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Non-almost periodic solutions of limit periodic and almost periodic homogeneous linear difference systems

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Abstract. We study limit periodic and almost periodic homogeneous linear difference systems. The coefficient matrices of the considered systems are taken from a given commutative group. We mention a condition on the group which ensures that, by arbitrarily small changes, the considered systems can be transformed to new systems, which do not possess any almost periodic solution other than the trivial one. The elements of the coefficient matrices are taken from an infinite field with an absolute value.

Keywords: limit periodicity, almost periodicity, almost periodic sequences, almost periodic solutions, limit periodic sequences, linear difference systems.

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1 Introduction

In this paper, for a commutative group X of square matrices over a field, we analyse the homogeneous linear difference systems

$$x_{k+1} = A_k x_k, \qquad k \in \mathbb{Z}, \tag{1.1}$$

where $\{A_k\}_{k\in\mathbb{Z}} \subseteq X$. We consider the case, when the sequence $\{A_k\}_{k\in\mathbb{Z}}$ is limit periodic or almost periodic. We continue in the research based on the results of papers [8, 9, 18, 22, 24].

In [18] (see also [16]), the unitary systems of the form (1.1) are considered. One of the main results of [18] says that the systems with non-almost periodic solutions form a dense subset of the space of all unitary systems. If one is interested in orthogonal difference systems and skew-Hermitian and skew-symmetric differential systems, the corresponding result can be found in [19], [21], and [23], respectively. Concerning almost periodic solutions of these systems, we refer to [12, 13, 17] as well.

In [8, 22], general almost periodic systems (1.1) are examined. There are found groups of matrices such that the homogeneous linear difference systems without any non-trivial almost

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periodic solution form a dense subset of the set of all considered systems. Transformable and strongly transformable groups of matrices are introduced. Based on this concept, the above-mentioned result of [18] is generalized for other matrix groups.

In papers [9, 24], the limit periodic systems of the form (1.1) are investigated, where matrices A_k are taken from a commutative group or from a bounded group. It is shown that any of the systems can be transformed to a new system, which does not possess any non-zero (asymptotically) almost periodic solution. Our goal is to improve the results of [9, 24] about systems of the form (1.1) with regard to their non-almost periodic solutions. Furthermore, we recall the corresponding Cauchy problem. Note that the presented results are new even for complex matrix groups.

The fundamental properties of limit periodic and almost periodic sequences or functions have been studied closely. One can easily find many relevant monographs. Here we point out only the books [3, 6, 15]. Concerning almost periodic solutions of linear almost periodic difference systems, we can refer to [4, 5, 25] (see also [7, 10, 26]). Other properties of (complex) almost periodic systems can be found in [1, 11, 14]. The properties of limit periodic homogeneous linear difference systems with respect to their almost periodic solutions are mentioned, e.g., in [9, 24].

This paper is divided into five sections as follows. First, in the next section, the definitions of limit and almost periodicity are recalled. In Section 3, we introduce the used notations. In Section 4, we collect auxiliary results, which we use in the proof of the main result. Finally, in Section 5, we formulate and prove our main result.

2 Limit and almost periodicity

In this section, we recall the definitions of limit periodic and almost periodic sequences in a metric space (M, ρ).

Definition 2.1. We say that a sequence $\{\varphi_k\}_{k\in\mathbb{Z}}$ is limit periodic if there exists a sequence of periodic sequences $\{\varphi_k^n\}_{k\in\mathbb{Z}} \subseteq M$, $n \in \mathbb{N}$, such that $\lim_{n\to\infty} \varphi_k^n = \varphi_k$ and the convergence is uniform with respect to $k \in \mathbb{Z}$.

Remark 2.2. The limit periodicity can be introduced in another equivalent way (see [2]).

Definition 2.3. A sequence $\{\varphi_k\}_{k\in\mathbb{Z}} \subseteq M$ is called almost periodic if for any $\varepsilon > 0$ there exists $r(\varepsilon) \in \mathbb{N}$ such that any set consisting of $r(\varepsilon)$ consecutive integers contains at least one number l satisfying

$$\rho\left(\varphi_{k+l}, \varphi_k\right) < \varepsilon, \qquad k \in \mathbb{Z}.$$

The definition mentioned above is the so-called Bohr definition of almost periodicity. The almost periodicity can be defined in another equivalent way (see the next theorem). This concept is the so-called Bochner definition.

Theorem 2.4. *Let $\{\varphi_k\}_{k\in\mathbb{Z}} \subseteq M$ be given. The sequence $\{\varphi_k\}_{k\in\mathbb{Z}}$ is almost periodic if and only if any sequence $\{l_n\}_{n\in\mathbb{N}} \subseteq \mathbb{Z}$ has a subsequence $\{\bar{l}_n\}_{n\in\mathbb{N}} \subseteq \{l_n\}_{n\in\mathbb{N}}$ such that, for any $\varepsilon > 0$, there exists $K(\varepsilon) \in \mathbb{N}$ satisfying*

$$\rho\left(\varphi_{k+\bar{l}_i}, \varphi_{k+\bar{l}_j}\right) < \varepsilon, \qquad i, j > K(\varepsilon),\ k \in \mathbb{Z}. \tag{2.1}$$

Proof. See, e.g., [20, Theorem 2.3]. □

3 Preliminaries

In the whole paper, we will consider an infinite field F with an absolute value $|\cdot|\colon F \to \mathbb{R}$. Let $m \in \mathbb{N}$ be arbitrarily given. We denote the set of all $m \times m$ matrices with elements in F by the symbol $Mat_m(F)$. The absolute value gives the norms $\|\cdot\|$ on F^m and $Mat_m(F)$ as the sum of the absolute values of elements. The absolute value and norms induce metrics on F and F^m, $Mat_m(F)$, respectively. We denote δ-neighbourhoods by symbol O_δ in all considered metric spaces.

Let $X \subseteq Mat_m(F)$. We repeat that X is a commutative group. The set of all limit periodic and almost periodic sequences with values in X will be denoted by $LP(X)$ and $AP(X)$, respectively. In these sets, we consider the metric

$$\sigma\left(\{A_k\}_{k\in\mathbb{Z}}, \{B_k\}_{k\in\mathbb{Z}}\right) := \sup_{k\in\mathbb{Z}} \|A_k - B_k\|.$$

For the reader's convenience, we also denote δ-neighbourhoods of sequences in $LP(X)$ and $AP(X)$ by symbol O_δ.

Instead of $\{Z_k\}_{k\in\mathbb{Z}}$, we will shortly write $\{Z_k\}$. If index k will be taken from another set, we will specify it at the corresponding place. The identity matrix will be denoted by I. The zero vector will be denoted by 0. Symbol ε stands for a positive real number.

In the definitions given below, we recall (and slightly generalize) the property P from [9].

Definition 3.1. We say that group X has property P if there exists $\zeta > 0$ such that for every $\delta > 0$ there exists $l \in \mathbb{N}$ such that for every $u \in F^m$ fulfilling $\|u\| \geq 1$ there exist matrices $M_1, M_2, \dots, M_l \in X$ with the property that

$$M_i \in O_\delta(I),\ i \in \{1,\dots,l\}, \qquad \|M_l \cdots M_1 u - u\| > \zeta.$$

Definition 3.2. Let $u \in F^m$ be an arbitrary non-zero vector. We say that group X has property P with respect to u if there exists $\zeta > 0$ such that for every $\delta > 0$ there exist matrices $M_1, M_2, \dots, M_l \in X$ with the property that

$$M_i \in O_\delta(I),\ i \in \{1,\dots,l\}, \qquad \|M_l \cdots M_1 u - u\| > \zeta.$$

4 Auxiliary results

Definition 3.1 can be apprehended in a little larger sense using the following lemma.

Lemma 4.1. *For any $a > 0$, there exists $f(a) \in F$ such that $\|f(a)\cdot v\| \geq 1$ for every $v \in F^m$ satisfying $\|v\| \geq a$.*

Proof. Let $a > 0$ be arbitrarily given. For $v \in F^m$, $\|v\| \geq a$, $\|v\| \geq 1$, we put $f(a) = e$, where e denotes the identity element of F. If there exists $v \in F^m$, $\|v\| \geq a$, $\|v\| < 1$, then there exists element $g \in F$ such that $|g| \leq \|v\| < 1$. Thus, $\lim_{n\to\infty} |g^n| = \lim_{n\to\infty} |g|^n = 0$ and, consequently, we have that $\lim_{n\to\infty} |g^{-n}| = \lim_{n\to\infty} |g^{-1}|^n = \infty$. It means that there exists $N \in \mathbb{N}$ such that $|g^{-N}| \geq 1/a$. It suffices to put $f(a) = g^{-N}$. □

Remark 4.2. According to Lemma 4.1, Definition 3.1 can be used to vectors satisfying $\|u\| \geq a$ in the following way. Instead of u, we apply the definition for vector $f(a)\cdot u$. Then the corresponding inequality takes the form $\|M_l \cdots M_1 f(a)\cdot u - f(a)\cdot u\| > \zeta$ which can be rewritten to $\|M_l \cdots M_1 u - u\| > \zeta/|f(a)| := \omega(a) > 0$.

