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Qualitative Properties of Monotone Linear Parabolic Operators*†

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Abstract

When we construct mathematical or numerical models in order to solve a real-life problem, it is important to preserve the characteristic properties of the original process. The models have to possess the equivalents of these properties. Parabolic partial equations generally serve as the mathematical models of heat conduction or diffusion processes, where the most important properties are the monotonicity, the nonnegativity preservation and the maximum principles. In this paper, the validity of the equivalents of these qualitative properties are investigated for the second order linear partial differential operator. Conditions are given that guarantee the qualitative properties. On some examples we investigate these conditions.

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1 Introduction, motivation

The classical theory of partial differential equations investigates the question of existence, uniqueness and the methods that produce the solutions of the equations. Qualitative investigations came into being from the mid-fifties. Researchers assumed that the solution of the problem is at hand and tried to answer the questions: What kind of properties does the solution have? What kind of class of functions does the solution belong to? The most representative result in this field is the well-known maximum principle. A comprehensive survey of the qualitative properties of the second order linear partial differential equations can be found e.g. in [2, 7].

Real-life phenomena possess a number of characteristic properties. For instance, let us consider the heat conduction of a physical body. When we increase the strength of the heat sources inside the body, the temperature on the boundary and the temperature in the initial state, then the temperature must not decrease inside the body. This property is called *monotonicity*. *Maximum principles* result in lower and upper bounds for the distribution of temperature in the body. One of the simplest form of them states that, if there are no heat sources and sinks present inside the body, then the maximum temperature appears also on the boundary of the body or in the initial state. The above example shows that when we construct a mathematical model of a phenomenon, it is important to investigate whether the mathematical model possesses the same properties as the original process. In this paper we investigate the validity of the monotonicity property and one of the maximum principles for the second order linear parabolic partial differential operator, and reveal the connections of the properties. The results of the qualitative theory of differential equations, albeit they have the importance on their own, help us to show that the qualitative properties of a mathematical model correspond to the qualitative properties of the modelled phenomenon.

The qualitative adequateness can be investigated also for numerical models, but this topic is beyond the scope of this paper. For more details consult e.g. [1, 4, 5].

The paper is organized as follows. In Section 2, we start with the general investigation of the qualitative properties of linear operators and we reveal their relations. The results are applied to second order parabolic differential operators in Section 3. Conditions are given that guarantee the qualitative properties. On some examples we investigate these conditions.

2 Qualitative Properties of Linear Operators

Let S, S_∂ be two arbitrary mutually disjoint sets in $\mathbb{R}^d$ $(1 \leq d \in \mathbb{N})$ and let T be an arbitrary positive number. Let X and Y denote given ordered vector spaces of bounded real valued functions, or some subspaces of them, defined on $(S \cup S_\partial) \times [0, T]$ and $S \times (0, T)$, respectively. We introduce the notations

$$K_\tau = S \times (0, \tau), \qquad \bar{K}_\tau = (S \cup S_\partial) \times [0, \tau],$$

$$K_{\bar{\tau}} = S \times (0, \tau], \qquad G_\tau = (S_\partial \times [0, \tau]) \cup (S \times \{0\})$$

for any arbitrary positive number τ.

Let $L : X \to Y$ be a linear operator. Next we define some important qualitative properties of L.

Definition 2.1. The operator L is said to be *monotone* if for all $t^\star \in (0, T)$ and $v_1, v_2 \in X$ such that $v_1|_{G_{t^\star}} \geq v_2|_{G_{t^\star}}$ and $(Lv_1)|_{K_{\bar{t}^\star}} \geq (Lv_2)|_{K_{\bar{t}^\star}}$, the relation $v_1|_{K_{\bar{t}^\star}} \geq v_2|_{K_{\bar{t}^\star}}$ holds.

Notice that the monotonicity property gives possibility to compare two elements on the entire $K_{\bar{t}^\star}$, knowing their image relation on this set and their relation on $G_{t^\star}$. This phenomenon is generally called *comparison principle*. From the practical point of view, the comparison principle is used for comparison of an unknown function with some other known function. On the base of the comparison principle, we can formulate some further qualitative properties.

