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Least energy nodal solutions for elliptic equations with indefinite nonlinearity

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Abstract. We prove the existence of a nodal solution with two nodal domains for the Dirichlet problem with indefinite nonlinearity

$$-\Delta_p u = \lambda |u|^{p-2}u + f(x)|u|^{\gamma-2}u$$

in a bounded domain $\Omega \subset \mathbb{R}^n$, provided $\lambda \in (-\infty, \lambda_1^*)$, where λ_1^* is a critical spectral value. The obtained solution has the least energy among all nodal solutions on the interval $(-\infty, \min\{\lambda_1^*, \lambda_2\})$, where λ_2 is the second Dirichlet eigenvalue of $-\Delta_p$ in Ω. Moreover, the obtained solution forms a branch with continuous energy on $(-\infty, \lambda_1^*)$.

Keywords: nodal solutions, indefinite nonlinearity, p-Laplacian, fibering method, critical spectral parameter.

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1 Introduction and main results

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with the smooth boundary $\partial\Omega$, $N \geq 1$. We consider the Dirichlet boundary value problem

$$\begin{cases} -\Delta_p u = \lambda |u|^{p-2}u + f(x)|u|^{\gamma-2}u, & x \in \Omega, \\ u|_{\partial\Omega} = 0, \end{cases} \tag{$\mathcal{D}$}$$

where $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian, $\lambda, p, \gamma \in \mathbb{R}$ and

$$1 < p < \gamma < p^*, \quad \text{where } p^* = \begin{cases} \frac{pN}{N-p} & \text{if } p < N, \\ +\infty & \text{if } p \geq N. \end{cases} \tag{1.1}$$

The function $f \in L^\infty(\Omega)$ is assumed to be sign-changing and therefore the nonlinearity of ($\mathcal{D}$) is called *indefinite*. Hereinafter we denote

$$\Omega^+ := \{x \in \Omega : f(x) > 0\}.$$

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The questions of existence, nonexistence and multiplicity of *positive* solutions to the problems of type ($\mathcal{D}$) have been comprehensively studied under various assumptions on differential operator, spatial domain, coefficients and structure of nonlinearity, see, e.g., [5, 11, 14, 15]. In particular, in [14] the explicit critical value λ^* was introduced, such that ($\mathcal{D}$) admits at least one positive solution for any $\lambda < \lambda^*$ and no positive solutions for $\lambda > \lambda^*$. In spite of plenty of references, the multiplicity of solutions on a local interval $(\lambda_1, \lambda_1 + \varepsilon)$ was proved in [11] using the fibering method, and this result was extended in [15] to the interval (λ_1, λ_1^*) (see Figure 1.1), where

$$\lambda_1^* := \inf_{u \in W_0^{1,p}} \left\{ \frac{\int_\Omega |\nabla u|^p \, dx}{\int_\Omega |u|^p \, dx} : \int_\Omega f(x) |u|^\gamma \, dx \geq 0 \right\}. \tag{1.2}$$

Note that $\lambda_1^* < +\infty$ if $\nu(\Omega^+) > 0$, where ν is the n-th Lebesgue measure, and under the assumption $\int_\Omega f(x) |\varphi_1|^\gamma \, dx < 0$ one can guarantee that $\lambda_1^* > \lambda_1$, whereas $\lambda_1^* = \lambda_1$ in the opposite case. Here by (λ_1, φ_1) we denote the first eigenpair of the operator $-\Delta_p$ on Ω with zero Dirichlet boundary conditions [3].

At the same time, in the last few decades the questions of existence, multiplicity and qualitative properties of *nodal* (sign-changing) solutions to the wide class of elliptic equations have attracted a lot of attention, cf. [1, 4, 6, 8, 9, 17] and survey [19] for historical overview and references. Nevertheless, to our best knowledge, there are only few articles concerning the existence of nodal solutions for the problems of type ($\mathcal{D}$). We can mention [1, 9, 17], where some of existence and multiplicity results have been proved using different topological and variational arguments. Note that these works deal mainly with the Laplace operator ($p = 2$). Moreover, to the best of our knowledge, the questions of the qualitative properties of nodal solutions to ($\mathcal{D}$) such as the precise number of nodal domains, property of the least energy among all nodal solutions, formation of branches, etc., have not been concerned.

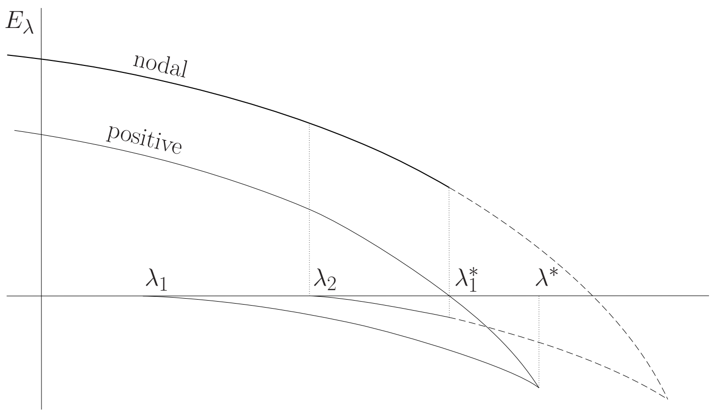

Figure 1.1: Branches of solutions w.r.t the energy E_λ; $\nu(\Omega^+) > 0$, $\lambda_2 < \lambda_1^*$.

In the present article we apply the constructive minimization technique of the Nehari manifolds with the fibering approach (see, e.g., [4, 8]) for the problem ($\mathcal{D}$), which allows us to prove the existence of a nodal solution with two nodal domains for any $\lambda \in (-\infty, \lambda_1^*)$ and the least energy among all nodal solutions on $(-\infty, \min\{\lambda_1^*, \lambda_2\})$ (see Figure 1.1). Here by λ_2 we denote the second eigenvalue of zero Dirichlet $-\Delta_p$ in Ω (see (1.4)).

A similar approach has been used in [6] to obtain the sign-changing solutions with positive energy for the elliptic equations with convex-concave nonlinearity

$$\begin{cases} -\Delta u = \lambda |u|^{q-2}u + |u|^{\gamma-2}u, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1.3}$$

where $1 < q < 2 < \gamma < 2^*$. The method of proof carries over to the corresponding problem with the p-Laplacian.

Finally, we note that the disadvantage of the Nehari manifolds method consists in the fact that it cannot be used in proving the existence of nodal solutions with negative energy for ($\mathcal{D}$) and (1.3), however the existence of such solutions is known [17].

Before introducing our main results let us recall some common notations.