Remark 4.3. We can assume that $\omega(a)$ is a non-decreasing function of a. It follows from Remark 4.2 and from the proof of Lemma 4.1.

Remark 4.4. Let $a > 0$. Let $v \in F^m$ satisfy $\|M_l \cdots M_1 v - v\| > \omega(a)$. Then, it is valid that $\|M_l \cdots M_1 v - w\| > \omega(a)/2$ if $\|w - v\| < \omega(a)/2$ holds. Indeed, it follows directly from

$$\omega(a) < \|M_l \cdots M_1 v - v + w - w\| \leq \|M_l \cdots M_1 v - w\| + \|w - v\|.$$

Now we recall two known lemmas, which we need to prove the main result of this paper.

Lemma 4.5. *Let $\{A_k\} \in AP(X)$. For any non-zero almost periodic solution $\{x_k\}$ of the system $x_{k+1} = A_k x_k$, it holds that $\inf_{k\in\mathbb{Z}} \|x_k\| > 0$.*

Proof. See [22, Lemma 3.10]. □

Lemma 4.6. *Let $\{A_k\} \in LP(X)$ and $\varepsilon > 0$ be arbitrarily given. Let $\{\delta_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$ be a decreasing sequence satisfying*

$$\lim_{n\to\infty} \delta_n = 0$$

and let $\{S_k^n\} \subset X$ be periodic sequences for $n \in \mathbb{N}$ such that

$$S_k^n \in O_{\delta_n}(I), \qquad k \in \mathbb{Z},\ n \in \mathbb{N}, \tag{4.1}$$

$$S_k^j = I \quad or \quad S_k^i = I, \qquad k \in \mathbb{Z},\ i \neq j,\ i,j \in \mathbb{N}. \tag{4.2}$$

If one puts

$$S_k := A_k \cdot S_k^1 \cdot S_k^2 \cdots S_k^n \cdots, \qquad k \in \mathbb{Z},$$

then $\{S_k\} \in LP(X)$. In addition, if

$$\delta_1 < \frac{\varepsilon}{\sup\limits_{k\in\mathbb{Z}} \| A_k \|}, \tag{4.3}$$

then $\{S_k\} \in O_\varepsilon(\{A_k\})$.

Proof. See [9, Lemma 5.1.] (and also [20, Theorem 3.5]). □

Remark 4.7. Lemma 4.6 remains true if $LP(X)$ is replaced by $AP(X)$ in the statement of this lemma, which gives [9, Lemma 5.8.].

5 Results

Now, we can prove the main result. We repeat that $X \subseteq Mat_m(F)$ is a commutative group.

Theorem 5.1. *Let X have property P and $\varepsilon > 0$ be arbitrary. Then, for every $\{A_k\} \in LP(X)$ and every sequence $\{u_n\}_{n\in\mathbb{N}}$ of non-zero vectors $u_n \in F^m$, there exists $\{S_k\} \in O_\varepsilon(\{A_k\}) \cap LP(X)$ such that the solution of*

$$x_{k+1} = S_k x_k, \qquad x_0 = u_n \tag{5.1}$$

is not almost periodic for any $n \in \mathbb{N}$.

Proof. Let $\varepsilon > 0$ be arbitrary. Let ζ be taken from Definition 3.1. We use the following construction.

In the first step of the construction, let us consider the initial problem

$$x_{k+1} = A_k x_k, \qquad x_0 = u_1. \tag{5.2}$$

Let $\{x_k^{(1,1,1)}\}$ be its solution and let $j(1,1,1) := 0$. For

$$\delta_1 := \frac{1}{1+\varepsilon} \cdot \frac{\varepsilon}{\sup\limits_{k\in\mathbb{Z}} \|A_k\|}, \tag{5.3}$$

there exists $l(\delta_1) \geq 2$ (see Definition 3.1). Denote $\triangle_1 := 2 \cdot l(\delta_1)$. Then, for vector $x^{(1,1,1)}_{j(1,1,1)+\triangle_1}$, there exist matrices $M_1^{(1,1,1)}, M_2^{(1,1,1)}, \ldots, M_{l(\delta_1)}^{(1,1,1)} \in O_{\delta_1}(I)$ given by property P in Definition 3.1. We define periodic sequence $\{S_k^{(1,1,1)}\}$ with period $p(1,1,1) := \triangle_1$ as follows. Denote $a_{(1,1,1)} := \left\| x^{(1,1,1)}_{j(1,1,1)} \right\|$. If $a_{(1,1,1)} \leq 1$, then we put

$$S_0^{(1,1,1)} = \cdots = S^{(1,1,1)}_{p(1,1,1)-1} = I.$$

If $a_{(1,1,1)} > 1$ and $\left\| x^{(1,1,1)}_{j(1,1,1)+\triangle_1} - x^{(1,1,1)}_{j(1,1,1)} \right\| > \omega(a_{(1,1,1)})/2$, then

$$S_0^{(1,1,1)} = \cdots = S^{(1,1,1)}_{p(1,1,1)-1} = I.$$

If $a_{(1,1,1)} > 1$ and $\left\| x^{(1,1,1)}_{j(1,1,1)+\triangle_1} - x^{(1,1,1)}_{j(1,1,1)} \right\| \leq \omega(a_{(1,1,1)})/2$, then

$$S_0^{(1,1,1)} = I, \qquad S_1^{(1,1,1)} = M_1^{(1,1,1)}, \qquad S_2^{(1,1,1)} = I, \qquad S_3^{(1,1,1)} = M_2^{(1,1,1)},$$

$$S_4^{(1,1,1)} = I, \ldots, S^{(1,1,1)}_{p(1,1,1)-1} = M^{(1,1,1)}_{l(\delta_1)}.$$

We denote $S_k^1 = S_k^{(1,1,1)}$ and $R_k^1 = A_k S_k^1$ for $k \in \mathbb{Z}$.

In the second step, we consider the Cauchy problem

$$x_{k+1} = R_k^1 x_k, \qquad x_0 = u_1.$$

Let $\{x_k^{(2,1,1)}\}$ be its solution. Then, there exists a positive integer $j(2,1,1)$ divisible by 4 satisfying $j(2,1,1) > p(1,1,1)$. For

$$\delta_2 := \frac{1}{4+\varepsilon} \cdot \frac{\varepsilon}{\sup\limits_{k\in\mathbb{Z}} \|A_k\|},$$

there exists $l(\delta_2)$ (see Definition 3.1). Without loss of generality, we can assume that $l(\delta_2) \geq l(\delta_1)$. Denote $\triangle_2 := 64 \cdot l(\delta_2) \cdot l(\delta_1)$. For $x^{(2,1,1)}_{j(2,1,1)+\triangle_2}$, there exist matrices

$$M_1^{(2,1,1)}, M_2^{(2,1,1)}, \ldots, M^{(2,1,1)}_{l(\delta_2)} \in O_{\delta_2}(I)$$

taken from Definition 3.1. We define the periodic sequence $\{S_k^{(2,1,1)}\}$ with period

$$p(2,1,1) := [j(2,1,1) + \triangle_2] p(1,1,1)$$

in the following way. Denote $a_{(2,1,1)} := \left\| x^{(2,1,1)}_{j(2,1,1)} \right\|$. If $a_{(2,1,1)} \leq 1/2$, then we define

$$S^{(2,1,1)}_0 = \cdots = S^{(2,1,1)}_{p(2,1,1)-1} = I.$$

If $a_{(2,1,1)} > 1/2$ and $\left\| x^{(2,1,1)}_{j(2,1,1)+\triangle_2} - x^{(2,1,1)}_{j(2,1,1)} \right\| > \omega(a_{(2,1,1)})/2$, then

$$S^{(2,1,1)}_0 = \cdots = S^{(2,1,1)}_{p(2,1,1)-1} = I.$$

If $a_{(2,1,1)} > 1/2$ and $\left\| x^{(2,1,1)}_{j(2,1,1)+\triangle_2} - x^{(2,1,1)}_{j(2,1,1)} \right\| \leq \omega(a_{(2,1,1)})/2$, then

$$S^{(2,1,1)}_0 = \cdots = S^{(2,1,1)}_{j(2,1,1)-1} = I,$$

$$S^{(2,1,1)}_{j(2,1,1)} = I, \qquad S^{(2,1,1)}_{j(2,1,1)+1} = I, \qquad S^{(2,1,1)}_{j(2,1,1)+2} = M^{(2,1,1)}_1,$$

$$S^{(2,1,1)}_{j(2,1,1)+3} = I, \qquad S^{(2,1,1)}_{j(2,1,1)+4} = I, \qquad S^{(2,1,1)}_{j(2,1,1)+5} = I, \qquad S^{(2,1,1)}_{j(2,1,1)+6} = M^{(2,1,1)}_2,$$

$$\vdots$$

$$S^{(2,1,1)}_{j(2,1,1)+4(l(\delta_2)-1))} = I, \qquad S^{(2,1,1)}_{j(2,1,1)+1+4(l(\delta_2)-1))} = I, \qquad S^{(2,1,1)}_{j(2,1,1)+2+4(l(\delta_2)-1)} = M^{(2,1,1)}_{l(\delta_2)},$$

$$S^{(2,1,1)}_{j(2,1,1)+3+4(l(\delta_2)-1)} = \cdots = S^{(2,1,1)}_{p(2,1,1)-1} = I.$$

We put $R^{(2,1,1)}_k = R^1_k S^{(2,1,1)}_k$, $k \in \mathbb{Z}$.

