Hopf Bifurcation of Integro-Differential Equations

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Abstract

A method reducing integro-differential equations (IDE's) to systems of ordinary differential equations is proposed. Stability and bifurcation phenomena in critical cases are studied using this method. An analog of Hopf bifurcation for scalar IDE's of first order is obtained. Conditions for existence of periodic solution are proposed. We conclude that phenomena typical for two dimensional systems of ODE's appear for scalar IDE's.

Key words: integral equations, bifurcation, periodic solution, stability[1]

1. Preliminaries

Asymptotic properties of integro-differential equations (IDE's) have been studied by many researchers (see, the monographs of N. V. Azbelev, V. P. Maksimov and L. F. Rakhmatullina [1], T. A. Burton [2], K. Corduneanu [3], R. K. Miller [4] and review of M. I. Imanaliev et al [5]. In our paper a new approach called Reduction Method is suggested. This method is to be applied to investigation of asymptotic properties of convolution IDE's. The method reduces an IDE to a system of ordinary differential equations (ODE's).

[1]This paper is in a final form and no version of it will be submitted for publication elsewhere

Let us consider the following IDE

$$u^{(r)}(t) = f(t, \varphi_0(u(t)), \varphi_1(u'(t)), \ldots, \varphi_{r-1}(u^{(r-1)}(t)), \int_0^t J_0(t-s)\xi_0(u(s))ds,$$

$$\int_0^t J_1(t-s)\xi_1(u'(s))ds, \ldots, \int_0^t J_{r-1}(t-s)\xi_{r-1}(u^{(r-1)}(s))ds), \qquad (A)$$

where $t \in [0, +\infty)$, f is a Caratheodory function, φ_i and ξ_i are polynomials, J_i are functions sufficiently smooth in $[0, +\infty)$ for $i = 0, 1, \ldots, r-1$, $f(t, 0, \ldots, 0) = 0$ for $t \in [0, +\infty)$.

Let us denote by n_k the minimal number such that $J_k^{(n_k-1)}(0) \neq 0$, and $A_k = J_k^{(n_k-1)}(0)$. Then we introduce the following system:

$$u^{(r)}(t) = f(t, u(t), u'(t), \ldots, u^{(r-1)}(t), A_0 v_0(t), A_1 v_1(t), \ldots, A_{r-1} v_{r-1}(t))$$

$t \in [0, +\infty)$,

$$\begin{array}{rll} (L_0 v_0)(t) = \xi_0(u(t)) & v_0^{(i-1)}(0) = 0 & i = 1, \ldots, n_0, \\ (L_1 v_1)(t) = \xi_1(u'(t)) & v_1^{(i-1)}(0) = 0 & i = 1, \ldots, n_1, \\ \cdots & \cdots & \cdots \\ (L_{r-1} v_{r-1})(t) = \xi_{r-1}(u^{(r-1)}(t)) & v_{r-1}^{(i-1)}(0) = 0 & i = 1, \ldots, n_{r-1} \end{array} \qquad (B)$$

where $L_0, \ldots, L_{r-1}$ are linear differential operators such that the functions $\frac{J_0(t-s)}{A_0}, \ldots, \frac{J_{r-1}(t-s)}{A_{r-1}}$ are the Cauchy functions (see [1]) of the equations $(L_i v)(t) = 0$, $t \in [0, +\infty)$, $i = 0, 1, \ldots, r-1$, respectively. The recent work [14] contains many examples of such operators L_i.

The connection between equation (A) and system of ODE's (B) can be formulated through the following assertion: solution $u(t)$ of equation (A) and component $u(t)$ of solution vector of system (B) coincide.

The Reduction Method can be extended, for instance, to the case of the kernels $K_i(t, s) = \sum_{j=1}^{m} J_{ij}(t-s)\alpha_{ij}(t)\beta_{ij}(s)$ instead of $J_i(t-s)$.

Let us apply the Reduction Method to investigate certain bifurcation phenomena that take place in nonlinear IDE's. Note that similar phenomena, analogous to Hopf bifurcation in systems of ODE's, for example, can be found in scalar IDE's.

