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DISCRETIZATION METHODS FOR NONCONVEX DIFFERENTIAL INCLUSIONS

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ABSTRACT. We prove the existence of solutions for the differential inclusion $\dot{x}(t) \in F(t, x(t)) + f(t, x(t))$ for a multifunction F upper semicontinuous with compact values contained in the generalized Clarke gradient of a regular locally Lipschitz function and f a Carathéodory function.

1. INTRODUCTION

In the present paper we consider the Cauchy problem for first order differential inclusion of the form

$$\dot{x}(t) \in F(x(t)) + f(t, x(t)), x(0) = x_0 \tag{1.1}$$

where F is a given set-valued map with nonconvex values and f is a Carathéodory function. The nonconvexity of the values of F do not permit the use of classical technique of convex analysis to obtain the existence of solution to this problem (see for instance [2]). One way to overcome this fact is to suppose F upper semicontinuous cyclically monotone, ie. the values of F are contained in the subdifferential of a proper convex lower semicontinuous function. The first result is due to [6] when $f \equiv 0$ and [1] for the problem (1.1) in the finite dimensional setting. As an extension of this results, [12], [11] have proved the existence of viable solutions under additive assumptions. An other extension of [6] is obtained by [3] and [4] in the finite and infinite dimensional setting, under the assumption that $F(x)$ is contained in the subdifferential of a locally Lipschitz and regular function. A different class of function has been used in [5] to solve the same problem, namely the authors take $F(x)$ in the proximal subdifferential of a locally Lipschitz uniformly regular function and prove that any convex lower semicontinuous function is uniformly regular. The present paper is a continuation of the above results. In Section 2 we recall some preliminary facts needed in the sequel and summarize some notions of regularity of functions; then we prove that, for locally Lipschitz functions, the class of convex functions, the class of lower-C^2 functions and the class of uniformly regular functions are strictly contained within the class of regular functions. in Section 3 we present an existence result to problem (1.1) in $\mathbf{R}^n$ which extend theorem (3.1) in [3]. A second result is obtained in the infinite dimensional Hilbert space by replacing the additional assumptions in [3] and [4] by weaker and more natural condition.

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2. Preliminaries and fundamental results

Let E be a separable Banach space, E' his topological dual. We denote by $B(x,r)$ the open ball centered at x with radius $r>0$, $\overline{B}(x,r)$ it'closure. For any function $V : E \to \mathbb{R}$ locally Lipschitz at $x \in E$ the generalized (Clarke) gradient of V at x, is given by

$$\partial^c V(x) := \{\xi \in E' : V^0(x,v) \geq \langle \xi, v\rangle \quad \forall v \in E\} \tag{2.1}$$

where

$$V^0(x,v) = limSup_{y\to x, h\downarrow 0}\frac{V(y+hv)-V(y)}{h} \tag{2.2}$$

is the generalized directional derivative of V at x. Let us recall (see [8]) that ∂V is a monotone and upper semicontinuous multifunction with nonempty convex weakly compact values in E' and for any $v \in E$ one has

$$V^0(x,v) = \max\{\langle \xi, v\rangle;\ \xi \in \partial^C V(x)\} \tag{2.3}$$

Definition 2.1. Let V be locally Lipschitz at $x \in E$, V is said to be regular at x provided that for all $v \in E$, the usual directional derivative $V'(x,v)$ exists and is equal to $V^0(x,v)$.

It's well known that any locally Lipschitz convex function is regular ([8], proposition 2.3.6).

We say that a multifunction $F : E \to 2^E$ is upper semicontinuous (u.s.c) at x_0 if for any open subset U of E containing $F(x_0)$, the set $\{y \in E : F(y) \subset U\}$ is a neighborhood of x_0; F is usc if F is u.s.c at each point. In the case of compact values multifunctions, this definition is equivalent to the one given in [l]: for every $x \in E$ and for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$F(y) \subset F(x) + B(0,\varepsilon)\ \forall y \in B(x,\delta) \tag{2.4}$$

For more details on multifunctions and differential inclusions, we refer to [2],[7] and [9].
For a function with bounded variation $f : I \to E$, we denote by Df the differential measure of f defined on the Borel tribe of I :

$$\forall [a,b] \subset I,\ Df([a,b]) = f^+(b) - f^-(a)$$

$f^+(t)$ (resp. $f^-(t)$) is the right (resp. left) derivative of f at t witch exists for all $t \in [0,T[$ (resp. $]0,T]$.) Let μ a positive Radon measure on $\mathbb{R}$ and $L^1_E(I,\mu)$ the Banach space of $\mu-$integrable functions from I to E.
The following proposition is crucial for the main result and proved by [3]