Next, we consider

$$x_{k+1} = R^{(2,1,1)}_k x_k, \qquad x_0 = u_1$$

with the solution $\{x^{(2,2,1)}_k\}$. There exists $j(2,2,1) \in \mathbb{N}$ divisible by 8 such that $j(2,2,1) > p(2,1,1)$. Property P (see Definition 3.1) used for vector $x^{(2,2,1)}_{j(2,2,1)+\triangle_2-\triangle_1}$ gives the matrices

$$M^{(2,2,1)}_1, M^{(2,2,1)}_2, \ldots, M^{(2,2,1)}_{l(\delta_2)} \in O_{\delta_2}(I).$$

Let us define the periodic sequence $\{S^{(2,2,1)}_k\}$ with period

$$p(2,2,1) := [j(2,2,1) + \triangle_2 - \triangle_1] p(2,1,1)$$

as follows. Denote $a_{(2,2,1)} := \left\| x^{(2,2,1)}_{j(2,2,1)} \right\|$. If $a_{(2,2,1)} \leq 1/2$, then

$$S^{(2,2,1)}_0 = \cdots = S^{(2,2,1)}_{p(2,2,1)-1} = I.$$

If $a_{(2,2,1)} > 1/2$ and $\left\| x^{(2,2,1)}_{j(2,2,1)+\triangle_2-\triangle_1} - x^{(2,2,1)}_{j(2,2,1)} \right\| > \omega(a_{(2,2,1)})/2$, then

$$S^{(2,2,1)}_0 = \cdots = S^{(2,2,1)}_{p(2,2,1)-1} = I.$$

If $a_{(2,2,1)} > 1/2$ and $\left\| x^{(2,2,1)}_{j(2,2,1)+\triangle_2-\triangle_1} - x^{(2,2,1)}_{j(2,2,1)} \right\| \leq \omega(a_{(2,2,1)})/2$, then

$$S^{(2,2,1)}_0 = \cdots = S^{(2,2,1)}_{j(2,2,1)-1} = I,$$

$$S^{(2,2,1)}_{j(2,2,1)} = \cdots = S^{(2,2,1)}_{j(2,2,1)+3} = I, \qquad S^{(2,2,1)}_{j(2,2,1)+4} = M^{(2,2,1)}_1,$$
$$S^{(2,2,1)}_{j(2,2,1)+5} = \cdots = S^{(2,2,1)}_{j(2,2,1)+11} = I, \qquad S^{(2,2,1)}_{j(2,2,1)+12} = M^{(2,2,1)}_2,$$
$$\vdots$$
$$S^{(2,2,1)}_{j(2,2,1)+8(l(\delta_2)-2)+5} = \cdots = S^{(2,2,1)}_{j(2,2,1)+8(l(\delta_2)-2)+11} = I, \qquad S^{(2,2,1)}_{j(2,2,1)+8(l(\delta_2)-1)+4} = M^{(2,2,1)}_{l(\delta_2)},$$
$$S^{(2,2,1)}_{j(2,2,1)+8(l(\delta_2)-1)+5} = \cdots = S^{(2,2,1)}_{p(2,2,1)-1} = I.$$

Again, we denote $R^{(2,2,1)}_k = R^{(2,1,1)}_k S^{(2,2,1)}_k$, $k \in \mathbb{Z}$.

Next, we consider the initial problem

$$x_{k+1} = R^{(2,2,1)}_k x_k, \qquad x_0 = u_2.$$

Let $\{x^{(2,1,2)}_k\}$ be its solution. Then there exists a positive integer $j(2,1,2)$ divisible by 16 satisfying $j(2,1,2) > p(2,2,1)$. For $x^{(2,1,2)}_{j(2,1,2)+\triangle_2}$, there exist matrices

$$M^{(2,1,2)}_1, M^{(2,1,2)}_2, \ldots, M^{(2,1,2)}_{l(\delta_2)} \in O_{\delta_2}(I)$$

taken from Definition 3.1. We define the periodic sequence $\{S^{(2,1,2)}_k\}$ with period

$$p(2,1,2) := [j(2,1,2) + \triangle_2]p(2,2,1)$$

in the following way. Denote $a_{(2,1,2)} := \left\| x^{(2,1,2)}_{j(2,1,2)} \right\|$. If $a_{(2,1,2)} \le 1/2$, then we put

$$S^{(2,1,2)}_0 = \cdots = S^{(2,1,2)}_{p(2,1,2)-1} = I.$$

If $a_{(2,1,2)} > 1/2$ and $\left\| x^{(2,1,2)}_{j(2,1,2)+\triangle_2} - x^{(2,1,2)}_{j(2,1,2)} \right\| > \omega(a_{(2,1,2)})/2$, then

$$S^{(2,1,2)}_0 = \cdots = S^{(2,1,2)}_{p(2,1,2)-1} = I.$$

If $a_{(2,1,2)} > 1/2$ and $\left\| x^{(2,1,2)}_{j(2,1,2)+\triangle_2} - x^{(2,1,2)}_{j(2,1,2)} \right\| \le \omega(a_{(2,1,2)})/2$, then

$$S^{(2,1,2)}_0 = \cdots = S^{(2,1,2)}_{j(2,1,2)-1} = I,$$

$$S^{(2,1,2)}_{j(2,1,2)} = \cdots = S^{(2,1,2)}_{j(2,1,2)+7} = I, \qquad S^{(2,1,2)}_{j(2,1,2)+8} = M^{(2,1,2)}_1,$$
$$S^{(2,1,2)}_{j(2,1,2)+9} = \cdots = S^{(2,1,2)}_{j(2,1,2)+23} = I, \qquad S^{(2,1,2)}_{j(2,1,2)+24} = M^{(2,1,2)}_2,$$
$$\vdots$$
$$S^{(2,1,2)}_{j(2,1,2)+16(l(\delta_2)-2)+9} = \cdots = S^{(2,1,2)}_{j(2,1,2)+16(l(\delta_2)-2)+23} = I, \qquad S^{(2,1,2)}_{j(2,1,2)+16(l(\delta_2)-1)+8} = M^{(2,1,2)}_{l(\delta_2)},$$
$$S^{(2,1,2)}_{j(2,1,2)+16(l(\delta_2)-1)+9} = \cdots = S^{(2,1,2)}_{p(2,1,2)-1} = I.$$

We put $R^{(2,1,2)}_k = R^{(2,2,1)}_k S^{(2,1,2)}_k$, $k \in \mathbb{Z}$.

We consider the system with the initial value

$$x_{k+1} = R^{(2,1,2)}_k x_k, \qquad x_0 = u_2.$$

Let $\{x_k^{(2,2,2)}\}$ be its solution. There exists a positive integer $j(2,2,2)$ divisible by 32 such that $j(2,2,2) > p(2,1,2)$. Again, for $x^{(2,2,2)}_{j(2,2,2)+\triangle_2-\triangle_1}$, there exist matrices (see Definition 3.1)

$$M_1^{(2,2,2)}, M_2^{(2,2,2)}, \dots, M_{l(\delta_2)}^{(2,2,2)} \in O_{\delta_2}(I).$$

Let us define the periodic sequence $\{S_k^{(2,2,2)}\}$ with period

$$p(2,2,2) := [j(2,2,2) + \triangle_2 - \triangle_1]p(2,1,2)$$

as follows. Denote $a_{(2,2,2)} := \left\| x^{(2,2,2)}_{j(2,2,2)} \right\|$. If $a_{(2,2,2)} \le 1/2$, then we define

$$S_0^{(2,2,2)} = \cdots = S^{(2,2,2)}_{p(2,2,2)-1} = I.$$

If $a_{(2,2,2)} > 1/2$ and $\left\| x^{(2,2,2)}_{j(2,2,2)+\triangle_2-\triangle_1} - x^{(2,2,2)}_{j(2,2,2)} \right\| > \omega(a_{(2,2,2)})/2$, then

$$S_0^{(2,2,2)} = \cdots = S^{(2,2,2)}_{p(2,2,2)-1} = I.$$

If $a_{(2,2,2)} > 1/2$ and $\left\| x^{(2,2,2)}_{j(2,2,2)+\triangle_2-\triangle_1} - x^{(2,2,2)}_{j(2,2,2)} \right\| \le \omega(a_{(2,2,2)})/2$, then

$$S_0^{(2,2,2)} = \cdots = S^{(2,2,2)}_{j(2,2,2)-1} = I,$$

$$S^{(2,2,2)}_{j(2,2,2)} = \cdots = S^{(2,2,2)}_{j(2,2,2)+15} = I, \qquad S^{(2,2,2)}_{j(2,2,2)+16} = M_1^{(2,2,2)},$$

$$S^{(2,2,2)}_{j(2,2,2)+17} = \cdots = S^{(2,2,2)}_{j(2,2,2)+47} = I, \qquad S^{(2,2,2)}_{j(2,2,2)+48} = M_2^{(2,2,2)},$$

$$\vdots$$

$$S^{(2,2,2)}_{j(2,2,2)+32(l(\delta_2)-2)+17} = \cdots = S^{(2,2,2)}_{j(2,2,2)+32(l(\delta_2)-2)+47} = I, \qquad S^{(2,2,2)}_{j(2,2,2)+32(l(\delta_2)-1)+16} = M^{(2,2,2)}_{l(\delta_2)},$$

$$S^{(2,2,2)}_{j(2,2,2)+32(l(\delta_2)-1)+17} = \cdots = S^{(2,2,2)}_{p(2,2,2)-1} = I.$$

We denote $R_k^2 = R_k^{(2,2,2)} = R_k^{(2,1,2)} S_k^{(2,2,2)}$ and $S_k^2 = S_k^{(2,1,1)} S_k^{(2,2,1)} S_k^{(2,1,2)} S_k^{(2,2,2)}$ for $k \in \mathbb{Z}$. It is the end of the second step.