Let us focus on the system of IDE's

$$u' + \varphi(\mu, u) + \int_0^t e^{A(\mu)(t-s)} \xi(\mu, u(s)) ds = 0, \tag{1.1}$$

where $u \in R^n$ and vector functions φ and ξ are sufficiently smooth and may be represented in the following form:

$$\varphi(\mu, u) = B(\mu)u + \varphi_2(\mu, u) + \varphi_3(\mu, u) + O(\|u\|^4),$$

$$\xi(\mu, u) = C(\mu)u + \xi_2(\mu, u) + \xi_3(\mu, u) + O(\|u\|^4). \tag{1.2}$$

In (1.1) and (1.2) $A(\mu)$, $B(\mu)$ and $C(\mu)$ are matrices, $\varphi_i(\mu, u)$ and $\xi_i(\mu, u)$ are vector-forms of $i-$th order in u and continuous in μ, where μ is a small parameter. Note that the initial condition $u(0) = 0$ is satisfied by the unique solution $u = 0$ of system (1.1), (1.2). Let us investigate stability of the trivial solution and any possible bifurcation phenomena in its neighborhood.

First of all we describe conditions on matrices $A(0)$, $B(0)$ and $C(0)$, under which the problem of stability is essentially nonlinear. Using the Reduction Method we consider a system of ODE's corresponding to (1.1), (1.2). It is clear that it looks like a $2n$-dimensional system ($(u, v) \in R^{2n}$)

$$u' = -v - B(\mu)u - \varphi_2(\mu, u) - \varphi_3(\mu, u) - O(\|u\|^4),$$

$$v' = A(\mu)v + C(\mu)u + \xi_2(\mu, u) + \xi_3(\mu, u) + O(\|u\|^4). \tag{1.3}$$

If the vector $(u(t), v(t))$ is a solution of system (1.3) such that $v(0) = 0$, then vector $u(t)$ is a solution of system of IDE's (1.1).

Let us introduce the $2n \times 2n$ matrix

$$D(\mu) = \begin{pmatrix} -B(\mu) & -E \\ C(\mu) & A(\mu) \end{pmatrix} \tag{1.4}$$

If for $\mu = 0$ any critical case of Lyapunov stability occurs in system (1.3) then it follows from the correlation between (1.1) and (1.3) that the problem of stability of the trivial solution of system (1.1) cannot be solved by linear approximation. In other words in the case when part of the spectrum of matrix $D(0)$ lies on the imaginary axis, and the noncritical part is found on the left from the imaginary axis, then the problem cannot be solved. In this case bifurcation phenomena for small μ are well known for system (1.3) (see V. I. Arnold [6] and J. K. Hale [11]) what is supposed to cause certain bifurcation phenomena in equation (1.10) as well. The classical theory of bifurcations (for example, theorems of Andronov, Hopf, Bogdanov-Takens) can be applied to the investigation of bifurcation phenomena for a system of type (1.3) in the case when the critical part of the spectrum contains a pair of pure imaginary eigenvalues as well as one or two zero eigenvalues (see V. I. Arnold [6], J. E. Marsden [7], B. D. Hassard, N. D. Kazarinof and Y. H. Wan [13]). To the more complicated cases, some of which are discussed by Yu. N. Bibikov [8] and Ya. M. Goltser [9,10], Hale's result on invariant manifolds [11] can be applied.

2. Scalar Case

Here we would like to show that even scalar IDE (1.1) can reflect bifurcation phenomena that are considered to be typical for two-dimension systems of ODE's. Let us consider the following equation with $u \in R^1$:

$$u' + \varphi(u) + \int_0^t e^{A(t-s)} \xi(u(s)) ds = 0, \tag{2.1}$$

where

$$\varphi(u) = Bu + lu^2 + du^3 + O(|u|^4), \ \xi(u) = Cu + \gamma u^2 + \delta u^3 + O(|u|^4), \tag{2.2}$$

matrix D looks like

$$D = \begin{bmatrix} -B & -1 \\ C & A \end{bmatrix}$$

with eigenvalues $\lambda_{1,2} = -\frac{(B-A)}{2} \pm \sqrt{\left(\frac{B-A}{2}\right)^2 - C + AB}$. Let us assume that $\mu = -\frac{(B-A)}{2}$ and $C > B^2$. Then (2.1) becomes

$$u' + \varphi(u) + \int_0^t e^{(B+2\mu)(t-s)} \xi(u(s)) ds = 0, \tag{2.1_μ}$$

and the system of ODE's corresponding to it will be

$$\begin{aligned} u' &= -Bu - v - lu^2 - du^3 - O(|u|^4), \\ v' &= Cu + (B + 2\mu)v + \gamma u^2 + \delta u^3 + O(|u|^4), \end{aligned} \tag{2.3}$$

with matrix

$$D(\mu) = \begin{bmatrix} -B & -1 \\ C & B+2\mu \end{bmatrix},$$

where $C > B^2$.