By a *weak solution* of ($\mathcal{D}$) we mean a critical point $u \in W_0^{1,p}(\Omega)$ of the energy functional

$$E_\lambda(u) := \frac{1}{p} H_\lambda(u) - \frac{1}{\gamma} F(u),$$

where

$$H_\lambda(u) := \int_\Omega |\nabla u|^p \, dx - \lambda \int_\Omega |u|^p \, dx, \qquad F(u) := \int_\Omega f(x) \, |u|^\gamma \, dx.$$

As usual, by $W_0^{1,p} := W_0^{1,p}(\Omega)$ we denote the standard Sobolev space equipped with the norm

$$\|u\| = \left(\int_\Omega |\nabla u|^p \, dx \right)^{1/p}.$$

By a weak *nodal* solution of ($\mathcal{D}$) we mean a critical point $u \in W_0^{1,p}$ of E_λ such that $u^\pm \not\equiv 0$ a.e. in Ω, where u^+ and u^- are the positive and negative parts of u, respectively. Note that $u^\pm \in W_0^{1,p}(\Omega)$ (see, e.g., [16, Corollary A.5, p. 54]). Moreover, using the classical bootstrap arguments (see, e.g., [10, Lemma 3.2, p. 114]) it is not hard to show that under assumptions (1.1) and $f \in L^\infty(\Omega)$ each weak solution of ($\mathcal{D}$) belongs to $L^\infty(\Omega)$, and therefore to $C^{1,\alpha}(\overline{\Omega})$, by [18]. By a *nodal domain* of a function $u \in C(\Omega)$ we denote any maximal connected open subset of $\{x \in \Omega : u(x) \neq 0\}$.

From the definition of a weak solution it follows that any weak solution $u \in W_0^{1,p}$ of ($\mathcal{D}$) satisfies

$$Q_\lambda(u) := \langle DE_\lambda(u), u \rangle = H_\lambda(u) - F(u) = 0,$$

and therefore any nontrivial solution belongs to the so-called *Nehari manifold*

$$\mathcal{N}_\lambda := \{u \in W_0^{1,p} \setminus \{0\} : Q_\lambda(u) = 0\}.$$

Clearly, each nodal solution of ($\mathcal{D}$) belongs to the *nodal Nehari set*

$$\mathcal{M}_\lambda := \{u \in W_0^{1,p} : u^\pm \in \mathcal{N}_\lambda\}.$$

The Nehari manifolds method [4, 8] enables one to find a nodal solution of ($\mathcal{D}$) as a minimum point of the energy functional E_λ on $\mathcal{M}_\lambda$. However, due to the fact that E_λ possesses critical points both with positive and nonpositive energy (see Lemma 2.2 below), a minimization sequence for E_λ over $\mathcal{M}_\lambda$ will converge, in general, to a positive solution. To overcome this difficulty, we distinguish critical points with the different signs of energy and seek for a nodal solution of ($\mathcal{D}$) as a minimum point of E_λ on the following subset of $\mathcal{M}_\lambda$:

$$\mathcal{N}_\lambda^1 := \{u \in W_0^{1,p} : u^\pm \in \mathcal{N}_\lambda, \, E_\lambda(u^\pm) > 0\}.$$

Our main result is the following.

Theorem 1.1. *Assume that* (1.1) *is satisfied and* $\lambda < \lambda_1^*$.

1. *If* $\nu(\Omega^+) > 0$, *then there exists a weak nodal solution* $u_\lambda \in \mathcal{N}_\lambda^1$ *of the problem* ($\mathcal{D}$) *with precisely two nodal domains. Moreover,* u_λ *has the least energy among all weak nodal solutions of* ($\mathcal{D}$) *on* $(-\infty, \min\{\lambda_1^*, \lambda_2\})$, *i.e.,*
$$-\infty < E_\lambda(u_\lambda) \leq E_\lambda(w_\lambda)$$
for any weak nodal solution w_λ *of* ($\mathcal{D}$) *on this interval.*

2. *If* $\nu(\Omega^+) = 0$, *then there are no weak nodal solutions of the problem* ($\mathcal{D}$) *for any* $\lambda \in (-\infty, \min\{\lambda_1^*, \lambda_2\})$.

In the proof it will be convenient to use the following variational characterization of the second eigenvalue λ_2 of the zero Dirichlet $-\Delta_p$ in Ω (see [12], p. 195):

$$\lambda_2 := \inf_{\mathcal{A} \in \mathcal{F}_2} \sup_{u \in \mathcal{A}} \int_\Omega |\nabla u|^p \, dx, \tag{1.4}$$

where

$$\mathcal{F}_2 := \left\{ \mathcal{A} \subset \mathcal{S} : \exists\, h \in C(S^1, \mathcal{A}) : h \text{ is odd} \right\}, \quad \mathcal{S} := \left\{ u \in W_0^{1,p} : \int_\Omega |u|^p \, dx = 1 \right\} \tag{1.5}$$

and S^1 represents the unit sphere in $\mathbb{R}^2$. By $\varphi_2 \in W_0^{1,p}$ we denote the corresponding second eigenfunction and note that $\varphi_2^\pm \not\equiv 0$.

The second result concerns the formation of branches by the nodal solutions to ($\mathcal{D}$). We say that the family $\{u_\lambda\}$ of critical points of E_λ forms *a continuous branch* on (a, b) (with respect to levels of E_λ) if the map
$$E_{(\cdot)}(u_{(\cdot)}) \colon (a, b) \longrightarrow \mathbb{R}$$
is a continuous function.

Theorem 1.2. *Assume that* (1.1) *is satisfied and* $\nu(\Omega^+) > 0$. *Then the set of nodal solutions* u_λ *for* ($\mathcal{D}$), *given by Theorem 1.1, forms a continuous branch on* $(-\infty, \lambda_1^*)$.

We note that the Nehari manifolds method leads to the similar results as above for more general class of problems of type ($\mathcal{D}$):

$$\begin{cases} -\Delta_p u = \lambda g(x)|u|^{p-2}u + f(x)|u|^{\gamma-2}u, & x \in \Omega, \\ u|_{\partial\Omega} = 0, \end{cases}$$

where $g(x) \in L^\infty(\Omega)$ also changes the sign. Nevertheless, we sacrifice this case for simplicity of exposition.

The paper is organized as follows. In Section 2, we give some auxiliary results concerning the properties of E_λ. Section 3 contains the proof of the existence of a weak nodal solution of ($\mathcal{D}$). In Section 4, we show that the obtained solutions have precisely two nodal domains and the the least energy property. Moreover, in Section 4 we show the nonexistence result. In Section 5, we prove that the set of such nodal solutions forms a continuous branch.

2 Auxiliary results

First we show the following lemma.

Lemma 2.1. *Assume that* (1.1) *is satisfied and* $Q_\lambda(u) = 0$ *for some* $u \in W_0^{1,p}$. *Then the following equivalences hold:*

1. $H_\lambda(u) > 0 \iff F(u) > 0 \iff E_\lambda(u) > 0$;
2. $H_\lambda(u) = 0 \iff F(u) = 0 \iff E_\lambda(u) = 0$;
3. $H_\lambda(u) < 0 \iff F(u) < 0 \iff E_\lambda(u) < 0$.