We continue the construction in the same way. Before the n-th step, we have $\{R_k^{n-1}\} = \{A_k S_k^1 S_k^2 \cdots S_k^{n-1}\}$ with period

$$p(n-1, n-1, n-1) := [j(n-1,n-1,n-1) + \triangle_{n-1} - \triangle_{n-2}]p(n-1, n-2, n-1).$$

We denote

$$d(x,y,z) := 2^{\frac{(x-1)x(2x-1)}{6} + y + (z-1)x}, \qquad x \in \mathbb{N}, \quad y, z \in \{1, 2, \dots, x\}, \tag{5.4}$$

$$\delta_j := \frac{1}{j^2 + \varepsilon} \cdot \frac{\varepsilon}{\sup\limits_{k \in \mathbb{Z}} \|A_k\|}, \qquad j \in \mathbb{N}, \tag{5.5}$$

$$\triangle_j := \prod_{i=1}^{j} d(i,i,i) l(\delta_i), \qquad j \in \mathbb{N}. \tag{5.6}$$

Let us consider the initial problem

$$x_{k+1} = R_k^{n-1} x_k, \qquad x_0 = u_1$$

and let $\{x_k^{(n,1,1)}\}$ be its solution. Then there exists $j(n,1,1) \in \mathbb{N}$ divisible by $d(n,1,1)$ such that $j(n,1,1) > p(n-1,n-1,n-1)$. From Definition 3.1, for $x_{j(n,1,1)+\triangle_n}^{(n,1,1)}$, there exist matrices

$$M_1^{(n,1,1)}, M_2^{(n,1,1)}, \dots, M_{l(\delta_n)}^{(n,1,1)} \in O_{\delta_n}(I), \tag{5.7}$$

where $l(\delta_n)$ can be taken in such a way that $l(\delta_n) \geq l(\delta_{n-1})$. Next, we define the periodic sequence $\{S_k^{(n,1,1)}\}$ with period

$$p(n,1,1) := [j(n,1,1) + \triangle_n]p(n-1,n-1,n-1).$$

Denote $a_{(n,1,1)} := \left\| x_{j(n,1,1)}^{(n,1,1)} \right\|$. If $a_{(n,1,1)} \leq 1/n$, then we put

$$S_0^{(n,1,1)} = \cdots = S_{p(n,1,1)-1}^{(n,1,1)} = I.$$

If $a_{(n,1,1)} > 1/n$ and $\left\| x_{j(n,1,1)+\triangle_n}^{(n,1,1)} - x_{j(n,1,1)}^{(n,1,1)} \right\| > \omega(a_{(n,1,1)})/2$, then

$$S_0^{(n,1,1)} = \cdots = S_{p(n,1,1)-1}^{(n,1,1)} = I.$$

If $a_{(n,1,1)} > 1/n$ and $\left\| x_{j(n,1,1)+\triangle_n}^{(n,1,1)} - x_{j(n,1,1)}^{(n,1,1)} \right\| \leq \omega(a_{(n,1,1)})/2$, then

$$S_0^{(n,1,1)} = \cdots = S_{j(n,1,1)-1}^{(n,1,1)} = I,$$

$$S_{j(n,1,1)}^{(n,1,1)} = \cdots = S_{j(n,1,1)+d(n,1,1)/2-1}^{(n,1,1)} = I, \qquad S_{j(n,1,1)+d(n,1,1)/2}^{(n,1,1)} = M_1^{(n,1,1)},$$
$$S_{j(n,1,1)+d(n,1,1)/2+1}^{(n,1,1)} = \cdots = S_{j(n,1,1)+d(n,1,1)+d(n,1,1)/2-1}^{(n,1,1)} = I,$$
$$S_{j(n,1,1)+d(n,1,1)+d(n,1,1)/2}^{(n,1,1)} = M_2^{(n,1,1)},$$
$$\vdots$$
$$S_{j(n,1,1)+d(n,1,1)(l(\delta_n)-2)+d(n,1,1)/2+1}^{(n,1,1)} = \cdots = S_{j(n,1,1)+d(n,1,1)(l(\delta_n)-1)+d(n,1,1)/2-1}^{(n,1,1)} = I,$$
$$S_{j(n,1,1)+d(n,1,1)(l(\delta_n)-1)+d(n,1,1)/2}^{(n,1,1)} = M_{l(\delta_n)}^{(n,1,1)},$$
$$S_{j(n,1,1)+d(n,1,1)(l(\delta_n)-1)+d(n,1,1)/2+1}^{(n,1,1)} = \cdots = S_{p(n,1,1)-1}^{(n,1,1)} = I.$$

We put $R_k^{(n,1,1)} = R_k^{n-1} S_k^{(n,1,1)}$, $k \in \mathbb{Z}$.

Let us have the problem

$$x_{k+1} = R_k^{(n,1,1)} x_k, \qquad x_0 = u_1$$

and its solution $\{x_k^{(n,2,1)}\}$. Let $j(n,2,1)$ be a positive integer divisible by $d(n,2,1)$ such that $j(n,2,1) > p(n,1,1)$. For vector $x_{j(n,2,1)+\triangle_n-\triangle_1}^{(n,2,1)}$, there exist matrices (see Definition 3.1)

$$M_1^{(n,2,1)}, M_2^{(n,2,1)}, \dots, M_{l(\delta_n)}^{(n,2,1)} \in O_{\delta_n}(I). \tag{5.8}$$

We define the sequence $\{S_k^{(n,2,1)}\}$ with period

$$p(n,2,1) := [j(n,2,1) + \triangle_n - \triangle_1]p(n,1,1)$$

in the following way. Denote $a_{(n,2,1)} := \left\| x^{(n,2,1)}_{j(n,2,1)} \right\|$. If $a_{(n,2,1)} \leq 1/n$, then

$$S^{(n,2,1)}_0 = \cdots = S^{(n,2,1)}_{p(n,2,1)-1} = I.$$

If $a_{(n,2,1)} > 1/n$ and $\left\| x^{(n,2,1)}_{j(n,2,1)+\triangle_n-\triangle_1} - x^{(n,2,1)}_{j(n,2,1)} \right\| > \omega(a_{(n,2,1)})/2$, then

$$S^{(n,2,1)}_0 = \cdots = S^{(n,2,1)}_{p(n,2,1)-1} = I.$$

If $a_{(n,2,1)} > 1/n$ and $\left\| x^{(n,2,1)}_{j(n,2,1)+\triangle_n-\triangle_1} - x^{(n,2,1)}_{j(n,2,1)} \right\| \leq \omega(a_{(n,2,1)})/2$, then

$$S^{(n,2,1)}_0 = \cdots = S^{(n,2,1)}_{j(n,2,1)-1} = I,$$

$$S^{(n,2,1)}_{j(n,2,1)} = \cdots = S^{(n,2,1)}_{j(n,2,1)+d(n,2,1)/2-1} = I, \qquad S^{(n,2,1)}_{j(n,2,1)+d(n,2,1)/2} = M^{(n,2,1)}_1,$$
$$S^{(n,2,1)}_{j(n,2,1)+d(n,2,1)/2+1} = \cdots = S^{(n,2,1)}_{j(n,2,1)+d(n,2,1)+d(n,2,1)/2-1} = I,$$
$$S^{(n,2,1)}_{j(n,2,1)+d(n,2,1)+d(n,2,1)/2} = M^{(n,2,1)}_2,$$
$$\vdots$$
$$S^{(n,2,1)}_{j(n,2,1)+d(n,2,1)(l(\delta_n)-2)+d(n,2,1)/2+1} = \cdots = S^{(n,2,1)}_{j(n,2,1)+d(n,2,1)(l(\delta_n)-1)+d(n,2,1)/2-1} = I,$$
$$S^{(n,2,1)}_{j(n,2,1)+d(n,2,1)(l(\delta_n)-1)+d(n,2,1)/2} = M^{(n,2,1)}_{l(\delta_n)},$$
$$S^{(n,2,1)}_{j(n,2,1)+d(n,2,1)(l(\delta_n)-1)+d(n,2,1)/2+1} = \cdots = S^{(n,2,1)}_{p(n,2,1)-1} = I.$$

Denote $R^{(n,2,1)}_k = R^{(n,1,1)}_k S^{(n,2,1)}_k$, $k \in \mathbb{Z}$.