Eigenvalues of $D(\mu)$ can be represented as $\lambda = \mu + i\omega(\mu)$, $\lambda^T = \mu - i\omega(\mu)$, where

$$\omega^2(\mu) = C - B^2 - 2B\mu - \mu^2. \tag{2.4}$$

Note that inequality $C > B^2$ guarantees the spectrum of matrix $D(0)$ to be pure imaginary for $\mu = 0$, and spectrum goes out of imaginary axis for $\mu \neq 0$. It is a typical Hopf bifurcation situation for system (2.3). If $C = B^2$, then the spectrum of matrix $D(0)$ has a multiple zero eigenvalue, with Jordan cell corresponding to it. For example, matrix

$$D(0) = \begin{bmatrix} -B & -1 \\ B^2 & B \end{bmatrix} \text{ is similar to matrix } \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Thus, necessary conditions of Bogdanov-Takens bifurcation (see V. I. Arnold [6]) are fulfilled. In addition, it is obvious that there are possible bifurcations caused by spectrum passing through its single zero eigenvalue upon $AB - C = \varepsilon$ where $|\varepsilon|$ is small and $A \neq B$.

Now let us restrict ourselves to the case of $C > B^2$.

Using system (2.3) we denote nonlinear part of function $\varphi(u)$ by $\eta(u)$: $\eta(u) = lu^2 + du^3 + O(u^4)$ and then consider the following two cases:
a) general case $\eta(u) \neq 0$,
b) conservative case $\eta(u) = 0$.

2.1. General Case

Investigating the problem of stability of the trivial solution of equation (2.1) for $\mu = 0$ and bifurcations in the neighborhood of this solution for $\mu \neq 0$ we can consider system (2.3) and normalize it to the third-order terms. A detailed description of the normalization procedure one can find in [6, 7, 12,13]. We give only scheme of calculations and final formulae. Normalization procedure consists of the following steps:

Step 1. Let us reduce matrix $D(\mu)$ to Jordan form using transformation matrix

$$S = \begin{bmatrix} 1 & 1 \\ -(B+\lambda(\mu)) & -(B+\lambda^T(\mu)) \end{bmatrix}$$

$$S^{-1} = \frac{1}{2i\omega(\mu)} \begin{bmatrix} -(B+\lambda^T(\mu)) & -1 \\ B+\lambda(\mu) & 1 \end{bmatrix}$$

$$S^{-1}DS = \begin{bmatrix} \lambda(\mu) & 0 \\ 0 & \lambda^T(\mu) \end{bmatrix} \equiv K(\mu),$$

where $\lambda(\mu)$ is defined by (2.4).

If real vector $x = (u, v)$ and complex vector $z = (y, y^T)$, where $z = S^{-1}x$ are introduced, we obtain the complex system

$$z' = K(\mu)z + F(\mu, z) \tag{2.5}$$

instead of (2.3). Here

$$F(\mu, z) = S^{-1}(\mu)(-\eta(Sz), \kappa(Sz)) \equiv (f(\mu, z, z^T), f^T(\mu, z, z^T)),$$

κ is the nonlinear part of ξ.

Let y^T denote the complex adjoint to y. Thus we obtain the equation

$$y' = K(\mu)y + f(\mu, z, z^T) \tag{2.6}$$

where

$$f(\mu, z, z^T) = L(z+z^T)^2 + M(z+z^T)^3 + O(|z|^4), \tag{2.7}$$

$L = \frac{1}{2i\omega(\mu)}((B+\lambda^T)l - \gamma)$, and $M = \frac{1}{2i\omega(\mu)}((B+\lambda^T)d - \delta)$.

Step 2. Let us introduce normalizing transformation

$$y = w + \sum_{j+k=2}^{3} c_{jk}(\mu) w^j w^{Tk} \tag{2.8}$$

so that equation for w is of the form

$$w' = \lambda(\mu)w + g_{21}(\mu)w^2 w^T + O(|w|^4). \tag{2.9}$$

By calculating the coefficients $c_{jk}(\mu)$ of the transformation (2.7) for $j+k=2$ we obtain

$$c_{11}(\mu) = \frac{L}{\lambda^T}, \quad c_{02}(\mu) = \frac{L}{2\lambda^T - \lambda}, \quad c_{20}(\mu) = \frac{L}{\lambda} \tag{2.10}$$

and for $g_{21}(\mu)$ we obtain

$$g_{21}(\mu) = 3M + 2L(c_{20} + c_{02}^T + 2c_{11} + 2c_{11}^T), \tag{2.11}$$

(while calculating $g_{21}(\mu)$, we assume that $c_{21} = 0$). From (2.10) and (2.11) we obtain

$$g_{21}(\mu) = 3M + 2L^2 \frac{2\lambda + \lambda^T}{|\lambda|^2} + \frac{2|L|^2}{2\lambda - \lambda^T} + \frac{4|L|^2}{\lambda}. \tag{2.12}$$