Definition 2.2. The operator L is said to be *nonnegativity preserving* if the relations $v|_{G_{t^\star}} \geq 0$ and $(Lv)|_{K_{\bar{t}^\star}} \geq 0$ imply that the relation $v|_{K_{\bar{t}^\star}} \geq 0$ also holds.

Clearly, the nonnegativity preservation property means the comparison of the function $v_1 = v$ with the function $v_2 = 0$. Moreover, we have

Corollary 2.3 *For linear operators $L : X \to Y$, the monotonicity and the nonnegativity preserving properties are equivalent.*

When an unknown function v can be compared with a function that is given by the values $(Lv)|_{K_{\bar{t}^\star}}$ and $v|_{G_{t^\star}}$, then we say that a *maximum principle* is defined. In the following we define four types of maximum principles, which differ in the definition of the comparison functions. More precisely, these comparison functions are constructed via $\sup_{G_{t^\star}} v$ and $\sup_{K_{\bar{t}^\star}} Lv$.

Definition 2.4. We say that the operator L satisfies the *weak maximum principle* if for any function $v \in X$ and $t^\star \in (0,T)$ the inequality

$$\sup_{\bar{K}_{t^\star}} v \le \max\{0, \sup_{G_{t^\star}} v\} + t^\star \cdot \max\{0, \sup_{K_{\bar{t}^\star}} Lv\} \tag{1}$$

is satisfied.

Definition 2.5. We say that the operator L satisfies the *strong maximum principle* if for any function $v \in X$ and $t^\star \in (0,T)$ the inequality

$$\sup_{\bar{K}_{t^\star}} v \le \sup_{G_{t^\star}} v + t^\star \cdot \max\{0, \sup_{K_{\bar{t}^\star}} Lv\} \tag{2}$$

is satisfied.

When the sign of Lv is known, then the comparison function is constructed only via $\sup_{G_{t^\star}} v$. Such type of maximum principles are called boundary maximum principles (c.f. [11]).

Definition 2.6. We say that the operator L satisfies the *weak boundary maximum principle* when for any function $v \in X$ and $t^\star \in (0,T)$ such that $Lv|_{K_{\bar{t}^\star}} \le 0$ the inequality

$$\sup_{\bar{K}_{t^\star}} v \le \max\{0, \sup_{G_{t^\star}} v\} \tag{3}$$

holds.

Definition 2.7. We say that the operator L satisfies the *strong boundary maximum principle* when for any function $v \in X$ and $t^\star \in (0,T)$ such that $Lv|_{K_{\bar{t}^\star}} \le 0$ the equality

$$\sup_{\bar{K}_{t^\star}} v = \sup_{G_{t^\star}} v \tag{4}$$

holds.

Remark 2.8 *Because $v \in X$ implies $-v \in X$ and the supremum of a real valued function v is minus one times the infimum of $-v$, the maximum principles can be written in an equivalent forms with minimums and infimums.*

Remark 2.9 *Assume that v has maximum (minimum) on $\bar{K}_{t^\star}$. Then, in case of validity of any boundary maximum principle, we can give the location of the maximum (minimum): it is taken on $G_{t^\star}$.*

As a first step, we investigate the relations of the above properties.

Theorem 2.10 *For a linear operator $L : X \to Y$, the following implications are valid*

i) the strong maximum principle implies the strong boundary maximum principle,

ii) the strong boundary maximum principle implies the weak boundary maximum principle,

iii) the weak maximum principle implies the weak boundary maximum principle,

iv) the weak boundary maximum principle implies the monotonicity of the operator.

Proof. In Implication i), the fact that the right-hand side of (4) is not greater than the left-hand side is obvious. The reverse relation follows from (2) and the non-positivity of Lv on $K_{\bar{t}^\star}$.

Implication ii) follows from the trivial relation $\max\{0, \sup_{G_{t^\star}} v\} \geq \sup_{G_{t^\star}} v$.

Implication iii) is trivial due to the non-positivity of Lv on $K_{\bar{t}^\star}$.

