))\,du, t \geq 0.
\end{aligned}$$

It follows from the continuity assumptions on a, h, C and C_t that $(T\phi)(t)$ is continuous in t.

Now

$$\begin{aligned}
|(T\phi)(t)| &\leq |a_0|e^{-\int_0^t C(s,s)\,ds} + \int_0^t e^{-\int_u^t C(s,s)\,ds}|a'(u)|\,du + \alpha||\phi|| \\
&< \infty
\end{aligned}$$

So $(T\phi)$ is bounded and $T : M \to M$.

For $\phi, \psi \in M$,

$$\begin{aligned}
|(T\phi)(t) - (T\psi)(t)| &\leq |\int_0^t e^{-\int_u^t C(s,s)\,ds}\int_0^u C_u(u,s)[\phi(s) - \psi(s) + h(s,\phi(s) - h(s,\psi(s))]\,ds| \\
&\quad + |\int_0^t e^{-\int_u^t C(s,s)\,ds} C(u,u)[h(u,\phi(u)) - h(u,\psi(u))]\,du| \\
&\leq \Big[(k+1)\int_0^t e^{-\int_u^t C(s,s)\,ds}\int_0^u |C_u(u,s)|\,ds\,du \\
&\quad + k\int_0^t e^{-\int_u^t C(s,s)\,ds}|C(u,u)|du\Big]||\phi - \psi|| \\
&\leq \alpha||\phi - \psi||.
\end{aligned}$$

Therefore T is a contraction map. So (2.4) has a unique bounded solution. Since (2.4) is equivalent to (2.1), where $x(0) = a(0)$, (2.1) has a unique bounded continuous solution.

One resolvent equation for

$$x'(t) = -C(t,t)x(t) - \int_0^t C_t(t,s)x(s)\,ds$$

is

$$Z_s(t,s) = Z(t,s)C(s,s) + \int_s^t Z(t,u)C_t(u,s)du\ , Z(t,t) = 1$$

with resolvent $Z(t,s)$. Then from (2.3), we obtain by the variation of parameters formula

$$x(t) = Z(t,0)a_0 + \int_0^t Z(t,s)[a'(s) - C(s,s)h(s,x(s)) - \int_0^s C_s(s,u)h(u,x(u))\,du]\,ds. \tag{2.5}$$

Theorem 2.4. Suppose h satisfies (H1). Assume $a'(t)$ is a bounded, continuous function, $Z(t,0)$ is bounded,

$$\sup_{t\ge 0} \int_0^t |Z(t,s)|\,ds < \infty, \tag{2.6}$$

and

$$\sup_{t\ge 0} k \int_0^t |Z(t,s)| \Big[|C(s,s)| + \int_0^s |C_s(s,u)|\,du \Big]\,ds \le \alpha < 1.$$

Then there exists a unique bounded, continuous solution of (2.1).

Proof. Let M be the Banach space of bounded continuous functions on $[0,\infty)$. For each $\phi \in M$, define

$$\begin{aligned}(T\phi)(t) &= Z(t,0)a_0 + \int_0^t Z(t,s)[a'(s) - C(s,s)h(s,\phi(s)) \\ &\quad - \int_0^s C_s(s,u)h(u,\phi(u))\,du]\,ds, t \ge 0.\end{aligned}$$

It follows from the continuity assumptions on a, h, R, C and C_t that $(T\phi)(t)$ is continuous in t.

Now

$$\begin{aligned}
|(T\phi)(t)| &= |Z(t,0)a_0 + \int_0^t Z(t,s)[a'(s) - [C(s,s)h(s,\phi(s)) \\
&\quad + \int_0^s C_s(s,u)h(u,\phi(u))\,du]]\,ds| \\
&\le |Z(t,0)||a_0| + \int_0^t |Z(t,s)||a'(s)|\,ds \\
&\quad + k\int_0^t |Z(t,s)||C(s,s)|\,ds||\phi|| \\
&\quad + k\int_0^t |Z(t,s)| \int_0^s |C_s(s,u)|\,du\,ds||\phi|| \\
&\le |Z(t,0)||a_0| + \int_0^t |Z(t,s)||a'(s)|\,ds + \alpha||\phi|| \\
&< \infty.
\end{aligned}$$

So $(T\phi)$ is bounded and $T : M \to M$.