From (2.12) we obtain

$$\operatorname{Re} g_{21}(\mu) = \alpha_{21}(\mu) = -\frac{3}{2}d + \tag{2.13}$$

$$\frac{1}{2(\mu^2+\omega^2(\mu))}\left[3\mu\left(l^2 - \frac{k(\mu)}{\omega^2(\mu)}\right) - 2lk(\mu) + \frac{\mu(3\mu^2+19\omega^2(\mu))}{\mu^2+9\omega^2(\mu)}\left(l^2 + \frac{k^2(\mu)}{\omega^2(\mu)}\right)\right],$$

where $k(\mu) = (B+\mu)l - \gamma$,

$$\alpha_{21}(0) = -\frac{3}{2}d - \frac{l(Bl-\gamma)}{\omega^2(0)}. \tag{2.14}$$

Using certain results of critical cases theory (see [12]) we come to the following conclusion regarding system (2.3).

Assertion 1. Let $\mu = 0$. *If* $\alpha_{21}(0) < 0$ ($\alpha_{21}(0) > 0$), *then the trivial solution* $x = (u, v) = (0, 0)$ *of system* (2.3) *is asymptotically stable* (*unstable*). *The point* $x = (0, 0)$ *of system* (2.3) *is a stable* (*unstable*) *focus.*

In other words, any trajectory $(u(t), v(t))$, starting from a sufficiently small neighborhood of $(0, 0)$ tends to $(0, 0)$ spiralling upon $t \to +\infty$ ($t \to -\infty$), when $\alpha_{21}(0) < 0$ ($\alpha_{21}(0) > 0$).

It follows from Assertion 1 that component $u(t)$ has the same limiting property for sufficiently small $u(0) = u_0$ and $v(0) = 0$: $u(t) \to 0$ for $t \to +\infty$ ($t \to -\infty$) if $\alpha_{21}(0) < 0$ ($\alpha_{21}(0) > 0$).

Now let $\mu \neq 0$ in system (2.3) be sufficiently small. It follows from Assertion 1 and formula (2.4) for λ that change of stability of the trivial solution of system (2.3) takes place if values of μ satisfy the condition

$$\operatorname{sign}\mu = -\operatorname{sign}\alpha_{21}(0). \tag{2.15}$$

For example, if $\alpha_{21}(0) < 0$ and $\mu > 0$, then the asymptotically stable trivial solution $x = (0, 0)$ for $\mu = 0$ becomes unstable for $\mu > 0$ and it remains asymptotically stable for $\mu < 0$.

Using the Hopf theorem (see, for instance, Theorem III from [13]) we come to the following assertion:

Assertion 2. Suppose that $\mu \neq 0$ in system (2.3) *and $\alpha_{21}(0) \neq 0$. Then for any sufficiently small μ, fulfilling condition* (2.15) *a limit cycle appears in system* (2.3). *This limit cycle is stable for $\alpha_{21}(0) < 0$ and unstable for $\alpha_{21}(0) > 0$.*

Remark 1. Let us consider the limit cycle of system (2.3) where its point of intersection with axis v is $(u_0(\mu), 0)$. The solution of system (2.3) starting from this point at $t = 0$ is periodic $(u(\mu, t),\ v(\mu, t))$, $u(\mu, 0) = u_0(\mu)$, $v(\mu, 0) = 0$. It follows from the Reduction Method that function $u(\mu, t)$, $u(\mu, 0) = u_0(\mu)$ is a periodic solution of IDE (2.1) for small μ, fulfilling condition (2.15). Summarizing the results described in Assertions 1, 2 and taking into account the remarks above we can state the following theorem on stability and occurence of periodic solutions of IDE (2.1):

Theorem 1. Suppose that for equation (2.1_μ) *the followings are fulfilled:*
a) $C > B^2$,
b) number $\alpha_{21}(0)$ specified by (2.14) *is different from zero.*

Then
1) the trivial solution $u = 0$ at $\mu = 0$ is asymptotically stable (*unstable*) *if $\alpha_{21}(0) < 0$ ($\alpha_{21}(0) > 0$);*
2) for every sufficiently small μ there is a periodic solution $u(\mu, t)$, where $u(\mu, 0)$ is the point of intersection of the limit cycle of system (2.3) *and the axis u.*

Remark 2. Note that the periodic solution in Assertion 2 appears when stability of the trivial solution is changed.