In order to prove Implication iv), let $v_1, v_2 \in X$ be two arbitrary functions and $t^\star \in (0, T)$ an arbitrary real number with the properties $v_1|_{G_{t^\star}} \geq v_2|_{G_{t^\star}}$ and $(Lv_1)|_{K_{\bar{t}^\star}} \geq (Lv_2)|_{K_{\bar{t}^\star}}$. Then, using the linearity of the operator L, we obtain that $(L(v_2 - v_1))|_{K_{\bar{t}^\star}} \leq 0$. It follows from the weak boundary maximum principle that

$$\sup_{\bar{K}_{t^\star}} (v_2 - v_1) \leq \max\{0, \sup_{G_{t^\star}} (v_2 - v_1)\} = 0,$$

which results in the relation $v_1|_{K_{\bar{t}^\star}} \geq v_2|_{K_{\bar{t}^\star}}$. This completes the proof. ■

Corollary 2.11 *If a linear operator $L : X \to Y$ satisfies any of the maximum principles, then it is monotone.*

Now we analyze the reverse implication: Under which conditions will the monotonicity imply the maximum principles? Because it causes no confusion, we denote the constant one function simply by 1 and the function $(\mathbf{x}, t) \mapsto t$ by t. These functions are supposed to be defined on K_T.

Theorem 2.12 *Let us suppose that the space X contains the functions 1 and t. Then, if a monotone operator L has the properties $L1 \geq 0$ and $Lt \geq 1$, then L possesses both the weak maximum principle and the weak boundary maximum principle.*

Proof. Because of Implication iii) in Theorem 2.10, it is enough to prove the validity of the weak maximum principle. Let $t^\star$ be a fixed number from the interval $(0,T)$. We choose an arbitrary function $v_2 \in X$ and we set $v_1 = \max\{0, \sup_{G_{t^\star}} v_2\} + t \cdot \max\{0, \sup_{K_{\bar{t}^\star}} Lv_2\}$, which function trivially belongs to X, because $1, t \in X$. Clearly, $v_1|_{G_{t^\star}} \geq v_2|_{G_{t^\star}}$, moreover

$$Lv_1 = \max\{0, \sup_{G_{t^\star}} v_2\}(L1) + \max\{0, \sup_{K_{\bar{t}^\star}} Lv_2\}(Lt) \geq \sup_{K_{\bar{t}^\star}} Lv_2 \geq Lv_2$$

on $K_{\bar{t}^\star}$. Hence, based on the monotonicity of the operator, we obtain that

$$v_1 = \max\{0, \sup_{G_{t^\star}} v_2\} + t \cdot \max\{0, \sup_{K_{\bar{t}^\star}} Lv_2\} \geq v_2$$

on $K_{\bar{t}^\star}$. This completes the proof. ■

Theorem 2.13 *Let us suppose that the space X contains the functions 1 and t. Then, if a monotone operator L has the properties $L1 = 0$ and $Lt \geq 1$, then L possesses all the investigated maximum principles.*

Proof. Because of Theorem 2.10 and Theorem 2.12, it is enough to prove the validity of the strong maximum principle. Let $t^\star$ be any fixed number from the interval $(0,T)$. We choose an arbitrary function $v_2 \in X$ and we set $v_1 = \sup_{G_{t^\star}} v_2 + t \cdot \max\{0, \sup_{K_{\bar{t}^\star}} Lv_2\}$. Clearly, $v_1|_{G_{t^\star}} \geq v_2|_{G_{t^\star}}$, moreover

$$Lv_1 = (\sup_{G_{t^\star}} v_2)(L1) + \max\{0, \sup_{K_{\bar{t}^\star}} Lv_2\}(Lt) \geq \sup_{K_{\bar{t}^\star}} Lv_2 \geq Lv_2$$

on $K_{\bar{t}^\star}$. Based on the monotonicity of the operator we obtain that

$$v_1 = \sup_{G_{t^\star}} v_2 + t \cdot \max\{0, \sup_{K_{\bar{t}^\star}} Lv_2\} \geq v_2$$

on $K_{\bar{t}^\star}$. This completes the proof. ■

3 Qualitative Properties of the Second Order Parabolic Operator

In this section, the results obtained for the general linear operators are applied to the second order linear partial differential operator.