For $\phi, \psi \in M$,

$$\begin{aligned}
|(T\phi)(t) - (T\psi)(t)| &\le |\int_0^t |Z(t,s)||C(s,s)||h(s,\phi(s)) - h(s,\psi(s))|\,ds \\
&\quad + \int_0^t |Z(t,s)| \int_0^s |C_s(s,u)||h(u,\phi(u)) - h(u,\psi(u))|\,duds \\
&\le k\int_0^t |Z(t,s)|\Big[|C(s,s)| + \int_0^s |C_s(s,u)|\,du\Big]\,ds||\phi - \psi|| \\
&\le \alpha||\phi - \psi||.
\end{aligned}$$

Therefore T is a contraction map, showing (2.5) has a unique bounded continuous solution. Since (2.5) is equivalent to (2.3), which is equivalent to (2.1), (2.1) has a unique bounded continuous solution.

We conclude this section by referring to a couple of known results relating to the integrability condition (2.6). Let $A(t) = -C(t,t)$ and $B(t,s) = -C_t(t,s)$. In [20, Example 2.1], it is shown that if there exists positive constants α and $K > 1$ such that

$$A(t) + K\int_0^t |B(t,s)|\,ds \le -\alpha, t \ge 0, \tag{2.7}$$

then

$$\sup_{t\ge 0} \int_0^t |Z(t,s)|(1 + |A(s)|)\,ds \le \frac{1}{k},$$

where $k = min\{(1 - \frac{1}{K}), \frac{\alpha}{K}\}$. This shows the integrability condition (2.6) holds if $A(s)$ is bounded, which is equivalent to

$$\sup_{t \geq 0} \int_0^t |B(t,s)| \, ds < \infty. \tag{2.8}$$

We remark that if (2.8) holds, then (2.7) is equivalent to

$$A(t) + \int_0^t |B(t,s)| \, ds \leq -\alpha, t \geq 0. \tag{2.9}$$

In [9, Theorem 2], it is shown that the integrability condition (2.6) holds if (2.9) holds.

3 L^p Solutions, Liapunov's Method.

In this section, we employ a technique outlined in [6]. Particularly, we construct Liapunov type functions that are suitable for integral equations. Various L^p properties of solutions are then obtained under appropriate assumptions on a, C, and g.

Theorem 3.1. Assume that equation (1.2) has a solution $x(t)$, $t \geq 0$. Suppose there exists a constant $k \geq 0$ such that

$$|g(t,x)| \leq k|x|,$$

and

$$k \int_0^\infty |C(u+t,t)| \, du \leq \alpha < 1.$$

Then the solution $x \in L^1[0,\infty)$ if $a \in L^1[0,\infty)$.

Proof. From (1.2) it follows that

$$\begin{aligned} |x(t)| &\leq |a(t)| + \int_0^t |C(t,s)||g(s,x(s))| \, ds \\ &\leq |a(t)| + k\int_0^t |C(t,s)||x(s)| \, ds. \end{aligned}$$

Therefore

$$-k\int_0^t |C(t,s)||x(s)| \, ds \leq |a(t)| - |x(t)|.$$

Let

$$V(t) = k \int_0^t \int_{t-s}^{\infty} |C(u+s,s)|\, du |x(s)|\, ds.$$

Then

$$\begin{aligned} V'(t) &= k \int_0^{\infty} |C(u+t,t)|\, du |x(t)| - \int_0^t k|C(t,s)||x(s)|\, ds \\ &\le \alpha |x(t)| - (|x(t)| - |a(t)|) \\ &= (\alpha - 1)|x(t)| + |a(t)|. \end{aligned}$$

Integrating from 0 to t,

$$V(t) - V(0) \le (\alpha - 1) \int_0^t |x(s)|\, ds + \int_0^t |a(s)|\, ds.$$

Since $V(t) \ge 0$, $V(0) = 0$ and $(\alpha - 1) < 0$,

$$(1-\alpha) \int_0^t |x(s)|\, ds \le \int_0^t |a(s)|\, ds.$$

This shows that $x \in L^1$ if $a \in L^1$.

Theorem 3.2 Assume that (1.2) has a nonnegative solution $x(t)$, $t \ge 0$. Also, assume there exists a constant $k > 0$ such that $0 \le g(t,x) \le kx$, for $x \ge 0$, $t \ge 0$. Let $C(t,s) \ge 0$, $C_s(t,s) \ge 0$, $C_{st}(t,s) \le 0$, and $C_t(t,0) \le 0$. Then $x \in L^2[0,\infty)$ if $a \in L^2[0,\infty)$.