Proof. Let $u \in W_0^{1,p}$ and $Q_\lambda(u) = 0$. Then

$$E_\lambda(u) = \frac{\gamma - p}{\gamma p} H_\lambda(u). \tag{2.1}$$

Since $1 < p < \gamma$, all the statements of the lemma are satisfied. □

Using this lemma it is easy to see that if $\lambda < \lambda_1$, then $E_\lambda(u) > 0$ for any nontrivial weak solution $u \in W_0^{1,p}$, whereas for $\lambda \geq \lambda_1$ the energy $E_\lambda(u)$ may be either positive, or nonpositive.

Let us consider the fibered version of the functional E_λ, given by

$$E_\lambda(tu) = \frac{t^p}{p} H_\lambda(u) - \frac{t^\gamma}{\gamma} F(u), \quad t > 0.$$

The next lemma describes the structure of critical points of $E_\lambda(tu)$ w.r.t. $t > 0$.

Lemma 2.2. *Assume that* (1.1) *is satisfied and* $u \in W_0^{1,p} \setminus \{0\}$.

1. *If* $H_\lambda(u), F(u) > 0$, *then there exists only one positive critical point* $t(u)$ *of* $E_\lambda(tu)$ *w.r.t.* $t > 0$, *which is a global maximum point, and* $E_\lambda(t(u)u) > 0$, $Q_\lambda(t(u)u) = 0$.
2. *If* $H_\lambda(u), F(u) < 0$, *then there exists only one positive critical point* $t(u)$ *of* $E_\lambda(tu)$ *w.r.t.* $t > 0$, *which is a global minimum point, and* $E_\lambda(t(u)u) < 0$, $Q_\lambda(t(u)u) = 0$.
3. *If* $H_\lambda(u) \cdot F(u) \leq 0$ *and* $(H_\lambda(u), F(u)) \neq (0, 0)$, *then* $E_\lambda(tu)$ *has no positive critical points.*

Proof. To obtain critical points of $E_\lambda(tu)$ w.r.t. $t > 0$, let us find roots of

$$\frac{\partial}{\partial t} E_\lambda(tu) = t^{p-1} H_\lambda(u) - t^{\gamma-1} F(u) = t^{p-1} \left(H_\lambda(u) - t^{\gamma-p} F(u) \right) = 0.$$

Hence, if $H_\lambda(u) \cdot F(u) \leq 0$ and $(H_\lambda(u), F(u)) \neq (0, 0)$, then $E_\lambda(tu)$ has no positive critical points, and if $H_\lambda(u) \cdot F(u) > 0$, then there exists exactly one positive critical point, given by

$$t(u) = \left(\frac{H_\lambda(u)}{F(u)} \right)^{\frac{1}{\gamma - p}} > 0. \tag{2.2}$$

Assume first that $H_\lambda(u), F(u) > 0$. Note that

$$\frac{\partial}{\partial t} E_\lambda(tu) = \frac{1}{t} Q_\lambda(tu), \qquad \frac{\partial^2}{\partial t^2} E_\lambda(tu) = \frac{1}{t^2} \left((p-1) H_\lambda(tu) - (\gamma - 1) F(tu) \right).$$

Hence, if $t(u) > 0$ is a critical point of $E_\lambda(tu)$, then $Q_\lambda(t(u)u) = 0$ and Lemma 2.1 implies that $E_\lambda(t(u)u) > 0$. Moreover,

$$\frac{\partial^2}{\partial t^2} E_\lambda(tu)\Big|_{t=t(u)} = -\frac{(\gamma - p)}{t^2(u)} H_\lambda(t(u)u) < 0.$$

Therefore, due to the fact that there is at most one critical point of $E_\lambda(tu)$ w.r.t. $t > 0$, we conclude that $t(u)$ is a point of global maximum of $E_\lambda(tu)$.

The case $H_\lambda(u), F(u) < 0$ of statement 2 may be handled in much the same way. □

In the next result we provide the criterion for nonemptiness of $\mathcal{N}_\lambda^1$.

Lemma 2.3. *The following statements hold:*

1. $\mathcal{N}_\lambda^1 \neq \emptyset$ *for all* $\lambda \in \mathbb{R}$, *whenever* $\nu(\Omega^+) > 0$;
2. $\mathcal{N}_\lambda^1 = \emptyset$ *for all* $\lambda \in \mathbb{R}$, *whenever* $\nu(\Omega^+) = 0$.

Proof. 1. Let $\lambda \in \mathbb{R}$ and $\nu(\Omega^+) > 0$. Then we are able to choose two open balls $B_1, B_2 \subset \Omega$ sufficiently small such that $\overline{B_1} \cap \overline{B_2} = \emptyset$, $\lambda_1(B_1)$, $\lambda_1(B_2) > \lambda$ and $\nu(B_1 \cap \Omega^+)$, $\nu(B_2 \cap \Omega^+) > 0$.

Consider now the characteristic function $\chi(B_1 \cap \Omega^+)$ of the set $B_1 \cap \Omega^+$. Since $\chi(B_1 \cap \Omega^+) \in L^\infty(\Omega)$, $\chi(B_1 \cap \Omega^+) \geq 0$ and $\operatorname{supp} \chi(B_1 \cap \Omega^+) \subseteq \overline{B_1}$, the standard approximation arguments (see, e.g., [13, Lemma 7.2, p. 148]) imply the existence of $u_\varepsilon \in C_0^\infty(\Omega)$, $u_\varepsilon \geq 0$, such that $u_\varepsilon \to \chi(B_1 \cap \Omega^+)$ in $L^\gamma(\Omega)$ as $\varepsilon \to 0$, and therefore

$$\int_\Omega f|u_\varepsilon|^\gamma \, dx \to \int_\Omega f|\chi(B_1 \cap \Omega^+)|^\gamma \, dx \equiv \int_\Omega f\chi(B_1 \cap \Omega^+) \, dx = \int_{B_1 \cap \Omega^+} f \, dx > 0,$$

i.e., $F(u_\varepsilon) > 0$ for sufficiently small $\varepsilon > 0$. The similar argumentation yields the existence of $v_\varepsilon \in C_0^\infty(\Omega)$, such that $F(v_\varepsilon) > 0$ for sufficiently small $\varepsilon > 0$. Moreover, due to the assumptions $\overline{B_1} \cap \overline{B_2} = \emptyset$ and $\lambda_1(B_1), \lambda_1(B_2) > \lambda$, we can take $\varepsilon > 0$ small enough to satisfy $\operatorname{supp} u_\varepsilon \cap \operatorname{supp} v_\varepsilon = \emptyset$ and $H_\lambda(u_\varepsilon)$, $H_\lambda(v_\varepsilon) > 0$. Hence, Lemma 2.2 implies the existence of $t(u_\varepsilon), t(v_\varepsilon) > 0$ such that

$$\begin{aligned} E_\lambda(t(u_\varepsilon)\, u_\varepsilon) > 0, \qquad Q_\lambda(t(u_\varepsilon)\, u_\varepsilon) = 0, \\ E_\lambda(t(v_\varepsilon)\, v_\varepsilon) > 0, \qquad Q_\lambda(t(v_\varepsilon)\, v_\varepsilon) = 0. \end{aligned}$$

Thus, $t(u_\varepsilon)\, u_\varepsilon - t(v_\varepsilon)\, v_\varepsilon \in \mathcal{N}_\lambda^1$.