Let Ω and $\partial\Omega$ denote, respectively, a bounded domain in $\mathbb{R}^d$ $(1 \le d \in \mathbb{N})$ and its boundary, and we introduce the sets

$$Q_\tau = \Omega \times (0,\tau), \qquad \bar{Q}_\tau = \bar{\Omega} \times [0,\tau],$$

$$Q_{\bar{\tau}} = \Omega \times (0,\tau], \qquad \Gamma_\tau = (\partial\Omega \times [0,\tau]) \cup (\Omega \times \{0\})$$

for any arbitrary positive number τ. In the sequel, Γ_τ is called *parabolic boundary.* For some fixed number $T > 0$, we consider the second order linear partial differential operator

$$L \equiv \frac{\partial}{\partial t} - \sum_{m,k=1}^{d} a_{m,k} \frac{\partial^2}{\partial x_m \partial x_k} - \sum_{m=1}^{d} a_m \frac{\partial}{\partial x_m} - a_0, \tag{5}$$

where the coefficient functions are defined and bounded on Q_T. We assume that the operator is parabolic, that is the matrix

$$\mathbf{S}(\mathbf{x},t) := [a_{m,k}(\mathbf{x},t)]_{m,k=1}^{d} \tag{6}$$

is positive definite at all points of Q_T. We define the domain of the operator L, denoted by $\operatorname{dom} L$, as the space of functions $v \in C(\bar{Q}_T)$, for which the derivatives $\partial v/\partial x_m$, $\partial^2 v/\partial x_m \partial x_k$ and $\partial v/\partial t$ exist in Q_T and they are bounded. It can be seen easily that Lv is bounded on $Q_{\bar{t}^\star}$ for each $v \in \operatorname{dom} L$ and $t^\star \in (0,T)$, which means that $\inf_{Q_{\bar{t}^\star}} Lv$ and $\sup_{Q_{\bar{t}^\star}} Lv$ are finite values.

The monotonicity and the maximum principles for operator (5) can be defined using the definitions of the previous section with the setting $S = \Omega$, $S_\partial = \partial\Omega$, $X = \operatorname{dom} L$ and

$$Y = \{w \in B(Q_T) \mid \text{there exists } v \in X \text{ such that } Lv = w \text{ in } Q_T\},$$

where $B(Q_T)$ denotes the space of bounded functions on Q_T. The fact $G_{t^\star} = \Gamma_{t^\star}$ motivates the earlier phrase "boundary" in the definitions of the previous section.

Remark 3.1 *For the sake of simplicity, we used the terminology strong maximum principle, but distinction should be made between this property and the following one (called also strong maximum principle in the literature): if* $Lv \leq 0$ *in* Q_T *and* v *assumes its positive maximum at an interior point* $(\mathbf{x}^0, t^0)$*, then* $v \equiv const.$ *in the set of all points* $(\mathbf{x}, t) \in Q_T$ *which can be connected to* $(\mathbf{x}^0, t^0)$ *by a simple continuous curve in* Q_T *along which the coordinate* t *is non-decreasing from* $(\mathbf{x}, t)$ *to* $(\mathbf{x}^0, t^0)$ *([10]). This property was extended also for boundary points in [6].*

Remark 3.2 *When operator* (5) *appears in a mathematical model of a physical phenomenon, then the physical units of the quantities must agree in the maximum principles. Now we check this agreement for the one-dimensional heat conduction operator. We use the units of SI, that is K=Kelvin, kg=kilogram, s=second, m=meter and J=Joule. Thus, let us consider the operator*

$$L = \frac{\partial}{\partial t} - \frac{\kappa}{c\varrho} \frac{\partial^2}{\partial x^2},$$

where κ *is the heat conduction coefficient (measured in Jm/(Ks),* ϱ *is the density (measured in kg/m), c is the specific heat (measured in* $J/(kgK)$*) and the function* v*, which the operator is applied to, is the temperature (measured in K). Because the temperature is estimated in the maximum principles from above, to the agreement of the physical units we need to check that the quantity* $t \sup_{K_{\bar{t}^\star}} Lv$ *can be measured in Kelvin. Indeed, the unit of this quantity results in*

$$s \cdot \left(\frac{K}{s} - \frac{\frac{J}{Ks}}{\frac{Jm}{kgK} \frac{kg}{m}} \frac{K}{m^2} \right) = K.$$

We turn to the investigation of the qualitative properties of operator (5). First we prove a basic property of operator L.