Proof. For $x(t)$, a nonnegative solution of (1.2), let

$$V(t) = \int_0^t C_s(t,s) \Big(\int_s^t g(u,x(u))\, du \Big)^2 ds + C(t,0) \Big(\int_0^t g(s,x(s))\, ds \Big)^2.$$

Then

$$\begin{aligned} V'(t) &= \int_0^t C_{st}(t,s) \Big(\int_s^t g(u,x(u))\, du \Big)^2 ds \\ &\quad + \int_0^t C_s(t,s) 2 \Big(\int_s^t g(u,x(u)\Big)\, du g(t,x(t))\, ds \\ &\quad + C_t(t,0) \Big(\int_0^t g(s,x(s))\, ds \Big)^2 + 2C(t,0) \int_0^t g(s,x(s))\, ds g(t,x(t)). \end{aligned}$$

Integrating the second term of $V'(t)$ by parts, we get

$$
\begin{aligned}
&2g(t,x)\Big[C(t,s)\int_s^t g(u,x(u))\,du\Big|_{s=0}^{s=t} + \int_0^t C(t,s)g(s,x(s))\,ds\Big] \\
&= 2g(t,x)\Big[0 - C(t,0)\int_0^t g(u,x(u))\,du + \int_0^t C(t,s)g(s,x(s))\,ds\Big].
\end{aligned}
$$

Therefore we get

$$
\begin{aligned}
V'(t) &= \int_0^t C_{st}(t,s)\Big(\int_s^t g(u,x(u))du\Big)^2\,ds \\
&\quad +2g(t,x)\int_0^t C(t,s)g(s,x(s))\,ds \\
&\quad +C_t(t,0)\Big(\int_0^t g(s,x(s))\,ds\Big)^2.
\end{aligned}
$$

Now from (1.2), $a(t) - x(t) = \int_0^t C(t,s)g(s,x(s))\,ds$. Notice that $a(t) - x(t) \geq 0$ by our positivity assumptions on C and g.

So

$$
\begin{aligned}
V'(t) &= \int_0^t C_{st}(t,s)\Big(\int_s^t g(u,x(u))\,du\Big)^2\,ds \\
&\quad +C_t(t,0)\Big(\int_0^t g(s,x(s))\,ds\Big)^2 \\
&\quad +2g(t,x)[a(t)-x(t)] \\
&\leq 2g(t,x)[a(t)-x(t)] \\
&\leq 2kx(t)[a(t)-x(t)] \\
&\leq k[a^2(t)+x^2(t)-2x^2(t)] \\
&= ka^2(t) - kx^2(t).
\end{aligned}
$$

Integrating from 0 to t, we obtain

$$V(t) \leq V(0) + k\int_0^t a^2(s)\,ds - k\int_0^t x^2(s)\,ds.$$

This implies $x \in L^2[0,\infty)$ if $a \in L^2[0,\infty)$.

Now suppose both C_t and $a'(t)$ are continuous. We can then write (1.2) as

$$x'(t) = a'(t) - C(t,t)g(t,x) - \int_0^t C_t(t,s)g(s,x(s))\,ds. \tag{3.1}$$

Theorem 3.3 Assume (3.1) has a nonnegative solution $x(t)$, $t \geq 0$. Suppose there exists a constant $m > 0$ such that $g(t,x) \geq mx^p$, for $x \geq 0$, $t \geq 0$, where p is a positive integer. Let $C_t(t,s) < 0$, and

$$-C(t,t) + \int_0^\infty |C_1(u+t,t)|\, du \leq -\alpha$$

for some $\alpha > 0$. Then $x \in L^p[0,\infty)$ if $a' \in L^1[0,\infty)$. Moreover, the solution $x(t)$ is bounded.

Proof. For $x(t)$, a nonnegative solution of (3.1), let

$$V(t) = x(t) + \int_0^t \int_{t-s}^\infty |C_1(u+s,s)|\, du g(s,x(s))\, ds.$$

Then

$$\begin{aligned}
V'(t) &= x'(t) + \int_0^\infty |C_1(u+t,t)|\, du g(t,x(t)) - \int_0^t |C_t(t,s)| g(s,x(s))\, ds \\
&= a'(t) - C(t,t)g(t,x(t)) - \int_0^t C_t(t,s)g(s,x(s))\, ds \\
&\quad + \int_0^\infty |C_1(u+t,t)|\, du g(t,x(t)) - \int_0^t |C_t(t,s)| g(s,x(s))\, ds \\
&= a'(t) + [-C(t,t) + \int_0^\infty |C_1(u+t,t)|\, du] g(t,x(t)) \\
&\quad - \int_0^t C_t(t,s)g(s,x(s))\, ds - \int_0^t |C_t(t,s)| g(s,x(s))\, ds \\
&= a'(t) + [-C(t,t) + \int_0^\infty |C_1(u+t,t)|\, du] g(t,x(t)) \\
&\leq |a'(t)| - \alpha m x^p(t).
\end{aligned}$$

Integrating the above relation from 0 to t yields

$$V(t) \leq V(0) + \int_0^t |a'(s)|\, ds - \alpha m \int_0^t x^p(s)\, ds. \tag{3.2}$$

Since $V(t) \geq 0$, we get

$$\alpha m \int_0^t x^p(s)\, ds \leq V(0) + \int_0^t |a'(s)|\, ds.$$

This shows that $x \in L^p[0,\infty)$ if $a' \in L^1[0,\infty)$. From the definition of $V(t)$, one can easily see that $x(t) \leq V(t)$. Thus the boundedness of $x(t)$ follows from (3.2).