Proposition 2.2. *Let Ω an open convex set in $E, V : \Omega \to \mathbb{R}$ regular and Lipschitz on any bounded set of Ω, $x : I \to \Omega$ a function with bounded variation, such that Dx has a density $\frac{Dx}{d\mu} \in L^1_E(I,\mu)$. Then, the function $V \circ x$ has bounded variation on I, $D(V\circ x)$ is absolutely continuous and $\mu-$almost everywhere:*

$$\langle \partial V(x(t)), \frac{Dx}{d\mu}(t)\rangle := \{\langle x', \frac{Dx}{d\mu}(t)\rangle : x' \in \partial V(x(t))\} = \{\frac{D(V\circ x)}{d\mu}(t)\} \tag{2.5}$$

In [5] and [10], a concept of uniformly regular functions is defined; they show that, for locally Lipschitz functions, the class of convex functions and the class of lower-C^2 functions are strictly contained within the class of uniformly regular functions. Let recall this concept

Definition 2.3. ([5]) Let $V : E \to R \cup \{+\infty\}$ be a lower semicontinuous function and let $\Omega \subset domV$ be a nonempty open subset. We will say that V is uniformly regular over Ω if there exists a positive number $\beta \geq 0$ such that for all $x \in \Omega$ and for all $\xi \in \partial^P V(x)$ one has

$$\langle \xi, x' - x \rangle \leq V(x') - V(x) + \beta \|x' - x\|^2 \quad for\ all\ x' \in \Omega.$$

We will say that V is uniformly regular over a closed set S if there exists an open set O containing S such that V is uniformly regular over O.

Here $\partial^P V(x)$ denotes the proximal subdifferential of V at x defined by

$$\partial^p V(x) = \left\{ \begin{array}{c} \xi \in E', \exists \delta, \sigma > 0 : \forall x' \in x + \delta \mathbb{B} \\ \langle \xi, x' - x \rangle \leq \sigma \|x' - x\|^2 + V(x') - V(x) . \end{array} \right\}$$

One has always $\partial^p V(x) \subset \partial^c V(x)$ and if V is uniformly regular then $\partial^p V(x)$ is closed and $\partial^p V(x) = \partial^c V(x)$.

Now, we state that, the class of regular functions is, for locally Lipschitz functions, larger than the class of uniformly regular functions and consequently the class of convex functions and the class of lower-C^2 functions.

Proposition 2.4. *Let V locally Lipschitz, if V is uniformly regular then V is regular*

Proof. Suppose that V is uniformly regular, evidently one has $V^0(x, v) \geq V'(x, v)$. To show that V is regular, one has to prove $V^0(x, v) \leq V'(x, v)$. By definition of uniform regularity, for a nonempty open subset S of E, there exists $\beta > 0$ such that for all $x \in S$, and for all $\xi \in \partial^p V(x)$ one has

$$\langle \xi, x' - x \rangle \leq V(x') - V(x) + \beta \|x' - x\|^2 \quad for\ all\ x' \in S$$

Choose a sufficiently small $h > 0$ such that $x' = x + hv \in S$, then

$$\langle \xi, hv \rangle \leq V(x + hv) - V(x) + \beta h^2 \|v\|^2$$

By (2.3) there exists $\xi_v \in \partial^c V(x)$ such that $V^0(x, v) = \langle \xi_v, v \rangle$. Since $\partial^p V(x) = \partial^c V(x)$, $\xi_v \in \partial^p V(x)$ and then

$$\langle \xi_v, hv \rangle \leq V(x + hv) - V(x) + \beta h^2 \|v\|^2$$

$$\langle \xi_v, v \rangle \leq \frac{1}{h}[V(x + hv) - V(x)] + \beta h \|v\|^2$$

$$\begin{aligned} V^0(x, v) &\leq \lim_{h \downarrow 0} \{\frac{1}{h}[V(x + hv) - V(x)] + \beta h \|v\|^2\} \\ &\leq V'(x, v) \end{aligned}$$

then V is regular. □

3. The main results

Our purpose is the study of existence of absolutely continuous solutions of problem (1.1), where the multifunction F has values contained in the subdifferential of the general class of regular functions. Consider first the finite dimensional case, let $E = \mathbf{R}^n$, $V : \mathbf{R}^n \to \mathbf{R}$ and $F : \mathbf{R}^n \to 2^{\mathbf{R}^n}$ such that