Theorem 3.3 *Operator* (5) *is monotone.*

Proof. We prove the statement in view of Corollary 2.3.
Let $v \in \operatorname{dom} L$ an arbitrary fixed function. Then the function

$$\hat{v}(\mathbf{x}, t) \equiv v(\mathbf{x}, t) e^{-\lambda t} \tag{7}$$

also belongs to $\operatorname{dom} L$ for any real parameter λ. Expressing v from (7) and applying operator (5) to it, we get

$$Lv = L(e^{\lambda t}\hat{v}) = e^{\lambda t}\left[\frac{\partial \hat{v}}{\partial t} - \sum_{m,k=1}^{d} a_{m,k}\frac{\partial^2 \hat{v}}{\partial x_m \partial x_k} - \sum_{m=1}^{d} a_m \frac{\partial \hat{v}}{\partial x_m} + (\lambda - a_0)\hat{v}\right]. \tag{8}$$

Let us fix the parameter $t^\star \in (0,T)$. Since $\hat{v}$ is a continuous function on $\bar{Q}_{t^\star}$, its minimum exists and it is taken at some point $(\mathbf{x}^0, t^0) \in \bar{Q}_{t^\star}$.

- First we assume that this point belongs to the parabolic boundary, i.e., $(\mathbf{x}^0, t^0) \in \Gamma_{t^\star}$. Then, due to the obvious relation

  $$\hat{v}(\mathbf{x}, t) \geq \hat{v}(\mathbf{x}^0, t^0) = \min_{\Gamma_{t^\star}} \hat{v}$$

  for all $(\mathbf{x}, t) \in \bar{Q}_{t^\star}$, we get the estimation

  $$\inf_{Q_{\bar{t}^\star}} \hat{v} \geq \min_{\Gamma_{t^\star}} \hat{v}. \tag{9}$$

- Assume now that $(\mathbf{x}^0, t^0) \in Q_{\bar{t}^\star}$. Then we get the relations

  $$\frac{\partial \hat{v}}{\partial t}(\mathbf{x}^0, t^0) \leq 0, \quad \frac{\partial \hat{v}}{\partial x_m}(\mathbf{x}^0, t^0) = 0, \tag{10}$$

  and, because $(\mathbf{x}^0, t^0)$ is a minimum point, the second derivative matrix

  $$\hat{\mathbf{V}}(\mathbf{x}^0, t^0) := \left[\frac{\partial^2 \hat{v}}{\partial x_m \partial x_k}(\mathbf{x}^0, t^0)\right]_{m,k=1}^{d}$$

  is positive semi-definite.

  Let us denote the Hadamard product

  $$\left(\mathbf{S}(\mathbf{x}^0, t^0) \circ \hat{\mathbf{V}}(\mathbf{x}^0, t^0)\right)_{m,k} = a_{m,k}(\mathbf{x}^0, t^0) \cdot \frac{\partial^2 \hat{v}}{\partial x_m \partial x_k}(\mathbf{x}^0, t^0) \tag{11}$$

  $(m, k = 1, \ldots, d)$ by $\mathbf{S}(\mathbf{x}^0, t^0) \circ \hat{\mathbf{V}}(\mathbf{x}^0, t^0) \in \mathbb{R}^{d \times d}$. Due to the assumptions, both the matrices $\mathbf{S}(\mathbf{x}^0, t^0)$ and $\hat{\mathbf{V}}(\mathbf{x}^0, t^0)$ are positive semi-definite, hence, according to the Schur theorem (e.g. Theorem 7.5.3 in [8]), the matrix $\mathbf{S}(\mathbf{x}^0, t^0) \circ \hat{\mathbf{V}}(\mathbf{x}^0, t^0)$ is also positive semi-definite.