In the next theorem we show the existence of nonnegative solutions of (1.2) using the contraction mapping principle.

Theorem 3.4. Assume a, g, and C satisfy the conditions stated in Theorem 2.1. Also assume $C(t,s) \geq 0$ for $0 \leq s \leq t < \infty$, and $g(t,x) > 0$ for $x > 0$. Suppose

$$a(t) - k\int_0^t C(t,s)a(s)ds \geq 0.$$

Then (1.2) has a unique nonnegative solution.

Proof. Let $BCP = \{x(t)$ such that $x(t)$ is bounded, continuous, and $x(t) \geq 0\}$. For each $x, y \in BCP$, let $\rho(x,y) = \sup_{t\geq 0}|x(t)-y(t)|$. Then ρ is a metric and BCP is a complete metric space. Now let $M = \{y \in BCP : a(t) - k\int_0^t C(t,s)y(s)\,ds \geq 0\}$. The set M is closed. To see this, let $\{y_n(t)\}$ be a sequence of functions in M and $y_n(t) \to y(t)$ as $n \to \infty$. By the definition of the metric, $y_n(t) \to y(t)$ uniformly on $[0,\infty)$ as $n \to \infty$. Therefore $y(t)$ is continuous, bounded, and $y(t) \geq 0$.

Now we shall show that

$$a(t) - \int_0^t C(t,s)y(s)\,ds \geq 0.$$

Since $y_n(t) \in M$ for every n, we have

$$a(t) - \int_0^t C(t,s)y_n(s)\,ds \geq 0. \tag{3.3}$$

For each fixed $t \geq 0$, the function $C(t,s)$ is a bounded function of s on $[0,t]$ because $C(t,s)$ is continuous. This means that when $y_n(s) \to y(s)$ uniformly on $[0,t]$, then $C(t,s)y_n(s) \to C(t,s)y(s)$ uniformly on $[0,t]$. Therefore taking the limit on (3.3), we obtain

$$a(t) - \int_0^t \lim_{n\to\infty} C(t,s)y_n(s)\,ds \geq 0,$$

which implies

$$a(t) - \int_0^t C(t,s)y(s)\,ds \geq 0.$$

This proves that M is closed.

Now define $T : M \to M$ by

$$(T\phi)(t) = a(t) - \int_0^t C(t,s)g(s,\phi(s))\, ds.$$

It follows from the continuity assumptions on a, C, and g that $(T\phi)(t)$ is continuous in t. Since $(T\phi)(t) \leq a(t)$ and $a(t)$ is bounded, then $(T\phi)(t)$ is bounded. Now we show $(T\phi)(t) \geq 0$.

$$\begin{aligned}
(T\phi)(t) &= a(t) - \int_0^t C(t,s)g(s,\phi(s))\, ds \\
&\geq a(t) - k\int_0^t C(t,s)\phi(s)\, ds \\
&\geq 0
\end{aligned}$$

by definition of M because $\phi \in M$.

We also need to show $(T\phi)$ satisfies the condition of M, i.e.

$$a(t) - k\int_0^t C(t,s)(T\phi)(s)\, ds \geq 0.$$

$$\begin{aligned}
a(t) - k\int_0^t C(t,s)(T\phi)(s)\, ds &= a(t) - k\int_0^t C(t,s)[a(s) \\
&\quad - \int_0^s C(s,u)g(u,\phi(u))du]\, ds \\
&= a(t) - k\int_0^t C(t,s)a(s)\, ds \\
&\quad + k\int_0^t C(t,s)\Big[\int_0^s C(s,u)g(u,\phi(u))\, du\Big]\, ds \\
&\geq 0.
\end{aligned}$$

Therefore $(T\phi)(t)$ satisfies all conditions of M, hence $T : M \to M$.

Now we show T is a contraction. Let $\phi, \psi \in M$. Then

$$\begin{aligned}
|(T\phi)(t) - (T\psi)(t)| &\leq \int_0^t C(t,s)|g(s,\phi(s)) - g(s,\psi(s))|\, ds \\
&\leq k\int_0^t C(t,s)||\phi - \psi||\, ds \\
&\leq \alpha||\phi - \psi||.
\end{aligned}$$

Therefore T is a contraction, which proves there exists a unique nonnegative solution $x(t)$ of (1.2).

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