(H_1) F is upper semicontinuous with compact values;

(H_2) V is regular such that

$$F(x) \subset \partial V(x), \ \forall x \in \mathbb{R}^n$$

(H_3) $f : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^n$ is a Carathéodory function, (i.e. for every $x \in \mathbf{R}^n, t \longmapsto f(t,x)$ is measurable, for $t \in \mathbb{R}^+, x \longmapsto f(t,x)$ is continuous) and for any bounded subset B of $\mathbf{R}^n$, there is a compact set K such that $f(t,x) \in K$ for all $(t,x) \in \mathbf{R}^+ \times B$.

First, we prove existence of "approximate solutions" to problem (1.1). Since V is locally Lipschitz, then there exists $r > 0$ and $M > 0$ such that V is $M-$Lipschitz on $B(x_0, r)$. F is locally bounded by ([2], Prop 1.1.3), so we can assume that

$$Sup\{\|y\| : y \in F(x), \ x \in B(x_0,r)\} \leq M \tag{3.1}$$

By our assumption (H_3), there is a positive constant m such that $f(t,x) \in K \subset m\mathbf{B}$ for all $(t,x) \in \mathbf{R}^+ \times \bar{\mathbf{B}}(x_0,r)$ Let $T > 0$ such that $T \leq \frac{r}{m+M}\}$ and $I := [0,T]$. For any $\varepsilon > 0$ and $s \in I$, we define the set

$$\begin{aligned} I_\varepsilon([0,s]) &= \{\theta : [0,s] \to [0,s], \ \text{ non decreasing}, \ \theta(0) = 0, \ \theta(s) = s, \\ &\qquad \forall t \in [0,s] \ \ \theta(\theta(t)) = \theta(t), \ 0 \leq t - \theta(t) \leq \varepsilon\} \end{aligned}$$

Note that $I_\varepsilon([0,s])$ contains the identity function.

Definition 3.1. For any $\varepsilon > 0$ and $s \in I$, a function $x : [0,s] \to \mathbb{R}^n$ is said an ε-approximate solution for the differential inclusion (1.1) if

1. $x(t) = x_0 + \int_0^t (\dot{u}(\tau) + f(\tau, x(\tau)))d\tau, \ \dot{u} \in L^\infty_{\mathbb{R}^n}([0,s], dt), \ x(t) \in B(x_0,r) \ \forall t \in [0,s]$
2. $\exists \theta \in I_\epsilon([0,s])$ s.t. $\dot{u}(t) \in F(x(\theta(t)))$ a. e. on $[0,s]$
3. $V(x(s)) - V(x_0) \geq \int_0^s \langle \dot{u}(\tau), \dot{u}(\tau) + f(\tau, x(\tau))\rangle d\tau \ - \ \varepsilon.s$

For every $\varepsilon > 0$, let us denote by $\mathcal{P}_\varepsilon([0,s])$ the set of all (θ, x) with x an ε-approximate solution for the differential inclusion (1.1) defined on $[0,s]$, and θ the element of $I_\epsilon([0,s])$ which correspond to x. Set $\mathcal{P}_\varepsilon := \bigcup_{s\in I} \mathcal{P}_\varepsilon([0,s])$. Observe that $\mathcal{P}_\varepsilon$ is not empty since $\mathcal{P}_\varepsilon([0,0]) = \{0, x_0\}$. Let us consider the order relation defined on $\mathcal{P}_\varepsilon$ by

$$\begin{aligned} (\theta_1, x_1) \leq (\theta_2, x_2) \quad &\Leftrightarrow \quad s_1 \leq s_2, \ \theta_2(s) = \theta_1(s) \text{ on } [0, s_1], \\ &\qquad \text{and } x_2(s) = x_1(s) \text{ on } [0, s_1] \end{aligned}$$

for every $(\theta_i, x_i) \in \mathcal{P}_\varepsilon([0,s_i]), \ i = 1,2$

Lemma 3.2. *For any $\varepsilon > 0$, there exists an ε-approximate solution for the differential inclusion (1.1) defined on I.*