We continue in the same way. Let us consider

$$x_{k+1} = R^{(n,n-1,1)}_k x_k, \qquad x_0 = u_1$$

with its solution $\{x^{(n,n,1)}_k\}$. We know that there exists $j(n,n,1) \in \mathbb{N}$ divisible by $d(n,n,1)$ such that $j(n,n,1) > p(n,n-1,1)$. Also for $x^{(n,n,1)}_{j(n,n,1)+\triangle_n-\triangle_{n-1}}$, there exist matrices

$$M^{(n,n,1)}_1, M^{(n,n,1)}_2, \ldots, M^{(n,n,1)}_{l(\delta_n)} \in O_{\delta_n}(I) \tag{5.9}$$

taken from Definition 3.1. Let us define the periodic sequence $\{S^{(n,n,1)}_k\}$ with period

$$p(n,n,1) := [j(n,n,1) + \triangle_n - \triangle_{n-1}]p(n,n-1,1).$$

Denote $a_{(n,n,1)} := \left\| x^{(n,n,1)}_{j(n,n,1)} \right\|$. If $a_{(n,n,1)} \leq 1/n$, then we define

$$S^{(n,n,1)}_0 = \cdots = S^{(n,n,1)}_{p(n,n,1)-1} = I.$$

If $a_{(n,n,1)} > 1/n$ and $\left\| x^{(n,n,1)}_{j(n,n,1)+\triangle_n-\triangle_{n-1}} - x^{(n,n,1)}_{j(n,n,1)} \right\| > \omega(a_{(n,n,1)})/2$, then

$$S^{(n,n,1)}_0 = \cdots = S^{(n,n,1)}_{p(n,n,1)-1} = I.$$

If $a_{(n,n,1)} > 1/n$ and $\left\| x^{(n,n,1)}_{j(n,n,1)+\triangle_n-\triangle_{n-1}} - x^{(n,n,1)}_{j(n,n,1)} \right\| \leq \omega(a_{(n,n,1)})/2$, then

$$S^{(n,n,1)}_0 = \cdots = S^{(n,n,1)}_{j(n,n,1)-1} = I,$$

$$S^{(n,n,1)}_{j(n,n,1)} = \cdots = S^{(n,n,1)}_{j(n,n,1)+d(n,n,1)/2-1} = I, \qquad S^{(n,n,1)}_{j(n,n,1)+d(n,n,1)/2} = M^{(n,n,1)}_1,$$
$$S^{(n,n,1)}_{j(n,n,1)+d(n,n,1)/2+1} = \cdots = S^{(n,n,1)}_{j(n,n,1)+d(n,n,1)+d(n,n,1)/2-1} = I,$$
$$S^{(n,n,1)}_{j(n,n,1)+d(n,n,1)+d(n,n,1)/2} = M^{(n,n,1)}_2,$$
$$\vdots$$
$$S^{(n,n,1)}_{j(n,n,1)+d(n,n,1)(l(\delta_n)-2)+d(n,n,1)/2+1} = \cdots = S^{(n,n,1)}_{j(n,n,1)+d(n,n,1)(l(\delta_n)-1)+d(n,n,1)/2-1} = I,$$
$$S^{(n,n,1)}_{j(n,n,1)+d(n,n,1)(l(\delta_n)-1)+d(n,n,1)/2} = M^{(n,n,1)}_{l(\delta_n)},$$
$$S^{(n,n,1)}_{j(n,n,1)+d(n,n,1)(l(\delta_n)-1)+d(n,n,1)/2+1} = \cdots = S^{(n,n,1)}_{p(n,n,1)-1} = I.$$

Denote $R^{(n,n,1)}_k = R^{(n,n-1,1)}_k S^{(n,n,1)}_k$, $k \in \mathbb{Z}$.

Let us consider the Cauchy problem

$$x_{k+1} = R^{(n,n,q-1)}_k x_k, \qquad x_0 = u_q, \ q \in \{2, \ldots, n-1\}$$

with solution $\{x^{(n,1,q)}_k\}$. We consider a positive integer $j(n,1,q)$ divisible by $d(n,1,q)$ such that $j(n,1,q) > p(n,n,q-1)$. Taken from Definition 3.1, for $x^{(n,1,q)}_{(n,1,q)+\triangle_n}$, there exist

$$M^{(n,1,q)}_1, M^{(n,1,q)}_2, \ldots, M^{(n,1,q)}_{l(\delta_n)} \in O_{\delta_n}(I). \tag{5.10}$$

Let us define the periodic sequence $\{S^{(n,1,q)}_k\}$ with period

$$p(n,1,q) := [j(n,1,q) + \triangle_n]p(n,n,q-1)$$

in the following way. Denote

$$a_{(n,1,q)} := \left\| x^{(n,1,q)}_{j(n,1,q)} \right\|. \tag{5.11}$$

If $a_{(n,1,q)} \leq 1/n$, then

$$S^{(n,1,q)}_0 = \cdots = S^{(n,1,q)}_{p(n,1,q)-1} = I.$$

If $a_{(n,1,q)} > 1/n$ and $\left\| x^{(n,1,q)}_{j(n,1,q)+\triangle_n} - x^{(n,1,q)}_{j(n,1,q)} \right\| > \omega(a_{(n,1,q)})/2$, then

$$S^{(n,1,q)}_0 = \cdots = S^{(n,1,q)}_{p(n,1,q)-1} = I. \tag{5.12}$$

If $a_{(n,1,q)} > 1/n$ and $\left\| x^{(n,1,q)}_{j(n,1,q)+\triangle_n} - x^{(n,1,q)}_{j(n,1,q)} \right\| \leq \omega(a_{(n,1,q)})/2$, then

$$S^{(n,1,q)}_0 = \cdots = S^{(n,1,q)}_{j(n,1,q)-1} = I,$$

$$S^{(n,1,q)}_{j(n,1,q)} = \cdots = S^{(n,1,q)}_{j(n,1,q)+d(n,1,q)/2-1} = I, \qquad S^{(n,1,q)}_{j(n,1,q)+d(n,1,q)/2} = M^{(n,1,q)}_1,$$
$$S^{(n,1,q)}_{j(n,1,q)+d(n,1,q)/2+1} = \cdots = S^{(n,1,q)}_{j(n,1,q)+d(n,1,q)+d(n,1,q)/2-1} = I,$$

$$S^{(n,1,q)}_{j(n,1,q)+d(n,1,q)+d(n,1,q)/2} = M^{(n,1,q)}_2,$$

$$\vdots$$

$$S^{(n,1,q)}_{j(n,1,q)+d(n,1,q)(l(\delta_n)-2)+d(n,1,q)/2+1} = \cdots = S^{(n,1,q)}_{j(n,1,q)+d(n,1,q)(l(\delta_n)-1)+d(n,1,q)/2-1} = I,$$

$$S^{(n,1,q)}_{j(n,1,q)+d(n,1,q)(l(\delta_n)-1)+d(n,1,q)/2} = M^{(n,1,q)}_{l(\delta_n)},$$

$$S^{(n,1,q)}_{j(n,1,q)+d(n,1,q)(l(\delta_n)-1)+d(n,1,q)/2+1} = \cdots = S^{(n,1,q)}_{p(n,1,q)-1} = I.$$

Denote $R^{(n,1,q)}_k = R^{(n,n,q-1)}_k S^{(n,1,q)}_k$, $k \in \mathbb{Z}$.

Let us have the problem

$$x_{k+1} = R^{(n,1,q)}_k x_k, \qquad x_0 = u_q$$

and its solution $\{x^{(n,2,q)}_k\}$. Let $j(n,2,q)$ be a positive integer divisible by $d(n,2,q)$ such that $j(n,2,q) > p(n,1,q)$. For vector $x^{(n,2,q)}_{j(n,2,q)+\triangle_n-\triangle_1}$, there exist matrices (see Definition 3.1)

$$M^{(n,2,q)}_1, M^{(n,2,q)}_2, \dots, M^{(n,2,q)}_{l(\delta_n)} \in O_{\delta_n}(I). \tag{5.13}$$