2. Let now $\nu(\Omega^+) = 0$. Then for any $u \in W_0^{1,p} \setminus \{0\}$ we have $F(u) \leq 0$, which is impossible for functions from $\mathcal{N}_\lambda^1$ in view of Lemma 2.1. □

Lemma 2.4. *Assume that* (1.1) *is satisfied,* $\lambda < \lambda_1^*$ *and* $u \in \mathcal{N}_\lambda^1$. *Then*

1. $E_\lambda(u^\pm) \to +\infty$ *as* $\|u^\pm\| \to +\infty$, *i.e.,* E_λ *is coercive on* $\mathcal{N}_\lambda^1$;
2. $\|u^\pm\| > c_1 > 0$ *and* $E_\lambda(u^\pm) > c_2 > 0$, *where the constants* c_1, c_2 *do not depend on* u.

Proof. 1. Let $u \in \mathcal{N}_\lambda^1$. From Lemma 2.1 it follows that $F(u^\pm) > 0$. Hence, $u^\pm$ are admissible functions for the minimization problem (1.2), and

$$\lambda_1^* \leq \frac{\int_\Omega |\nabla u^\pm|^p \, dx}{\int_\Omega |u^\pm|^p \, dx}. \tag{2.3}$$

Using this fact and (2.1) we get

$$E_\lambda(u^\pm) = \frac{\gamma - p}{\gamma p} H_\lambda(u^\pm) \geq \frac{\lambda_1^* - \lambda}{\lambda_1^*} \frac{\gamma - p}{\gamma p} \int_\Omega |\nabla u^\pm|^p \, dx, \tag{2.4}$$

if $\lambda \geq 0$, and

$$E_\lambda(u^\pm) = \frac{\gamma - p}{\gamma p} H_\lambda(u^\pm) \geq \frac{\gamma - p}{\gamma p} \int_\Omega |\nabla u^\pm|^p \, dx \tag{2.5}$$

for $\lambda < 0$. Therefore, by the assumption $\lambda < \lambda_1^*$, we conclude that $E_\lambda(u^\pm) \to +\infty$ as $\|u^\pm\| \to +\infty$.

2. Using (2.3) and the Sobolev embedding theorem we have the following chain for the case $\lambda \geq 0$:

$$\frac{\lambda_1^* - \lambda}{\lambda_1^*} \|u^\pm\|^p \leq H_\lambda(u^\pm) = F(u^\pm) \leq C_\gamma \|u^\pm\|^\gamma.$$

Since $\gamma > p$ and $\lambda < \lambda_1^*$ we get

$$\|u^\pm\| \geq \left(\frac{\lambda_1^* - \lambda}{C_\gamma \lambda_1^*} \right)^{\frac{1}{\gamma - p}} = c_1 > 0. \tag{2.6}$$

Combining this estimation with (2.4) we get the desired result. The case $\lambda < 0$ can be handled in the same way using the estimation (2.5). □

3 Existence of nodal solution

In this section we prove the existence of a nodal solution for the problem ($\mathcal{D}$). As noted above, we seek for a solution of ($\mathcal{D}$) as a minimizer of the problem

$$\begin{cases} E_\lambda(w) \to \inf, \\ w \in \mathcal{N}_\lambda^1. \end{cases} \tag{3.1}$$

Lemma 3.1. *Assume that* (1.1) *is satisfied,* $\nu(\Omega^+) > 0$ *and* $\lambda < \lambda_1^*$. *Then there exists a minimizer* $u \in W_0^{1,p}$ *of* (3.1) *and* $u \in \mathcal{N}_\lambda^1$.

Proof. Since $\nu(\Omega^+) > 0$, Lemma 2.3 implies that $\mathcal{N}_\lambda^1 \neq \emptyset$ for any $\lambda \in \mathbb{R}$ and therefore there exists a minimizing sequence $u_k \in \mathcal{N}_\lambda^1$, $k \in \mathbb{N}$ for (3.1). Let us denote

$$c_\lambda := \inf\{E_\lambda(w) : w \in \mathcal{N}_\lambda^1\}.$$

We have $c_\lambda \in (0, +\infty)$, since $E_\lambda > c_2 > 0$ on $\mathcal{N}_\lambda^1$ by Lemma 2.4. Hence, using the coercivity of E_λ on $\mathcal{N}_\lambda^1$, given by Lemma 2.4, we conclude that $u_k^\pm$ are bounded in $W_0^{1,p}$. Hence, there exist $u, v, w \in W_0^{1,p}$ such that, up to subsequence,

$$u_k \to u, \qquad u_k^+ \to v, \qquad u_k^- \to w,$$

weakly in $W_0^{1,p}$ and strongly in $L^{p^*}(\Omega)$.

Let us introduce the map $h \colon L^\gamma \to L^\gamma$ by $h(u) = u^+$. From [8, Lemma 2.3, p. 1046] it follows that $h \in C(L^\gamma, L^\gamma)$. Hence, $u^+ = v \geq 0$ and $u^- = w \leq 0$ in Ω. Moreover, $u^\pm \not\equiv 0$ in Ω.

Indeed, using Lemma 2.4 and Nehari constraints $Q_\lambda(u_k^\pm) = 0$ we get

$$
\begin{aligned}
0 < c_0 < \lim_{k\to+\infty} \int_\Omega |\nabla u_k^\pm|^p \, dx \\
&= \lim_{k\to+\infty} \left(\lambda \int_\Omega |u_k^\pm|^p \, dx + \int_\Omega f |u_k^\pm|^\gamma \, dx \right) \\
&= \lambda \int_\Omega |u^\pm|^p \, dx + \int_\Omega f |u^\pm|^\gamma \, dx.
\end{aligned}
$$

Now we show that $u_k^\pm \to u^\pm$ strongly in $W_0^{1,p}$. For this end note that $F(u^\pm) > 0$. Indeed, since $F(u_k^\pm) > 0$, $u^\pm$ are admissible functions for (1.2). Combining this fact with the weak lower semi-continuity of the norm in $W_0^{1,p}$, we get

$$
\begin{aligned}
0 < (\lambda_1^* - \lambda) \int_\Omega |u^\pm|^p \, dx &\le \int_\Omega |\nabla u^\pm|^p \, dx - \lambda \int_\Omega |u^\pm|^p \, dx \\
&\le \liminf_{k\to+\infty} \left(\int_\Omega |\nabla u_k^\pm|^p \, dx - \lambda \int_\Omega |u_k^\pm|^p \, dx \right) = \int_\Omega f |u^\pm|^\gamma \, dx.
\end{aligned}
$$

From here it follows also that $H_\lambda(u^\pm) > 0$.