Proof. Let $\mathcal{T} = \{(\theta_j, x_j), j \in J\}$ a chain of $(\mathcal{P}_\varepsilon, \leq)$ with $(\theta_j, x_j) \in \mathcal{P}_\varepsilon([0, s_j])$ for $j \in J$. Set $s = \sup_{j\in J} s_j$, if $s = s_{j_0}, j_0 \in J$ then $\mathcal{T}$ is bounded below by (θ_{j_0}, x_{j_0}). Suppose now $s > s_j$ $\forall j \in J$, and consider the functions $x : [0, s[\to \mathbf{R}^n,\ \theta : [0, s] \to [0, s]$ defined by

$$x(t) = x_j(t)\ ,\ \theta(t) = \theta_j(t)\ \forall t \in [0, s_j[\ j \in J$$

and $\theta(s) = s$. For any $j \in J$ and any $t, t' \in [0, s_j],\ t \leq t'$ we have

$$\|x_j(t') - x_j(t)\| \leq \int_t^{t'} \|\dot{u}(\tau) + f(\tau, x(\tau))\| d\tau \leq (M + m)(t' - t)$$

then

$$\|x(t') - x(t)\| \leq (M + m)(t' - t)\ \forall t, t' \in [0, s[, t \leq t' \tag{3.2}$$

so x admits a left limit ω at s, prolong then x on $[0, s]$ by $x(s) = \omega$. We have to show that $(\theta, x) \in \mathcal{P}_\varepsilon([0, s])$. Let $(s_{j_n})_{n\in\mathbf{N}}$ a non decreasing sequence converging to s and let $\dot{u} : [0, s] \to \mathbf{R}^n$ defined by

$$\dot{u}(t) = \begin{cases} \dot{u}_{j_n}(t) & \text{if } t \in [s_{j_{n-1}}, s_{j_n}[,\ n \in \mathbf{N}^* \\ 0 & \text{if } t = s \end{cases} \tag{3.3}$$

since $\mathcal{T}$ is a chain, we have

$$\dot{u}(t) = \dot{u}_{j_n}(t) \ \text{ a.e on } [0, s_{j_n}]$$

and then for every $j \in J$

$$\dot{u}(t) = \dot{u}_j(t) \ \text{ a.e on } [0, s_j] \tag{3.4}$$

It follows by the definition of x that for every $t \in [0, s[$

$$x(t) = x_0 + \int_0^t (\dot{u}(\tau) + f(\tau, x(\tau)))d\tau \tag{3.5}$$

with $\dot{u}(t) \in F(x(\theta(t)))$ a.e on $[0, s]$ and

$$\begin{aligned} V(x(s)) - V(x_0) &= \lim_{n\to\infty} [V(x(s_{j_n})) - V(x_0)] \\ &\geq \overline{lim}_{n\to\infty} [\int_0^{s_{j_n}} \langle \dot{u}_{j_n}(\tau), \dot{u}_{j_n}(\tau), \dot{u}_{j_n}(\tau) + f(\tau, x_{j_n}(\tau))\rangle d\tau - \varepsilon s_{j_n}] \\ &= \int_0^s \langle \dot{u}(\tau), \dot{u}(\tau) + f(\tau, x(\tau))\rangle d\tau - \varepsilon s \end{aligned}$$

So (θ, x) is an element of $\mathcal{P}_\varepsilon$. By Zorn lemma, $\mathcal{P}_\varepsilon$ admits a maximal element $(\theta_\varepsilon, x_\varepsilon) \in \mathcal{P}_\varepsilon([0, s_\varepsilon])$. Let us prove that $s_\varepsilon = T$. Suppose that $s_\varepsilon < T$, by the definition of the directional derivative, for a fixed element y_ε of $F(x_\varepsilon(s_\varepsilon))$, there exists $\delta > 0,\ \delta < \inf\{\varepsilon, T - s_\varepsilon\}$ such that

$$\frac{1}{\delta}[V(x_\varepsilon(s_\varepsilon) + \delta y_\varepsilon) - V(x_\varepsilon(s_\varepsilon))] \geq V'(x_\varepsilon(s_\varepsilon), y_\varepsilon) - \varepsilon \tag{3.6}$$