We investigate (8) in the rearranged form

$$e^{-\lambda t} Lv + \sum_{m=1}^{d} a_m \frac{\partial \hat{v}}{\partial x_m} - (\lambda - a_0)\hat{v} = \frac{\partial \hat{v}}{\partial t} - \sum_{m,k=1}^{d} a_{m,k} \frac{\partial^2 \hat{v}}{\partial x_m \partial x_k}. \quad (12)$$

Introducing the notation $\mathbf{e} = [1, 1, \ldots, 1]^\top \in \mathbb{R}^d$, the relation

$$\sum_{m,k=1}^{d} a_{m,k}(\mathbf{x}^0, t^0) \frac{\partial^2 \hat{v}}{\partial x_m \partial x_k}(\mathbf{x}^0, t^0) = ((\mathbf{S}(\mathbf{x}^0, t^0) \circ \hat{\mathbf{V}}(\mathbf{x}^0, t^0))\mathbf{e}, \mathbf{e}) \geq 0. \quad (13)$$

is valid. On the base of (10) and (13), the right-hand side of (12) is nonpositive at the point $(\mathbf{x}^0, t^0)$. Hence, the inequality

$$e^{-\lambda t^0}(Lv)(\mathbf{x}^0, t^0) - (\lambda - a_0(\mathbf{x}^0, t^0))\hat{v}(\mathbf{x}^0, t^0) \leq 0 \quad (14)$$

holds. Let us introduce the notations $a_{\inf} := \inf_{Q_T} a_0$ and $a_{\sup} := \sup_{Q_T} a_0$, which are well-defined because of the boundedness of the coefficient function a_0. For any $\lambda > a_{\sup}$, we have

$$\hat{v}(\mathbf{x}^0, t^0) \geq \frac{e^{-\lambda t^0}(Lv)(\mathbf{x}^0, t^0)}{\lambda - a_0(\mathbf{x}^0, t^0)} \geq \frac{e^{-\lambda t^0}(Lv)(\mathbf{x}^0, t^0)}{\lambda - a_{\inf}} \geq$$
$$\geq \frac{1}{\lambda - a_{\inf}} \inf_{Q_{\bar{t}^\star}} (e^{-\lambda t}(Lv)(\mathbf{x}, t)). \quad (15)$$

Since the function $\hat{v}$ takes its minimum at the point $(\mathbf{x}^0, t^0)$, therefore the estimation (15) shows the validity of the inequality

$$\inf_{Q_{\bar{t}^\star}} \hat{v} \geq \frac{1}{\lambda - a_{\inf}} \inf_{Q_{\bar{t}^\star}} (e^{-\lambda t}(Lv)(\mathbf{x}, t)). \quad (16)$$

Clearly, the estimates of the two different cases, namely (9) and (16) together, imply that

$$\inf_{\bar{Q}_{t^\star}} \hat{v} \geq \min\{\inf_{\Gamma_{t^\star}} \hat{v};\ \frac{1}{\lambda - a_{\inf}} \inf_{Q_{\bar{t}^\star}} \left(e^{-\lambda t}(Lv)(\mathbf{x}, t)\right)\}, \quad (17)$$

for any $\lambda > a_{\sup}$. From (17) and from the definition of the function $\hat{v}$ in (7), we obtain that if $v|_{\Gamma_{t^\star}} \geq 0$ and $(Lv)|_{Q_{\bar{t}^\star}} \geq 0$, then $v(\mathbf{x}, t^\star) \geq 0$ on $Q_{\bar{t}^\star}$. This completes the proof. ■

For simpler operators, similar statements can be found in [3, 9].

Theorem 3.4 *If $a_0 \le 0$, then operator* (5) *possesses both the weak maximum principle and the weak boundary maximum principle.*

Proof. It is trivial that $1, t \in \operatorname{dom} L$, thus we can apply Theorem 2.12. The statement of the theorem follows from the facts that, under the conditions of the theorem, $L1 = -a_0 \ge 0$ and $Lt = 1 - a_0 t \ge 1$. ∎

Theorem 3.5 *If $a_0 = 0$, then operator* (5) *possesses all the maximum principles.*

Proof. We apply Theorem 2.13. The statement of the theorem follows from the facts that $L1 = -a_0 = 0$ and $Lt = 1 - a_0 t = 1$. ∎

In order to analyse the necessity of the condition in Theorem 3.4, we introduce the following notation. The set of those real values ω for which the condition $a_0 \le \omega$ implies the weak maximum principle for all operators in the form (5), will be denoted by M_W.