Suppose now, by contradiction to the strong convergence in $W_0^{1,p}$, without loss of generality, that $\|u^+\| < \liminf_{k\to+\infty} \|u_k^+\|$. Since $F(u^+) > 0$ and $H_\lambda(u^+) > 0$, Lemma 2.2 implies the existence of exactly one critical point $t(u^+) > 0$ of $E_\lambda(tu^+)$ w.r.t. $t > 0$, such that

$$
E_\lambda(t(u^+)u^+) > 0, \qquad Q_\lambda(t(u^+)u^+) = 0.
$$

By the same reason there exists $t(u^-) > 0$, possibly equals to 1, such that

$$
E_\lambda(t(u^-)u^-) > 0, \qquad Q_\lambda(t(u^-)u^-) = 0.
$$

Therefore, $t(u^+)u^+ + t(u^-)u^- \in \mathcal{N}_\lambda^1$, and since $u_k \in \mathcal{N}_\lambda^1$, we get

$$
\begin{aligned}
E_\lambda(t(u^+)u^+ + t(u^-)u^-) &< \liminf_{k\to+\infty} \left(E_\lambda(t(u^+)u_k^+) + E_\lambda(t(u^-)u_k^+) \right) \\
&\le \liminf_{k\to+\infty} \left(E_\lambda(u_k^+) + E_\lambda(u_k^+) \right) = \inf\{E_\lambda(w) : w \in \mathcal{N}_\lambda^1\} = c_\lambda.
\end{aligned}
$$

Thus, we get a contradiction. Consequently, $u_k^\pm \to u^\pm$ strongly in $W_0^{1,p}$ and $u \in \mathcal{N}_\lambda^1$. □

Now we adapt the proof of [4, Proposition 3.1, p. 8] to show that the minimizer $u \in \mathcal{N}_\lambda^1$ of (3.1) is, in fact, a solution of ($\mathcal{D}$).

Lemma 3.2. *Assume that* (1.1) *is satisfied. If $u \in \mathcal{N}_\lambda^1$ is a solution of* (3.1), *then $DE_\lambda(u) = 0$ in $W^{-1,p'}(\Omega)$, i.e., u is a critical point of E_λ in $W_0^{1,p}$.*

Proof. Let $u \in \mathcal{N}_\lambda^1$ is a solution of (3.1), i.e.,

$$
E_\lambda(u) = c_\lambda := \inf\{E_\lambda(w) : w \in \mathcal{N}_\lambda^1\} > 0.
$$

By Lemma 2.2, $t(u^\pm) = 1$ are the global maximum points of $E_\lambda(tu^\pm)$ w.r.t. $t > 0$ and hence

$$
E_\lambda(ru^+ + su^-) = E_\lambda(ru^+) + E_\lambda(su^-) < E_\lambda(u^+) + E_\lambda(u^-) = E_\lambda(u) \tag{3.2}
$$

for all $(r,s) \in \mathbb{R}^2_+ \setminus \{(1,1)\}$. Moreover, due to the fact that $E_\lambda(u^\pm) > 0$, we are able to choose $\kappa > 0$ small enough, such that

$$\min_{t\in[1-\kappa,1+\kappa]} E_\lambda(tu^\pm) > 0. \tag{3.3}$$

Consider now the function

$$g\colon A := (1-\kappa, 1+\kappa)^2 \subset \mathbb{R}^2 \to W_0^{1,p}, \quad g(r,s) = ru^+ + su^-.$$

Hence, from (3.3) and (3.2) it follows that

$$0 < c_0 := \max_{(r,s)\in\partial A} E_\lambda(g(r,s)) < c_\lambda.$$

Assume now, by contradiction, that $DE_\lambda(u) \neq 0$. Hence, using the continuity of DE_λ we conclude, that there exist some constants $\alpha, \delta > 0$, such that $\|DE_\lambda(v)\| \geq \alpha$ for all $v \in U_{3\delta}(u) := \{w \in W_0^{1,p} : \|u-w\| < 3\delta\}$.

Let us take some $\varepsilon < \min\left\{\frac{c_\lambda - c_0}{2}, \frac{\alpha\delta}{8}\right\}$ and denote $S_\delta := U_{2\delta}(u)$. Then the deformation lemma (see [20, Lemma 2.3, Parts (i), (v), (vi), p. 38]) implies the existence of homotopy $\eta \in C([0,1] \times W_0^{1,p}, W_0^{1,p})$, such that

1) $\eta(t,v) = v$ for all $t \in [0,1]$, if $E_\lambda(v) < c_\lambda - 2\varepsilon$,

2) $E_\lambda(\eta(t,v)) \leq E_\lambda(v)$ for all $v \in W_0^{1,p}$ and $t \in [0,1]$;

3) $E_\lambda(\eta(t,v)) < c_\lambda$ for all $v \in \{w \in S_\delta : E_\lambda(w) \leq c\}$ and $t \in (0,1]$.

From 3) it follows that

$$\max_{\{(r,s)\in A\colon g(r,s)\in S_\delta\}} E_\lambda(\eta(t, g(r,s))) < c_\lambda, \quad \forall t \in (0,1]. \tag{3.4}$$

On the other hand, 2) and (3.2) imply that for all $t \in [0,1]$

$$\max_{\{(r,s)\in A\colon g(r,s)\notin S_\delta\}} E_\lambda(\eta(t, g(r,s))) \leq \max_{\{(r,s)\in A\colon g(r,s)\notin S_\delta\}} E_\lambda(g(r,s)) < c_\lambda. \tag{3.5}$$

Furthermore, from 1) it follows that $\eta(t, g(r,s)) = g(r,s)$ for $(r,s) \in \partial A$ and all $t \in [0,1]$, since $c_0 < c_\lambda - 2\varepsilon$.

Now, due to the continuity of η and E_λ, (3.3) implies the existence of $t_0 \in (0,1]$, such that $E_\lambda(\eta^\pm(t, g(r,s))) > 0$ for all $t \in [0, t_0]$ and $(r,s) \in A$.