Put $s' = s_\varepsilon + \delta \in I$ and let define the functions $\gamma : [0, s'] \to [0, s'],\ y : [0, s'] \to \mathbf{R}^n$ by

$$\gamma(t) = \begin{cases} \theta_\varepsilon(t) & \text{if} \quad t \in [0, s_\varepsilon] \\ s_\varepsilon & \text{if} \quad t \in [s_\varepsilon, s'[\\ s' & \text{if} \quad t = s' \end{cases} \tag{3.7}$$

$$y(t)=\begin{cases} x_\varepsilon(t) & \text{if} \quad t\in[0,s_\varepsilon] \\ x_\varepsilon(s_\varepsilon)+(t-s_\varepsilon)(y_\varepsilon+f(s_\varepsilon,x_\varepsilon(s_\varepsilon))) & \text{if} \quad t\in]s_\varepsilon,s']\end{cases} \tag{3.8}$$

then $y(t)=x_0+\int_0^t \dot{y}(\tau)d\tau \ \forall t\in[0,s']$ with

$$\dot{y}(t)=\begin{cases} \dot{x}_\varepsilon(t) & \text{if } \ t\in[0,s_\varepsilon] \\ y_\varepsilon+f(s_\varepsilon,x_\varepsilon(s_\varepsilon)) & \text{if} \quad t\in]s_\varepsilon,s']\end{cases} \tag{3.9}$$

Since $\dot{x}_\varepsilon(t)=\dot{u}_\varepsilon(t)+f(t,x_\varepsilon(t))$ put

$$\dot{z}(t)=\begin{cases} \dot{u}_\varepsilon(t) & \text{if} \quad t\in[0,s_\varepsilon] \\ y_\varepsilon & \text{if} \quad t\in]s_\varepsilon,s']\end{cases} \tag{3.10}$$

and

$$\widetilde{f}(t,y(t))=\begin{cases} f(t,x_\varepsilon(t)) & \text{if} \quad t\in[0,s_\varepsilon] \\ f(s_\varepsilon,x_\varepsilon(s_\varepsilon)) & \text{if} \quad t\in]s_\varepsilon,s']\end{cases} \tag{3.11}$$

By construction, we have

$$\dot{z}(t)\in F(y(\gamma(t))) \ \text{ a.e on } [0,s'] \tag{3.12}$$

and

$$y(t)=x_0+\int_0^t(\dot{z}(\tau)+\widetilde{f}(\tau,y(\tau)))d\tau. \tag{3.13}$$

Moreover for every $t\in]s_\varepsilon,s']$

$$\begin{aligned} y(t)-x_0 &= \int_0^{s_\varepsilon}\dot{x}_\varepsilon(\tau)d\tau+[y_\varepsilon+f(s_\varepsilon,x_\varepsilon(s_\varepsilon))](t-s_\varepsilon) \\ &\le (M+m)t<r. \end{aligned}$$

By hypotheses, $y_\varepsilon\in\partial V(x_\varepsilon(s_\varepsilon))$ and by (3.6) we have

$$\begin{aligned} V(y(s'))-V(x_0) &= V(y(s'))-V(x_\varepsilon(s_\varepsilon))+V(x_\varepsilon(s_\varepsilon))-V(x_0) \\ &\ge (s'-s_\varepsilon)V'(x_\varepsilon(s_\varepsilon),y_\varepsilon+f(s_\varepsilon,x_\varepsilon(s_\varepsilon)))-\varepsilon(s'-s_\varepsilon)+ \\ &+ \int_0^{s_\varepsilon}\langle\dot{u}_\varepsilon(\tau),\dot{u}_\varepsilon(\tau)+f(\tau,x_\varepsilon(\tau))\rangle d\tau \ - \ \varepsilon.s_\varepsilon \\ &\ge (s'-s_\varepsilon)\langle y_\varepsilon,y_\varepsilon+f(s_\varepsilon,x_\varepsilon(s_\varepsilon))\rangle+ \\ &+ \int_0^{s_\varepsilon}\langle\dot{u}_\varepsilon(\tau),\dot{u}_\varepsilon(\tau)+f(\tau,x_\varepsilon(\tau))\rangle d\tau \ -\varepsilon(s'-s_\varepsilon+s_\varepsilon) \\ &= \int_0^{s'}\langle\dot{z}(\tau)+\widetilde{f}(\tau,y(\tau))\rangle d\tau \ - \ \varepsilon s'. \end{aligned}$$