Remark 3. A periodic solution can be actually found in the construction of the family of periodic solutions, corresponding to the limit cycle of system (2.3), by techniques developed in bifurcation theory of Hopf (see [13]).

2.2. Conservative Case

It follows from (2.14) that in case $l = d = 0$ we obtain $\alpha_{21}(0) = 0$. In order to investigate properties of equation (2.1_μ) one has to continue normalization up to terms of order higher than 3. If $\mu = 0$, the problem is to distinguish between centre and focus. Here we would like to show that, if nonlinear part $\eta(u)$ of $\varphi(u)$ is zero, a point (0,0) of system (2.3) is always a center. It means that all real parts of normal form coefficients are equal to zero in the normalization process (nonlinear part of $\xi(u)$ is assumed to be an analytical function).

Assertion 3. Let $\mu = 0$, $\eta(u) = 0$ in system (2.3). *Then*
1) system (2.3) *has sign definite first integral;*

2) point $(0,0)$ *is a center;*
3) the solutions, starting from a sufficiently small neighborhood of point $(0,0)$ *form a two-parameter family of periodic solutions.*

Proof. According to Assertion 3 system (2.3) looks like

$$u' = -Bu - v, \quad v' = Cu + Bv + \kappa(u), \tag{2.17}$$

where $\kappa(u)$ is the nonlinear part of $\xi(u)$, $C > B^2$.

Let us consider the function

$$2H(u,v) = Cu^2 + 2Buv + v^2 + 2\int_0^u \kappa(s)ds.$$

It is easy to check that function H is an integral of system (2.17). It follows from inequality $C > B^2$ that H is a sign definite first integral, what proves Part 1. Assertions in Parts 2 and 3 follows from theorem of Lyapunov-Poincaret on the problem of centre and focus (e.g. see [12]).

Now let us consider equation (2.1_μ) for $\mu = 0$ and $\eta(u) = 0$:

$$u' + Bu + \int_0^t e^{B(t-s)}\xi(u(s))ds = 0. \tag{2.18}$$

The following result follows from Assertion 3:

Theorem 2. Let $\xi(u) = Cu + \kappa(u)$, *and suppose that* $C > B^2$ *in equation* (2.18). *Then all solutions of equation* (2.18) *are periodic for sufficiently small* $u_0 = u(0)$.

In other words, we can say that equation (2.18) has a one-parameter family of periodic solutions.

Remark 4. It is obvious that equation (2.18) is equivalent to the second order ODE

$$u'' + (B^2 - C)u = -\kappa(u), \quad u(0) = u_0, \quad u'(0) = -Bu_0,$$

or to the system

$$x_1' = x_2, \qquad x_2' = (C - B^2)x_1 - \kappa(x_1). \tag{2.19}$$

The point $x_1 = x_2 = 0$ is known to be center (e.g. see J. K. Hale [11]). We separate the one-parameter family out of bi-parametric family of periodic solutions that satisfies initial conditions $x_1(0) = u_0$, $x_2(0) = -Bu_0$ for any small u_0. Introducing a small parameter into (2.19) and using well known techniques (see J. K. Hale [11], N. N. Bogolubov and Yu. A. Mitropolsky [15]), we can obtain asymptotic representation of periodic solutions and their period.

3. Some General Remarks

It has been shown above that the Reduction Method combined with the Method of Normal Forms allows to investigate certain nonlinear phenomena in scalar IDE's. By using an analogous technique one can also study multi-dimensional IDE's of type (1.1), (1.2). In particular, this approach makes it possible to study problems on stability and bifurcations of invariant torus in the case when the spectrum of matrix $D(\mu)$ (see (1.4)) contains a few pairs of pure imaginary eigenvalues for $\mu = 0$. Similar problems for ODE's were investigated, for example, in [8-10].

The theory of invariant manifolds developed by J. K. Hale [11] for FDE's and combined with the method of Normal Forms allows to study similar bifurcation phenomena for delay equations with parameters [16-18] as well. The Reduction Method combined with the method of Normal Forms enables to study bifurcation phenomena in more complicated systems containing both integral terms and delay.

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