Let us denote for simplicity

$$h(r,s) := \eta(t_0, g(r,s)),$$

and consider the maps

$$\psi_1\colon A \to \mathbb{R}^2, \quad \psi_1(r,s) := \left(Q_\lambda(h^+(r,s)), Q_\lambda(h^-(r,s))\right),$$
$$\psi_2\colon A \to \mathbb{R}^2, \quad \psi_2(r,s) := \left(Q_\lambda(ru^+), Q_\lambda(su^-)\right).$$

Note that $\psi_1(r,s) = (0,0)$ if and only if $h^\pm(r,s) \in \mathcal{N}_\lambda$. On the one hand, $\deg(\psi_2, 0, A) = 1$, since there exists only one point $(r,s) = (1,1) \in A$ such that $Q_\lambda(ru^+)$, $Q_\lambda(su^-) = 0$ and the Jacobian determinant

$$\det J_{\psi_2(1,1)} = \left.\frac{\partial Q(ru^+)}{\partial r}\right|_{r=1} \cdot \left.\frac{\partial Q(su^-)}{\partial s}\right|_{s=1} > 0.$$

On the other hand, since $h(r,s) = g(r,s)$ for all $(r,s) \in \partial A$, we get

$$\psi_1(r,s) \equiv \psi_2(r,s), \quad (r,s) \in \partial A.$$

Consequently, using the homotopy invariance property of the degree (see [2, Theorem 3, (iv), p. 190 and Remark 7, (a), p. 192]), we get $\deg(\psi_1, 0, A) = \deg(\psi_2, 0, A) = 1$. Hence, there exists $(r_0, s_0) \in A$ such that $Q_\lambda(h^\pm(r_0,s_0)) = 0$. Furthermore, from the fact that $E_\lambda(\eta^\pm(t_0, g(r_0,s_0))) > 0$, we conclude that $h(r_0,s_0) \in \mathcal{N}_\lambda^1$.

Finally, from (3.4) and (3.5) we obtain

$$E_\lambda(h(r_0,s_0)) < c_\lambda = \inf\{E_\lambda(w) : w \in \mathcal{N}_\lambda^1\},$$

but it is a contradiction. Thus, $DE_\lambda(u) = 0$ in $W^{-1,p'}$, i.e., $u \in \mathcal{N}_\lambda^1$ is a critical point of E_λ on $W_0^{1,p}$. $\square$

4 Least energy and number of nodal domains

Let us consider other subsets of the nodal Nehari set $\mathcal{M}_\lambda$:

$$\mathcal{N}_\lambda^2 = \{u \in W_0^{1,p} : u^\pm \in \mathcal{N}_\lambda,\ E_\lambda(u^+) \cdot E_\lambda(u^-) \le 0\},$$
$$\mathcal{N}_\lambda^3 = \{u \in W_0^{1,p} : u^\pm \in \mathcal{N}_\lambda,\ E_\lambda(u^\pm) < 0\}.$$

It is easy to see that $\mathcal{M}_\lambda = \mathcal{N}_\lambda^1 \cup \mathcal{N}_\lambda^2 \cup \mathcal{N}_\lambda^3$.

Lemma 4.1. *$\mathcal{N}_\lambda^2 = \emptyset$ for all $\lambda < \lambda_1^*$ and $\mathcal{N}_\lambda^3 = \emptyset$ for all $\lambda < \lambda_2$.*

Proof. 1. First we show that $\mathcal{N}_\lambda^2 = \emptyset$ for $\lambda < \lambda_1^*$. Assume, contrary to our claim, that for some $\lambda < \lambda_1^*$ there exists $w \in \mathcal{N}_\lambda^2$. Suppose first that $E_\lambda(w^+) \cdot E_\lambda(w^-) < 0$. From Lemma 2.1 it follows that $F(w^+) \cdot F(w^-) < 0$. Therefore, there exists $t > 0$, such that

$$F(tw^+ + w^-) = t^\gamma F(w^+) + F(w^-) = 0.$$

This implies that $tw^+ + w^-$ is an admissible function for minimization problem (1.2), which yields a contradiction, since $\lambda < \lambda_1^*$.

Suppose now, without loss of generality, that $E_\lambda(w^+) = 0$. Lemma 2.1 implies that $F(w^+) = 0$, and consequently w^+ is also an admissible function for (1.2), a contradiction.

2. Let us show that $\mathcal{N}_\lambda^3 = \emptyset$ for $\lambda < \lambda_2$. For this end we consider the critical point

$$\mu_2 := \inf_{\substack{u \in W_0^{1,p}(\Omega) \\ u_\pm \not\equiv 0}} \left[\max\left\{ \frac{\int |\nabla u^+|^p\,dx}{\int |u^+|^p\,dx}, \frac{\int |\nabla u^-|^p\,dx}{\int |u^-|^p\,dx} \right\} \right]. \tag{4.1}$$

Proposition 4.2. $\mu_2 = \lambda_2$.

Proof. Note first that $\mu_2 \le \lambda_2$. Indeed, using the second eigenfunction $\varphi_2 \in W_0^{1,p}$, which corresponds to λ_2, as an admissible function for (4.1), we get

$$\mu_2 \le \max\left\{ \frac{\int |\nabla \varphi_2^+|^p\,dx}{\int |\varphi_2^+|^p\,dx}, \frac{\int |\nabla \varphi_2^-|^p\,dx}{\int |\varphi_2^-|^p\,dx} \right\} = \frac{\int |\nabla \varphi_2^\pm|^p\,dx}{\int |\varphi_2^\pm|^p\,dx} = \lambda_2.$$

Now we show that $\lambda_2 \leq \mu_2$. Arguing as in the proof of [7, Proposition 4.2, p. 8] it is not hard to obtain a nonzero minimizer $\psi_2 \in W_0^{1,p}$ of (4.1), such that $\psi_2^{\pm} \neq 0$ and, due to the homogeneity of (4.1), $\int_\Omega |\psi_2|^p \, dx = 1$. Consider the set

$$\mathcal{A} := \left\{ u \in W_0^{1,p} : u = s\psi_2^+ + t\psi_2^-, \text{ where } s,t \in \mathbb{R}, \text{ such that } \int_\Omega |s\psi_2^+ + t\psi_2^-|^p \, dx = 1 \right\}.$$