Then $(\gamma,y)\in\mathcal{P}_\varepsilon([0,s'])$ with $(\theta_\varepsilon,x_\varepsilon)\le(\gamma,y)$ and $(\theta_\varepsilon,x_\varepsilon)\ne(\gamma,y)$ which contradicts the maximality property of $(\theta_\varepsilon,x_\varepsilon)$. We deduce that $s_\varepsilon=T$, ie. x_ε is an ε-approximate solution of (1.1) defined on $I=[0,T]$. □

Now, we are able to prove our first result

Theorem 3.3. *Under the assumptions (H1)-(H3), for every $x_0\in\mathbf{R}^n$ there exists $T>0$ and an absolutely continuous function $x:[0,T]\to\mathbf{R}^n$ solution of (1.1).*

Proof. Let $(\epsilon_n)_{n\in\mathbf{N}}$ be a positive sequence converging to 0. By Lemma 3.2, let for any ϵ_n, (θ_n, x_n) a corresponding solution ie. x_n an ϵ_n-solution of (1.1) and θ_n the element of $I_{\epsilon_n}(I)$ corresponding to x_n. By construction of the approximate functions, we have for $s = T$

$$x_n(t) \in B(x_0, r) \quad \forall t \in I := [0, T] \tag{3.14}$$

$$\dot{u}_n(t) \in F(x_n(\theta_n(t))) \ a.e \ on \ [0, T] \tag{3.15}$$

and

$$|\theta_n(t) - t| \leq \epsilon_n \quad \forall t \ \in [0, T] \tag{3.16}$$

$(\dot{u}_n)$ is relatively $\sigma(L^\infty(I, \mathbb{R}^n), L^1(I, \mathbb{R}^n))$compact since by (3.15) and (3.1) we have

$$\dot{u}_n(t) \in M\overline{B}(0, 1) \tag{3.17}$$

Moreover (x_n) is equi-continuous since

$$\|x_n(t) - x_n(s)\| \leq \int_s^t \|\dot{u}_n(\tau) + f(\tau, x_n(\tau))\| d\tau \leq (M + m)|t - s| \tag{3.18}$$

By (3.14) and (3.18), (x_n) is relatively compact in $\mathcal{C}(I, \mathbf{R}^n)$. Consequently, there exists $x \in \mathcal{C}(I, \mathbb{R}^n), \dot{u} \in L^\infty(I, \mu; \mathbb{R}^n)$ and there exists subsequences, still denoted by (x_n) and $(\dot{x}_n)$ such that x_n converges uniformly to x and $\dot{u}_n$ converges $\sigma(L^\infty, L^1)$ to $\dot{u}$. We have also $f(., x_n(.)) \to f(., x(.))$ for the norm of $L^1([0, T], \mathbb{R}^n)$ Moreover, for any $t \in I$, one has

$$\begin{aligned} x(t) = \lim_{n\to\infty} x_n(t) &= \lim_{n\to\infty} x_0 + \int_0^t (\dot{u}_n(\tau) + f(\tau, x_n(\tau)))d\tau \\ &= x_0 + \int_0^t (\dot{u}(\tau) + f(\tau, x(\tau)))d\tau \end{aligned}$$

Further, by (3.16), $\theta_n(t) \to t$ uniformly on $[0, T]$ and thus $x_n(\theta_n(t)) \to x(t)$ uniformly on $[0, T]$. Now, by (3.15), (3.17) and since ∂V is upper semicontinuous with convex weakly compact values, we obtain

$$\dot{u}(t) \in \bigcap \overline{co}\{\dot{u}_k(t), \ k \geq n\} \subset \bigcap_{n\in\mathbb{N}} \overline{co} \bigcup_{k\geq n} \partial V(x_n(\theta_n(t))) \subset \partial V(x(t))$$

a.e. on $[0, T]$. ie.

$$\dot{u}(t) \in \partial V(x(t)) \tag{3.19}$$

Applying Proposition 2.2, $V \circ x$ is of bounded variation on I, the measure $D(V \circ x)$ is absolutely continuous with regard to the Lebesgue measure on I and a.e. on I

$$\langle \partial V(x(t)), \dot{x}(t)\rangle = \{\frac{D(V \circ x)}{dt}(t)\}$$

By (3.19) we have

$$\frac{D(V \circ x)}{dt}(t) = \langle \dot{u}(t)), \dot{x}(t)\rangle = \langle \dot{u}(t)), \dot{u}(t) + f(t, x(t))\rangle \tag{3.20}$$