We define sequence $\{S^{(n,2,q)}_k\}$ with period

$$p(n,2,q) := [j(n,2,q) + \triangle_n - \triangle_1]p(n,1,q)$$

in the following way. Denote $a_{(n,2,q)} := \left\| x^{(n,2,q)}_{j(n,2,q)} \right\|$. If $a_{(n,2,q)} \le 1/n$, then

$$S^{(n,2,q)}_0 = \cdots = S^{(n,2,q)}_{p(n,2,q)-1} = I.$$

If $a_{(n,2,q)} > 1/n$ and $\left\| x^{(n,2,q)}_{j(n,2,q)+\triangle_n-\triangle_1} - x^{(n,2,q)}_{j(n,2,q)} \right\| > \omega(a_{(n,2,q)})/2$, then

$$S^{(n,2,q)}_0 = \cdots = S^{(n,2,q)}_{p(n,2,q)-1} = I.$$

If $a_{(n,2,q)} > 1/n$ and $\left\| x^{(n,2,q)}_{j(n,2,q)+\triangle_n-\triangle_1} - x^{(n,2,q)}_{j(n,2,q)} \right\| \le \omega(a_{(n,2,q)})/2$, then

$$S^{(n,2,q)}_0 = \cdots = S^{(n,2,q)}_{j(n,2,q)-1} = I,$$

$$S^{(n,2,q)}_{j(n,2,q)} = \cdots = S^{(n,2,q)}_{j(n,2,q)+d(n,2,q)/2-1} = I, \qquad S^{(n,2,q)}_{j(n,2,q)+d(n,2,q)/2} = M^{(n,2,q)}_1,$$

$$S^{(n,2,q)}_{j(n,2,q)+d(n,2,q)/2+1} = \cdots = S^{(n,2,q)}_{j(n,2,q)+d(n,2,q)+d(n,2,q)/2-1} = I,$$

$$S^{(n,2,q)}_{j(n,2,q)+d(n,2,q)+d(n,2,q)/2} = M^{(n,2,q)}_2,$$

$$\vdots$$

$$S^{(n,2,q)}_{j(n,2,q)+d(n,2,q)(l(\delta_n)-2)+d(n,2,q)/2+1} = \cdots = S^{(n,2,q)}_{j(n,2,q)+d(n,2,q)(l(\delta_n)-1)+d(n,2,q)/2-1} = I,$$

$$S^{(n,2,q)}_{j(n,2,q)+d(n,2,q)(l(\delta_n)-1)+d(n,2,q)/2} = M^{(n,2,q)}_{l(\delta_n)},$$

$$S^{(n,2,q)}_{j(n,2,q)+d(n,2,q)(l(\delta_n)-1)+d(n,2,q)/2+1} = \cdots = S^{(n,2,q)}_{p(n,2,q)-1} = I.$$

Denote $R_k^{(n,2,q)} = R_k^{(n,1,q)} S_k^{(n,2,q)}$, $k \in \mathbb{Z}$. Let us continue in the same manner.

Consider the initial problem

$$x_{k+1} = R_k^{(n,n-1,q)} x_k, \qquad x_0 = u_q$$

with $\{x_k^{(n,n,q)}\}$ as its solution. There exists a positive integer $j(n,n,q)$ divisible by $d(n,n,q)$ such that $j(n,n,q) > p(n,n-1,q)$. For $x^{(n,n,q)}_{j(n,n,q)+\triangle_n-\triangle_{n-1}}$, there exist (see Definition 3.1)

$$M_1^{(n,n,q)}, M_2^{(n,n,q)}, \dots, M_{l(\delta_n)}^{(n,n,q)} \in O_{\delta_n}(I). \tag{5.14}$$

Let us define a sequence $\{S_k^{(n,n,q)}\}$ with period

$$p(n,n,q) := [j(n,n,q) + \triangle_n - \triangle_{n-1}]\, p(n,n-1,q).$$

Denote

$$a_{(n,n,q)} := \left\| x^{(n,n,q)}_{j(n,n,q)} \right\|. \tag{5.15}$$

If $a_{(n,n,q)} \leq 1/n$, then we put

$$S_0^{(n,n,q)} = \cdots = S^{(n,n,q)}_{p(n,n,q)-1} = I.$$

If $a_{(n,n,q)} > 1/n$ and $\left\| x^{(n,n,q)}_{j(n,n,q)+\triangle_n-\triangle_{n-1}} - x^{(n,n,q)}_{j(n,n,q)} \right\| > \omega(a_{(n,n,q)})/2$, then

$$S_0^{(n,n,q)} = \cdots = S^{(n,n,q)}_{p(n,n,q)-1} = I. \tag{5.16}$$

If $a_{(n,n,q)} > 1/n$ and $\left\| x^{(n,n,q)}_{j(n,n,q)+\triangle_n-\triangle_{n-1}} - x^{(n,n,q)}_{j(n,n,q)} \right\| \leq \omega(a_{(n,n,q)})/2$, then

$$S_0^{(n,n,q)} = \cdots = S^{(n,n,q)}_{j(n,n,q)-1} = I,$$

$$S^{(n,n,q)}_{j(n,n,q)} = \cdots = S^{(n,n,q)}_{j(n,n,q)+d(n,n,q)/2-1} = I, \qquad S^{(n,n,q)}_{j(n,n,q)+d(n,n,q)/2} = M_1^{(n,n,q)},$$

$$S^{(n,n,q)}_{j(n,n,q)+d(n,n,q)/2+1} = \cdots = S^{(n,n,q)}_{j(n,n,q)+d(n,n,q)+d(n,n,q)/2-1} = I,$$

$$S^{(n,n,q)}_{j(n,n,q)+d(n,n,q)+d(n,n,q)/2} = M_2^{(n,n,q)},$$

$$\vdots$$

$$S^{(n,n,q)}_{j(n,n,q)+d(n,n,q)(l(\delta_n)-2)+d(n,n,q)/2+1} = \cdots = S^{(n,n,q)}_{j(n,n,q)+d(n,n,q)(l(\delta_n)-1)+d(n,n,q)/2-1} = I,$$

$$S^{(n,n,q)}_{j(n,n,q)+d(n,n,q)(l(\delta_n)-1)+d(n,n,q)/2} = M^{(n,n,q)}_{l(\delta_n)},$$

$$S^{(n,n,q)}_{j(n,n,q)+d(n,n,q)(l(\delta_n)-1)+d(n,n,q)/2+1} = \cdots = S^{(n,n,q)}_{p(n,n,q)-1} = I.$$

Denote $R_k^{(n,n,q)} = R_k^{(n,n-1,q)} S_k^{(n,n,q)}$, $k \in \mathbb{Z}$.

We consider the initial problem

$$x_{k+1} = R_k^{(n,n,n-1)} x_k, \qquad x_0 = u_n$$

with $\{x_k^{(n,1,n)}\}$ as its solution. There exists a positive integer $j(n,1,n)$ divisible by $d(n,1,n)$ such that $j(n,1,n) > p(n,n,n-1)$. For $x_{j(n,1,n)+\triangle_n}^{(n,1,n)}$, there exist (see Definition 3.1)

$$M_1^{(n,1,n)}, M_2^{(n,1,n)}, \dots, M_{l(\delta_n)}^{(n,1,n)} \in O_{\delta_n}(I). \tag{5.17}$$

Let us define a sequence $\{S_k^{(n,1,n)}\}$ with period

$$p(n,1,n) := [j(n,1,n) + \triangle_n] p(n,n,n-1).$$

Denote $a_{(n,1,n)} := \left\| x_{j(n,1,n)}^{(n,1,n)} \right\|$. If $a_{(n,1,n)} \le 1/n$, then we put

$$S_0^{(n,1,n)} = \cdots = S_{p(n,1,n)-1}^{(n,1,n)} = I.$$

If $a_{(n,1,n)} > 1/n$ and $\left\| x_{j(n,1,n)+\triangle_n}^{(n,1,n)} - x_{j(n,1,n)}^{(n,1,n)} \right\| > \omega(a_{(n,1,n)})/2$, then

$$S_0^{(n,1,n)} = \cdots = S_{p(n,1,n)-1}^{(n,1,n)} = I.$$

If $a_{(n,1,n)} > 1/n$ and $\left\| x_{j(n,1,n)+\triangle_n}^{(n,1,n)} - x_{j(n,1,n)}^{(n,1,n)} \right\| \le \omega(a_{(n,1,n)})/2$, then

$$S_0^{(n,1,n)} = \cdots = S_{j(n,1,n)-1}^{(n,1,n)} = I,$$

$$S_{j(n,1,n)}^{(n,1,n)} = \cdots = S_{j(n,1,n)+d(n,1,n)/2-1}^{(n,1,n)} = I, \qquad S_{j(n,1,n)+d(n,1,n)/2}^{(n,1,n)} = M_1^{(n,1,n)},$$
$$S_{j(n,1,n)+d(n,1,n)/2+1}^{(n,1,n)} = \cdots = S_{j(n,1,n)+d(n,1,n)+d(n,1,n)/2-1}^{(n,1,n)} = I,$$
$$S_{j(n,1,n)+d(n,1,n)+d(n,1,n)/2}^{(n,1,n)} = M_2^{(n,1,n)},$$
$$\vdots$$
$$S_{j(n,1,n)+d(n,1,n)(l(\delta_n)-2)+d(n,1,n)/2+1}^{(n,1,n)} = \cdots = S_{j(n,1,n)+d(n,1,n)(l(\delta_n)-1)+d(n,1,n)/2-1}^{(n,1,n)} = I,$$
$$S_{j(n,1,n)+d(n,1,n)(l(\delta_n)-1)+d(n,1,n)/2}^{(n,1,n)} = M_{l(\delta_n)}^{(n,1,n)},$$
$$S_{j(n,1,n)+d(n,1,n)(l(\delta_n)-1)+d(n,1,n)/2+1}^{(n,1,n)} = \cdots = S_{p(n,1,n)-1}^{(n,1,n)} = I.$$

Denote $R_k^{(n,1,n)} = R_k^{(n,n,n-1)} S_k^{(n,1,n)}$, $k \in \mathbb{Z}$.