By construction, $\mathcal{A} \subset \mathcal{S}$, where $\mathcal{S}$ is defined in (1.5). Moreover, taking

$$h(x,y) := |x|^{\frac{2}{p}-1} x \frac{\psi_2^+}{\left(\int |\psi_2^+|^p \, dx\right)^{\frac{1}{p}}} + |y|^{\frac{2}{p}-1} y \frac{\psi_2^-}{\left(\int |\psi_2^-|^p \, dx\right)^{\frac{1}{p}}},$$

we conclude that $h : S^1 \to \mathcal{A}$ is continuous and odd, and consequently $\mathcal{A} \in \mathcal{F}_2$. Therefore,

$$\begin{aligned}
\lambda_2 \leq \sup_{u \in \mathcal{A}} \int_\Omega |\nabla u|^p \, dx &= \sup_{\substack{s,t \in \mathbb{R}: \\ \int_\Omega |s\psi_2^+ + t\psi_2^-|^p \, dx = 1}} \left(|s|^p \int_\Omega |\nabla \psi_2^+|^p \, dx + |t|^p \int_\Omega |\nabla \psi_2^-|^p \, dx \right) \\
&\leq \mu_2 \sup_{\substack{s,t \in \mathbb{R}: \\ \int_\Omega |s\psi_2^+ + t\psi_2^-|^p \, dx = 1}} \left(|s|^p \int_\Omega |\psi_2^+|^p \, dx + |t|^p \int_\Omega |\psi_2^-|^p \, dx \right) = \mu_2.
\end{aligned}$$

Hence, $\mu_2 = \lambda_2$. □

To finish the proof of Lemma 4.1 suppose a contradiction, i.e., there exists $w \in \mathcal{N}_\lambda^3$ for some $\lambda < \lambda_2$. Lemma 2.1 implies that $H_\lambda(w^{\pm}) < 0$, and therefore

$$\frac{\int |\nabla w^{\pm}|^p \, dx}{\int |w^{\pm}|^p \, dx} < \lambda < \lambda_2 = \mu_2 \leq \max \left[\frac{\int |\nabla w^+|^p \, dx}{\int |w^+|^p \, dx}, \frac{\int |\nabla w^-|^p \, dx}{\int |w^-|^p \, dx} \right],$$

which is impossible. □

The property of the least energy is given in the following lemma.

Lemma 4.3. *Assume that* (1.1) *is satisfied,* $\nu(\Omega^+) > 0$ *and* $u_\lambda \in \mathcal{N}_\lambda^1$ *is a nodal solution of* ($\mathcal{D}$) *given by Lemma* 3.2*. Then* u_λ *has the least energy among all nodal solutions of* ($\mathcal{D}$) *on* $(-\infty, \min\{\lambda_1^*, \lambda_2\})$*, i.e.,*

$$-\infty < E_\lambda(u_\lambda) \leq E_\lambda(w_\lambda),$$

for any nodal solution w_λ *of* ($\mathcal{D}$) *on this interval.*

Proof. Lemma 4.1 implies that $\mathcal{N}_\lambda^2, \mathcal{N}_\lambda^3 = \emptyset$ for $\lambda < \min\{\lambda_1^*, \lambda_2\}$. Therefore, $\mathcal{M}_\lambda = \mathcal{N}_\lambda^1 \neq \emptyset$ for such λ, due to Lemma 2.3, and thus any nodal solution of ($\mathcal{D}$) belongs to $\mathcal{N}_\lambda^1$. Since u_λ is obtained by minimization of E_λ over $\mathcal{N}_\lambda^1$, we get the desired result. □

In the next lemma we prove nonexistence result for ($\mathcal{D}$).

Lemma 4.4. *If* $\nu(\Omega^+) = 0$*, then there are no weak nodal solutions of the problem* ($\mathcal{D}$) *for any* $\lambda \in (-\infty, \min\{\lambda_1^*, \lambda_2\})$*.*

Proof. From the proof of Lemma 4.3 it follows that $\mathcal{M}_\lambda = \mathcal{N}_\lambda^1$. However, Lemma 2.3 implies that $\mathcal{N}_\lambda^1 = \emptyset$ for any $\lambda \in \mathbb{R}$, whenever $\nu(\Omega^+) = 0$. Thus, $\mathcal{M}_\lambda = \emptyset$, which implies the nonexistence of weak nodal solutions for ($\mathcal{D}$) on $(-\infty, \min\{\lambda_1^*, \lambda_2\})$. □

The next result gives the information about the precise number of nodal domains for solutions of (3.1).

Lemma 4.5. *Assume that* (1.1) *is satisfied and* $\lambda < \lambda_1^*$*. Then any solution* $u \in \mathcal{N}_\lambda^1$ *of* (3.1) *has precisely two nodal domains.*

Proof. Let $u \in \mathcal{N}_\lambda^1$ be a solution of (3.1) and consequently a solution of ($\mathcal{D}$). Recall that any solution of ($\mathcal{D}$) is, in fact, of class $C^{1,\alpha}(\overline{\Omega})$, $\alpha \in (0,1)$ (see Section 1). Suppose, by contradiction, that there exist three nodal domains D_i, $i = 1..3$, and, without loss of generality, $u > 0$ in D_1 and D_3. We denote $u = u_1 + u_2 + u_3$, where

$$u_i(x) = \begin{cases} u(x) & \text{if } \ x \in D_i, \\ 0 & \text{if } \ x \in \Omega \backslash D_i, \end{cases} \qquad i = 1..3.$$

Hence, $u_i \in C^{1,\alpha}(\overline{\Omega})$ and $u_1, u_3 > 0$, $u_2 < 0$ in their supports. Moreover, testing ($\mathcal{D}$) by u_i one can get $Q_\lambda(u_i) = 0$ for all $i = 1..3$.

Assume first that $E_\lambda(u_i) > 0$, $i = 1..3$. However, $u_1 + u_2 \in \mathcal{N}_\lambda^1$ and $E_\lambda(u_1 + u_2) < E_\lambda(u) = c_\lambda$. Hence, we get a contradiction.

Suppose now, without loss of generality, that $E_\lambda(u_1) \leq 0$. Since $E_\lambda(u_2) > 0$, we conclude that $u_1 + u_2 \in \mathcal{N}_\lambda^2$, which contradicts Lemma 4.1. □

5 Continuous branch

Let $u_\lambda \in \mathcal{N}_\lambda^1$ be a nodal solution of the problem ($\mathcal{D}$) given by a minimizer of (3.1). First we show that for any $\lambda \in (-\infty, \lambda_1^*)$ and for any sequence $\Delta\lambda \to 0$, the corresponding sequence of *solutions* $u_{\lambda+\Delta\lambda} \in \mathcal{N}_{\lambda+\Delta\lambda}^1$ converges strongly in $W_0^{1,p}$, up to subsequence, to some $u_0 \in \mathcal{N}_\lambda^1$. It is not hard to see that for the sequence $u_{\lambda+\Delta\lambda}$ we have

$$E_{\lambda+\Delta\lambda}(u_{\lambda+\Delta\lambda}) \to c \quad \text{and} \quad DE_{\lambda+\Delta\lambda}(u_{\lambda+\Delta\lambda}) = 0.$$