Theorem 3.6 *The set M_W is identical with the set $\mathbb{R}_0^-$.*

Proof. In view of Theorem 3.4, the inclusion $\mathbb{R}_0^- \subset M_W$ is trivial. In order to show the inclusion in the opposite direction, let us choose an arbitrary positive constant γ and consider the one-dimensional operator

$$L \equiv \frac{\partial}{\partial t} - \frac{\gamma}{2}\frac{\partial^2}{\partial x^2} - \gamma, \tag{18}$$

where the set Q_T is defined to be $Q_T = (0,\pi) \times (0,T)$. Clearly, this operator has the form (5). Moreover $a_0 = \gamma > 0$.

Let us choose the function $v(x,t) = (\gamma/2)e^{\gamma t/2}\sin x$, for which function the relation $Lv(x,t) = 0$ is true. Thus, we have

$$\sup_{\bar{Q}_{t^\star}} v = \frac{\gamma}{2}e^{\gamma t^\star/2} > \frac{\gamma}{2} = \max\{0, \sup_{\Gamma_{t^\star}} v\}$$

for any $t^\star \in (0,T)$. This shows that the weak boundary maximum principle does not hold for (18). We note that, due to the implications in Theorem 2.10, any of the maximum principles cannot be valid. Thus, the weak maximum principle does not hold either. ∎

We investigate the condition of Theorem 3.5. Let us consider the set of those non-positive real values μ for which the condition $\mu \le a_0 \le 0$ implies all the maximum principles for all operators in the form (5). This set will be denoted by M_A. The zero upper bound for the function a_0 is justifiable by the previous theorem.

Theorem 3.7 *The set M_A is identical with the set $\{0\}$.*

Proof. Because of the relation $\mu \le a_0 \le 0$, the operator L possesses both the weak maximum principle and the weak boundary maximum principle by all the choices of μ (see Theorem 3.4). Thus, it is enough to investigate the validity of the strong boundary maximum principle.

In view of Theorem 3.5, the inclusion $0 \in M_A$ is trivial. Now let γ be an arbitrary negative number and consider the operator

$$L \equiv \frac{\partial}{\partial t} + \frac{\gamma}{2}\frac{\partial^2}{\partial x^2} - \gamma, \tag{19}$$

where the set Q_T is defined again to be $Q_T = (0,\pi) \times (0,T)$. Here $a_0 = \gamma < 0$. We set $v(x,t) = -\frac{\gamma}{2}e^{\gamma t/2}(\sin x - 2)$, for which

$$Lv(x,t) = \frac{\gamma^2}{2}e^{\gamma t/2}(\sin x - 1) \le 0.$$

With this function v, we get the relation

$$\max_{\bar{Q}_{t^\star}} v = \frac{\gamma}{2}e^{\gamma t^\star/2} > \max\{\gamma e^{\gamma t^\star/2}, \gamma/2\} = \max_{\Gamma_{t^\star}} v$$

for any $t^\star \in (0,T)$. Thus, the strong boundary maximum principle is not satisfied. Theorem 2.10 Implication i) gives that the strong maximum principle cannot be valid either. This completes the proof. ■

Remark 3.8 *The operator in the proof of Theorem 3.7 shows that the validity of the weak (resp. weak boundary) maximum principle does not imply the strong (resp. strong boundary) one. Namely, while the strong maximum principle and the strong boundary maximum principle break to hold for operator* (19)*, the weak maximum principle and the weak boundary maximum principle are valid. Indeed,*

$$\sup_{\bar{Q}_{t^\star}} v = \frac{\gamma}{2}e^{\gamma t^\star/2} \le 0 = \max\{0, \frac{\gamma}{2}, \gamma e^{\gamma t^\star/2}\} =$$

$$= \max\{0, \sup_{\Gamma_{t^\star}} v\} + t^\star \max\{0, \sup_{Q_{\bar{t}^\star}} Lv\}$$

and

$$\sup_{\bar{Q}_{t^\star}} v = \frac{\gamma}{2} e^{\gamma t^\star/2} \leq 0 = \max\{0, \frac{\gamma}{2}, \gamma e^{\gamma t^\star/2}\} = \max\{0, \sup_{\Gamma_{t^\star}} v\}.$$

After analyzing the condition in Theorem 3.4, we investigate a special operator

$$L = \frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2} - \gamma, \tag{20}$$

which is widely used in applications: the so-called one-dimensional heat conduction operator with linear source term.