Let us have the problem

$$x_{k+1} = R_k^{(n,1,n)} x_k, \; x_0 = u_n$$

and its solution $\{x_k^{(n,2,n)}\}$. Let $j(n,2,n)$ be a positive integer divisible by $d(n,2,n)$ such that $j(n,2,n) > p(n,1,n)$. For vector $x_{j(n,2,n)+\triangle_n-\triangle_1}^{(n,2,n)}$, there exist matrices (see Definition 3.1)

$$M_1^{(n,2,n)}, M_2^{(n,2,n)}, \dots, M_{l(\delta_n)}^{(n,2,n)} \in O_{\delta_n}(I). \tag{5.18}$$

We define the sequence $\{S_k^{(n,2,n)}\}$ with period

$$p(n,2,n) := [j(n,2,n) + \triangle_n - \triangle_1] p(n,1,n)$$

in the following way. Denote $a_{(n,2,n)} := \left\| x^{(n,2,n)}_{j(n,2,n)} \right\|$. If $a_{(n,2,n)} \leq 1/n$, then

$$S^{(n,2,n)}_0 = \cdots = S^{(n,2,n)}_{p(n,2,n)-1} = I.$$

If $a_{(n,2,n)} > 1/n$ and $\left\| x^{(n,2,n)}_{j(n,2,n)+\triangle_n-\triangle_1} - x^{(n,2,n)}_{j(n,2,n)} \right\| > \omega(a_{(n,2,n)})/2$, then

$$S^{(n,2,n)}_0 = \cdots = S^{(n,2,n)}_{p(n,2,n)-1} = I.$$

If $a_{(n,2,n)} > 1/n$ and $\left\| x^{(n,2,n)}_{j(n,2,n)+\triangle_n-\triangle_1} - x^{(n,2,n)}_{j(n,2,n)} \right\| \leq \omega(a_{(n,2,n)})/2$, then

$$S^{(n,2,n)}_0 = \cdots = S^{(n,2,n)}_{j(n,2,n)-1} = I,$$

$$S^{(n,2,n)}_{j(n,2,n)} = \cdots = S^{(n,2,n)}_{j(n,2,n)+d(n,2,n)/2-1} = I, \qquad S^{(n,2,n)}_{j(n,2,n)+d(n,2,n)/2} = M^{(n,2,n)}_1,$$

$$S^{(n,2,n)}_{j(n,2,n)+d(n,2,n)/2+1} = \cdots = S^{(n,2,n)}_{j(n,2,n)+d(n,2,n)+d(n,2,n)/2-1} = I,$$

$$S^{(n,2,n)}_{j(n,2,n)+d(n,2,n)+d(n,2,n)/2} = M^{(n,2,n)}_2,$$

$$\vdots$$

$$S^{(n,2,n)}_{j(n,2,n)+d(n,2,n)(l(\delta_n)-2)+d(n,2,n)/2+1} = \cdots = S^{(n,2,n)}_{j(n,2,n)+d(n,2,n)(l(\delta_n)-1)+d(n,2,n)/2-1} = I,$$

$$S^{(n,2,n)}_{j(n,2,n)+d(n,2,n)(l(\delta_n)-1)+d(n,2,n)/2} = M^{(n,2,n)}_{l(\delta_n)},$$

$$S^{(n,2,n)}_{j(n,2,n)+d(n,2,n)(l(\delta_n)-1)+d(n,2,n)/2+1} = \cdots = S^{(n,2,n)}_{p(n,2,n)-1} = I.$$

Denote $R^{(n,2,n)}_k = R^{(n,1,n)}_k S^{(n,2,n)}_k$, $k \in \mathbb{Z}$. We continue in the same way.

Let us have the system

$$x_{k+1} = R^{(n,n-1,n)}_k x_k, \quad x_0 = u_n$$

and let $\{x^{(n,n,n)}_k\}$ be its solution. Then, there exists a positive integer $j(n,n,n)$ divisible by $d(n,n,n)$ such that $j(n,n,n) > p(n,n-1,n)$. Also for vector $x^{(n,n,n)}_{j(n,n,n)+\triangle_n-\triangle_{n-1}}$, there exist

$$M^{(n,n,n)}_1, M^{(n,n,n)}_2, \ldots, M^{(n,n,n)}_{l(\delta_n)} \in O_{\delta_n}(I) \tag{5.19}$$

taken from Definition 3.1. Next, we define periodic the sequence $\{S^{(n,n,n)}_k\}$ with period

$$p(n,n,n) := [j(n,n,n) + \triangle_n - \triangle_{n-1}]p(n,n-1,n)$$

as follows. Denote $a_{(n,n,n)} := \left\| x^{(n,n,n)}_{j(n,n,n)} \right\|$. If $a_{(n,n,n)} \leq 1/n$, then we put

$$S^{(n,n,n)}_0 = \cdots = S^{(n,n,n)}_{p(n,n,n)-1} = I.$$

If $a_{(n,n,n)} > 1/n$ and $\left\| x^{(n,n,n)}_{j(n,n,n)+\triangle_n-\triangle_{n-1}} - x^{(n,n,n)}_{j(n,n,n)} \right\| > \omega(a_{(n,n,n)})/2$, then

$$S^{(n,n,n)}_0 = \cdots = S^{(n,n,n)}_{p(n,n,n)-1} = I.$$

If $a_{(n,n,n)} > 1/n$ and $\left\| x^{(n,n,n)}_{j(n,n,n)+\triangle_n-\triangle_{n-1}} - x^{(n,n,n)}_{j(n,n,n)} \right\| \le \omega(a_{(n,n,n)})/2$, then

$$S^{(n,n,n)}_0 = \cdots = S^{(n,n,n)}_{j(n,n,n)-1} = I,$$

$$S^{(n,n,n)}_{j(n,n,n)} = \cdots = S^{(n,n,n)}_{j(n,n,n)+d(n,n,n)/2-1} = I, \qquad S^{(n,n,n)}_{j(n,n,n)+d(n,n,n)/2} = M^{(n,n,n)}_1,$$

$$S^{(n,n,n)}_{j(n,n,n)+d(n,n,n)/2+1} = \cdots = S^{(n,n,n)}_{j(n,n,n)+d(n,n,n)+d(n,n,n)/2-1} = I,$$

$$S^{(n,n,n)}_{j(n,n,n)+d(n,n,n)+d(n,n,n)/2} = M^{(n,n,n)}_2,$$

$$\vdots$$

$$S^{(n,n,n)}_{j(n,n,n)+d(n,n,n)(l(\delta_n)-2)+d(n,n,n)/2+1} = \cdots = S^{(n,n,n)}_{j(n,n,n)+d(n,n,n)(l(\delta_n)-1)+d(n,n,n)/2-1} = I,$$

$$S^{(n,n,n)}_{j(n,n,n)+d(n,n,n)(l(\delta_n)-1)+d(n,n,n)/2} = M^{(n,n,n)}_{l(\delta_n)},$$

$$S^{(n,n,n)}_{j(n,n,n)+d(n,n,n)(l(\delta_n)-1)+d(n,n,n)/2+1} = \cdots = S^{(n,n,n)}_{p(n,n,n)-1} = I.$$

We put $R^n_k = R^{(n,n,n)}_k = R^{(n,n-1,n)}_k S^{(n,n,n)}_k$ and $S^n_k = S^{(n,1,1)}_k \cdots S^{(n,n,1)}_k S^{(n,1,2)}_k \cdots S^{(n,n,n)}_k$ for $k \in \mathbb{Z}$. We continue in the construction.

According to Lemma 4.6, for

$$S_k := A_k S^1_k \cdots S^n_k \cdots, \qquad k \in \mathbb{Z},$$

we have $S_k \in O_\varepsilon(\{A_k\}) \cap LP(X)$. Of course, sequence $\{\delta_n\}_{n\in\mathbb{N}}$ defined by (5.5) is decreasing and tends to 0. One can verify (see (5.7)–(5.10), (5.13), (5.14), and (5.17)–(5.19)) that (4.1) holds. From the structure of indices in the construction and the definition of d (see (5.4)), one can see that possible positions for non-identity matrices in sequences $\{S^{(\cdot,\cdot,\cdot)}_k\}$ are never the same for different sequences. Therefore, (4.2) holds. One can easily verify (see (5.3)) that (4.3) holds as well.