Moreover, using Lemma 2.4 it is not hard to show that there exist constants K_1, K_2, such that for all sufficiently small $\Delta\lambda$ it holds $0 < K_1 < \|u_{\lambda+\Delta\lambda}\| < K_2 < \infty$. Hence, Sobolev's embedding theorem and the Eberlein–Smulian theorem imply the existence of $u_0 \in W_0^{1,p}$ such that, up to subsequence, $u_{\lambda+\Delta\lambda} \to u_0$ strongly in $L^{p^*}(\Omega)$ and $u_{\lambda+\Delta\lambda} \rightharpoonup u_0$ weakly in $W_0^{1,p}$. Since $DE_{\lambda+\Delta\lambda}(u_{\lambda+\Delta\lambda}) = 0$ for any $\Delta\lambda$ small enough, we have

$$\begin{aligned} \langle DE_{\lambda+\Delta\lambda}(u_{\lambda+\Delta\lambda}), u_0 - u_{\lambda+\Delta\lambda} \rangle = & \int_\Omega |\nabla u_{\lambda+\Delta\lambda}|^{p-2} \nabla u_{\lambda+\Delta\lambda} \nabla (u_0 - u_{\lambda+\Delta\lambda})\, dx \\ & - (\lambda + \Delta\lambda) \int_\Omega |u_{\lambda+\Delta\lambda}|^{p-2} u_{\lambda+\Delta\lambda} (u_0 - u_{\lambda+\Delta\lambda})\, dx \\ & - \int_\Omega f |u_{\lambda+\Delta\lambda}|^{\gamma-2} u_{\lambda+\Delta\lambda} (u_0 - u_{\lambda+\Delta\lambda})\, dx = 0. \end{aligned}$$

From here, using the strong convergence $u_{\lambda+\Delta\lambda} \to u_0$ in $L^{p^*}(\Omega)$, we obtain

$$\int_\Omega |\nabla u_{\lambda+\Delta\lambda}|^{p-2} \nabla u_{\lambda+\Delta\lambda} \nabla u_0\, dx - \int_\Omega |\nabla u_{\lambda+\Delta\lambda}|^p\, dx \to 0,$$

which implies that $u_k \to u_0$ strongly in $W_0^{1,p}$. Moreover, since

$$E_\lambda(u_0) = \lim_{\Delta\lambda \to 0} E_{\lambda+\Delta\lambda}(u_{\lambda+\Delta\lambda}) > c_2 > 0, \qquad Q_\lambda(u_0) = \lim_{\Delta\lambda \to 0} Q_{\lambda+\Delta\lambda}(u_{\lambda+\Delta\lambda}) = 0,$$

we conclude that $u_0 \in \mathcal{N}^1_\lambda$. Obviously, u_0 is a critical point of E_λ.

Let us prove now that u_0 is also a solution of (3.1). Recall the definition $c_\lambda := E_\lambda(u_\lambda)$ and define, additionally, $c^\pm_\lambda := E_\lambda(u^\pm_\lambda)$. Note that $c_\lambda, c^\pm_\lambda \in (0, +\infty)$. Assume, by contradiction, that $c_\lambda < E_\lambda(u_0)$. Hence, we have $\delta^+ + \delta^- > 0$, where

$$\delta^+ := E_\lambda(u_0^+) - c^+_\lambda, \qquad \delta^- := E_\lambda(u_0^-) - c^-_\lambda.$$

Note first that the continuity of $H_{\lambda+\Delta\lambda}$ w.r.t. $\Delta\lambda$ implies that for sufficiently small $\Delta\lambda$ the sign of $H_{\lambda+\Delta\lambda}(u^\pm_\lambda)$ is positive, and at the same time $F(u^\pm_\lambda) > 0$. Hence, Lemma 2.2 yields the existence of points of global maximum $t^\pm_{\lambda+\Delta\lambda} := t_{\lambda+\Delta\lambda}(u^\pm_\lambda)$ of $E_{\lambda+\Delta\lambda}(tu^\pm_\lambda)$ w.r.t. $t > 0$. Moreover, $t^\pm_{\lambda+\Delta\lambda}$ tends to 1 as $\Delta\lambda \to 0$. Indeed, using (2.2) we obtain

$$\begin{aligned} t^\pm_{\lambda+\Delta\lambda} &= \left(\frac{H_{\lambda+\Delta\lambda}(u^\pm_\lambda)}{F(u^\pm_\lambda)}\right)^{\frac{1}{\gamma-p}} = \left(\frac{H_\lambda(u^\pm_\lambda) - \Delta\lambda\int |u^\pm_\lambda|^p\,dx}{F(u^\pm_\lambda)}\right)^{\frac{1}{\gamma-p}} \\ &= \left(1 - \Delta\lambda\frac{\int |u^\pm_\lambda|^p\,dx}{F(u^\pm_\lambda)}\right)^{\frac{1}{\gamma-p}} \to 1 \quad \text{as} \quad \Delta\lambda \to 0. \end{aligned}$$

Using this fact and strong convergence $u_{\lambda+\Delta\lambda} \to u_0$ in $W^{1,p}_0$ it is not hard to see that for any $\varepsilon > 0$ there exists $\delta > 0$, such that for all $|\Delta\lambda| < \delta$ we have

$$\begin{aligned} |E_{\lambda+\Delta\lambda}(t^\pm_{\lambda+\Delta\lambda}u^\pm_\lambda) - E_\lambda(u^\pm_\lambda)| < \varepsilon, \\ |E_\lambda(u_0^\pm) - E_{\lambda+\Delta\lambda}(u^\pm_{\lambda+\Delta\lambda})| < \varepsilon. \end{aligned}$$

From these estimations we get

$$E_{\lambda+\Delta\lambda}(t^+_{\lambda+\Delta\lambda}u^+_\lambda) < E_{\lambda+\Delta\lambda}(u^+_{\lambda+\Delta\lambda}) + 2\varepsilon - \delta^+, \tag{5.1}$$
$$E_{\lambda+\Delta\lambda}(t^-_{\lambda+\Delta\lambda}u^-_\lambda) < E_{\lambda+\Delta\lambda}(u^-_{\lambda+\Delta\lambda}) + 2\varepsilon - \delta^-. \tag{5.2}$$

Combining (5.1) and (5.2) with the assumption $\delta^+ + \delta^- > 0$ we conclude that for $v_{\lambda+\Delta\lambda} = t^+_{\lambda+\Delta\lambda}u^+_\lambda + t^-_{\lambda+\Delta\lambda}u^-_\lambda$ and sufficiently small $\varepsilon > 0$ the inequality

$$E_{\lambda+\Delta\lambda}(v_{\lambda+\Delta\lambda}) < E_{\lambda+\Delta\lambda}(u_{\lambda+\Delta\lambda})$$

holds. However, by construction, $v_{\lambda+\Delta\lambda} \in \mathcal{N}^1_{\lambda+\Delta\lambda}$, which implies a contradiction, since $u_{\lambda+\Delta\lambda}$ is a minimizer of $E_{\lambda+\Delta\lambda}$ over $\mathcal{N}^1_{\lambda+\Delta\lambda}$.

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