Theorem 3.9 *The operator L in* (20) *does not posses the weak boundary maximum principle for all constant $\gamma \geq 1$.*

Proof. We set $Q_T = (0, \pi) \times (0, T)$ again, and $t^\star$ is an arbitrary value from the interval $(0, T)$.

Let $\gamma > 1$. We consider the function

$$v(x, t) = e^{(\gamma-1)t} \sin x,$$

for which trivially $Lv = 0$ in Q_T, and $v(0, t) = v(\pi, t) = 0$ for all $t \in [0, T]$. For this function we have

$$\sup_{\bar{Q}_{t^\star}} v = e^{(\gamma-1)t^\star} > 1 = \max\{0, \sup_{\Gamma_{t^\star}} v\}.$$

Thus, the operator (20) does not possess the weak boundary maximum principle for the values $\gamma > 1$.

Let $\gamma = 1$, and we consider the function

$$v(x, t) = \frac{8}{\pi} \sum_{\substack{k=1 \\ k \text{ is odd}}}^{\infty} \frac{1}{k^3} e^{(1-k^2)t} \sin(kx)$$

for which $Lv = 0$ in Q_T, and $v(0, t) = v(\pi, t) = 0$ for all $t \in [0, T]$. It is known from the theory of Fourier series that

$$v(x, 0) = \frac{8}{\pi} \sum_{\substack{k=1 \\ k \text{ is odd}}}^{\infty} \frac{1}{k^3} \sin(kx) = x(\pi - x),$$

that is

$$\max_{x \in [0,\pi]} \{v(x, 0)\} = \pi^2/4.$$

Supposing that $t^\star > 1/10$, we obtain that

$$v(\pi/2,t^\star) = \frac{8}{\pi} + \frac{8}{\pi}e^{t^\star} \sum_{\substack{k=3 \\ k \text{ is odd}}}^{\infty} \frac{(-1)^{(k-1)/2}}{k^3}\, e^{-k^2 t^\star} \geq$$

$$\geq \frac{8}{\pi} - \frac{8}{\pi}e^{t^\star} \sum_{\substack{k=3 \\ k \text{ is odd}}}^{\infty} \frac{1}{k^3}\, e^{-k^2 t^\star} \geq \frac{8}{\pi} - \frac{8}{27\pi}e^{-8t^\star} \sum_{\substack{k=3 \\ k \text{ is odd}}}^{\infty} e^{-8(k-3)t^\star} =$$

$$= \frac{8}{\pi} - \frac{8}{27\pi}e^{-8t^\star}\sum_{l=0}^{\infty}(e^{-16t^\star})^l = \frac{8}{\pi} - \frac{8}{27\pi}e^{-8t^\star}\frac{1}{1-e^{-16t^\star}} \geq$$

$$\geq \frac{8}{\pi} - \frac{8}{27\pi}e^{-4/5}\frac{1}{1-e^{-8/5}} = 2.4934.$$

This yields that

$$\sup_{\bar{Q}_{t^\star}} v \geq 2.4934 > 2.4674 = \frac{\pi^2}{4} = \max\{0, \sup_{\Gamma_{t^\star}} v\}.$$

This shows that the weak boundary maximum principle does not hold for the operator (20) for the value $\gamma = 1$ either. ∎

Remark 3.10 *The operator given in Theorem 3.9 would serve as a good example in Theorem 3.6 for the case* $\gamma \geq 1$*. However, we note that Theorem 3.6 does not imply directly Theorem 3.9 for arbitrary positive* γ *values.*

4 Summary

In this paper, we analyzed the qualitative properties of second order linear parabolic partial differential operators. We showed that these operators are monotone. If the coefficient function of v is less or equal to zero, then the operator possesses both the weak boundary maximum principle and the weak maximum principle. If the coefficient function is zero, then the operator fulfills all the discussed maximum principles. We gave examples that show that the obtained conditions are not only sufficient but they are necessary, too. In view of the generality of Section 2, our results can be applied not only to the investigated parabolic operator but also to other linear operators.

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