Let $c \in \mathbb{N}$ be arbitrarily given. We show that the non-zero solution of the initial problem

$$x_{k+1} = S_k x_k, \qquad x_0 = u_c \tag{5.20}$$

is not almost periodic. By contradiction, we suppose that the solution is almost periodic and we denote it as $\{x_k\}$. Using Lemma 4.5, there exists $b \in \mathbb{N}$, $b > c$, such that

$$\|x_k\| > \frac{1}{b}, \qquad k \in \mathbb{Z}. \tag{5.21}$$

It follows from the construction that, for given $k \in \mathbb{N}$, there exists $K \in \mathbb{N}$ such that $S^K_k = S^{K+1}_k = \cdots = I$. More precisely,

$$S^{i+j}_k = I, \qquad k \in \{0, 1, \ldots, p(i,i,i) - 1\},\ i, j \in \mathbb{N}.$$

Considering the construction in the b-th step, we get

$$\begin{aligned}\left\| x_{j(b,1,c)} - x_{j(b,1,c)+\triangle_b} \right\| &= \left\| S_{j(b,1,c)-1} \cdots S_1 S_0 u_c - S_{j(b,1,c)+\triangle_b-1} \cdots S_1 S_0 u_c \right\| \\ &= \left\| S^{(b,1,c)}_{j(b,1,c)-1} \cdots S^{(b,1,c)}_1 S^{(b,1,c)}_0 x^{(b,1,c)}_{j(b,1,c)} - S^{(b,1,c)}_{j(b,1,c)+\triangle_b-1} \cdots S^{(b,1,c)}_1 S^{(b,1,c)}_0 x^{(b,1,c)}_{j(b,1,c)+\triangle_b} \right\|.\end{aligned}$$

If $\left\| x^{(b,1,c)}_{j(b,1,c)} - x^{(b,1,c)}_{j(b,1,c)+\triangle_b} \right\| > \omega(a_{(b,1,c)})/2$, then (see (5.12))

$$\left\| x_{j(b,1,c)} - x_{j(b,1,c)+\triangle_b} \right\| = \left\| I \cdot x^{(b,1,c)}_{j(b,1,c)} - I \cdot x^{(b,1,c)}_{j(b,1,c)+\triangle_b} \right\| > \frac{\omega(a_{(b,1,c)})}{2}.$$

If $\left\| x^{(b,1,c)}_{j(b,1,c)} - x^{(b,1,c)}_{j(b,1,c)+\triangle_b} \right\| \leq \omega(a_{(b,1,c)})/2$, then (see Remark 4.4)

$$\left\| x_{j(b,1,c)} - x_{j(b,1,c)+\triangle_b} \right\| = \left\| I \cdot x^{(b,1,c)}_{j(b,1,c)} - M^{(b,1,c)}_{l(\delta_b)} \cdots M^{(b,1,c)}_1 x^{(b,1,c)}_{j(b,1,c)+\triangle_b} \right\| > \frac{\omega(a_{(b,1,c)})}{2}.$$

In both cases, we have (for the last inequality, see Remark 4.3, (5.11) and (5.21))

$$\left\| x_{j(b,1,c)} - x_{j(b,1,c)+\triangle_b} \right\| > \omega(a_{(b,1,c)})/2 \geq \frac{\omega(1/b)}{2}. \tag{5.22}$$

We continue in the same manner. We have

$$\begin{aligned} &\left\| x_{j(b,b,c)} - x_{j(b,b,c)+\triangle_b-\triangle_{b-1}} \right\| \\ &\quad = \left\| S_{j(b,b,c)-1} \cdots S_1 S_0 u_c - S_{j(b,b,c)+\triangle_b-\triangle_{b-1}-1} \cdots S_1 S_0 u_c \right\| \\ &\quad = \left\| S^{(b,b,c)}_{j(b,b,c)-1} \cdots S^{(b,b,c)}_0 x^{(b,b,c)}_{j(b,b,c)} - S^{(b,b,c)}_{j(b,b,c)+\triangle_b-\triangle_{b-1}-1} \cdots S^{(b,b,c)}_0 x^{(b,b,c)}_{j(b,b,c)+\triangle_b-\triangle_{b-1}} \right\|. \end{aligned}$$

If $\left\| x^{(b,b,c)}_{j(b,b,c)} - x^{(b,b,c)}_{j(b,b,c)+\triangle_b-\triangle_{b-1}} \right\| > \omega(a_{(b,b,c)})/2$, then (see (5.16))

$$\left\| x_{j(b,b,c)} - x_{j(b,b,c)+\triangle_b-\triangle_{b-1}} \right\| = \left\| I \cdot x^{(b,b,c)}_{j(b,b,c)} - I \cdot x^{(b,b,c)}_{j(b,b,c)+\triangle_b-\triangle_{b-1}} \right\| > \frac{\omega(a_{(b,b,c)})}{2}.$$

If $\left\| x^{(b,b,c)}_{j(b,b,c)} - x^{(b,b,c)}_{j(b,b,c)+\triangle_b-\triangle_{b-1}} \right\| \leq \omega(a_{(b,b,c)})/2$, then (see Remark 4.4)

$$\begin{aligned} &\left\| x_{j(b,b,c)} - x_{j(b,b,c)+\triangle_b-\triangle_{b-1}} \right\| \\ &\quad = \left\| I \cdot x^{(b,b,c)}_{j(b,b,c)} - M^{(b,b,c)}_{l(\delta_b)} \cdots M^{(b,b,c)}_1 x^{(b,b,c)}_{j(b,b,c)+\triangle_b-\triangle_{b-1}} \right\| > \frac{\omega(a_{(b,b,c)})}{2}. \end{aligned}$$

Again, we have (see Remark 4.3, (5.15) and (5.21))

$$\left\| x_{j(b,b,c)} - x_{j(b,b,c)+\triangle_b-\triangle_{b-1}} \right\| > \omega(a_{(b,b,c)})/2 \geq \frac{\omega(1/b)}{2}. \tag{5.23}$$

We can continue in the same way, when we obtain

$$\left\| x_{j(b+n,1,c)} - x_{j(b+n,1,c)+\triangle_{b+n}} \right\| > \frac{\omega(1/b)}{2}, \tag{5.24}$$

$$\vdots$$

$$\left\| x_{j(b+n,b+n,c)} - x_{j(b+n,b+n,c)+\triangle_{b+n}-\triangle_{b+n-1}} \right\| > \frac{\omega(1/b)}{2} \tag{5.25}$$

for any $n \in \mathbb{N}$.

Now we use Theorem 2.4. Let us put $l_1 := 0, \ldots, l_n := \triangle_{n-1}, \ldots$ Considering previous inequalities (5.22), (5.23), (5.24), (5.25), it is seen that, for all large $i, j \in \mathbb{N}$, $i \neq j$, there exists $l \in \mathbb{Z}$ such that

$$\| x_{l+l_i} - x_{l+l_j} \| > \frac{\omega(1/b)}{2},$$

which is a contradiction with (2.1). Number c is arbitrarily given. Therefore, the initial problem (5.20) does not have an almost periodic solution for any $c \in \mathbb{N}$. $\square$

At the end of this section, we mention some results nearly related with Theorem 5.1.

Corollary 5.2. *Let X have property P with respect to a vector u. For any $\{A_k\} \in LP(X)$ and $\varepsilon > 0$, there exists a system $\{S_k\} \in O_\varepsilon(\{A_k\}) \cap LP(X)$ whose fundamental matrix is not almost periodic.*

Remark 5.3. The previous corollary is the main result of [9]. It shows how our result generalizes the result which is the basic motivation.

The next theorem is a modification of Theorem 5.1, where coefficient matrices are taken from $AP(X)$ instead of $LP(X)$.

Theorem 5.4. *Let X have property P and $\varepsilon > 0$ be arbitrary. Then, for every $\{A_k\} \in AP(X)$ and every sequence $\{u_n\}_{n\in\mathbb{N}}$ of non-zero vectors $u_n \in F^m$, there exists $\{S_k\} \in O_\varepsilon(\{A_k\})$ such that the solution of*

$$x_{k+1} = S_k x_k, \qquad x_0 = u_n$$

is not almost periodic for any $n \in \mathbb{N}$.

Proof. The proof of this theorem can be obtained in the same way as the one of Theorem 5.1. In fact, the same construction can be used. It suffices to modify Lemma 4.6 (see Remark 4.7). □

In addition, we obtain the following result (from the proof of Theorem 5.1).

Theorem 5.5. *Let $\varepsilon > 0$ be arbitrary. Let X have the property that there exist $\zeta > 0$ and open sets U_i, $i \in \mathbb{N}$, fulfilling $U_i \subseteq F^m$, $F^m \setminus O_1(0) \subseteq \bigcup_{i\in\mathbb{N}} U_i \subseteq F^m \setminus \{0\}$, such that, for every $\delta > 0$, there exists $l \in \mathbb{N}$ such that, for every $j \in \mathbb{N}$, there exist matrices $M_1, M_2, \dots, M_l \in X$ with the property that*

$$M_i \in O_\delta(I),\ i \in \{1, \dots, l\}, \qquad \|M_l \cdots M_1 u - u\| > \zeta,\ u \in U_j.$$

For every $\{A_k\} \in LP(X)$, there exists $\{S_k\} \in O_\varepsilon(\{A_k\}) \cap LP(X)$ such that all non-zero solutions of $x_{k+1} = S_k x_k$ are not almost periodic. For every $\{B_k\} \in AP(X)$, there exists $\{R_k\} \in O_\varepsilon(\{B_k\})$ such that all non-zero solutions of $x_{k+1} = R_k x_k$ are not almost periodic.

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