By the construction of the x_n, one has

$$V(x_n(T)) - V(x_0) \geq \int_0^T \langle \dot{u}_n(\tau), \dot{u}_n(\tau) + f(\tau, x_n(\tau))\rangle d\tau - \epsilon_n.T$$

Then

$$\limsup_{n\to\infty}\int_0^T \langle \dot{u}_n(\tau),\dot{u}_n(\tau)+f(\tau,x_n(\tau))\rangle d\tau \leq V(x(T))-V(x_0)=\int_0^T \frac{D(V\circ x)}{dt}(\tau)d\tau$$
$$= \int_0^T \langle \dot{u}(\tau),\dot{u}(\tau)+f(\tau,x(\tau))\rangle d\tau$$

Since $(\dot{u}_n)$ converges weakly in $L^2_{\mathbb{R}^n}(I,dt)$ to $\dot{u}$, we get

$$Lim_{n\to\infty}\int_0^T \|\dot{u}_n(\tau)\|^2 d\tau=\int_0^T \|\dot{u}(\tau)\|^2 d\tau \tag{3.21}$$

Thus, $(\dot{u}_n)$ converges in norm in $L^2_{\mathbb{R}^n}(I,dt)$ to $\dot{u}$. By extracting a subsequence, still denoted $\dot{u}_n$, one can suppose that

$$\dot{u}_n(t)\to\dot{u}(t) \quad a.e\ on\ I \tag{3.22}$$

By (H_1), the graph of F is closed in $\mathbb{R}^n\times\mathbb{R}^n$, so we can conclude that

$$(x(t),\dot{u}(t))=lim_{n\to\infty}(x_n(\theta_n(t)),\dot{u}_n(t)))\in gr(F)$$

ie. $\dot{x}(t)-f(t,x(t))=\dot{u}(t)\in F(x(t))$ *a.e on* I. □

The study of the same problem in the infinite dimensional setting presents some difficulties, essentially the passage from the weak convergence of the derivatives of approximate solutions to the strong convergence. To overcome this fact, the authors in [4] replace the condition (H_2) by :

$F(x)\subset A(\partial V(x))\ \ \forall x\in E$ with V convex and continuous, $A:E'\to E$ a linear operator satisfying the conditions

(a) the restriction of A to any bounded subset B in E' is continuous for the weak topology $\sigma(E',E)$ and the norm topology in E.

(b) $\forall x\in E'\backslash\{0\},\ \langle x,\gamma(x)\rangle>0$.

In what follows, we present the same result but with more natural and weaker hypotheses, namely we consider a separable Hilbert space H, $F:H\to 2^H$ such that

$(H_2^{'})$ $F(x)\subset\partial V(x),\ F(x)\subset(1+\|x\|)K$ with V regular, K a convex compact in H

and $f:\mathbb{R}^+\times H\to H$ verifying

$(H_3^{'})$ f is Lipschitz for the second variable and for any bounded subset B of $\mathbb{H}$, there is a compact set K_1 such that $f(t,x)\in K_1$ for all $(t,x)\in\mathbb{R}^+\times B$.

Theorem 3.4. *Let F and f be such that (H_1), $(H_2^{'})$ and $(H_3^{'})$ are satisfied, then for every $x_0\in H$, there exists $T>0$ and an absolutely continuous function $x:[0,T]\to H$ solution of problem (1.1).*

Proof. Let $r>0$ be such that V is $L-$Lipschitz on $\bar{\mathbb{B}}(x_0,r)$. Then we have that $\partial^C V(y)\subset L\bar{\mathbb{B}}$, whenever $x\in\bar{\mathbb{B}}(x_0,r)$. By our assumption $(H_3^{'})$, there exists a positive constant m_1 such that $f(t,x)\in K_1\subset m_1\mathbb{B}$ for all $(t,x)\in\mathbb{R}^+\times\bar{\mathbb{B}}(x_0,r)$.

Moreover, by (H_2') there exists a positive constant m such that for any $x \in \bar{\mathbb{B}}(x_0, r),\ F(x) \subset (1 + \|x_0\| + r)K \subset m\mathbb{B}$. Choose T such that

$$0 < T < \frac{r}{m_1 + m}$$

Set $I := [0, T]$. For each integer $n \geq 1$ and for $1 \leq i \leq n-1$ we set $t_i^n := \frac{iT}{n}$, and let define the following approximate sequences

$$x_n(t) = x_n(t_i^n) + \int_{t_i^n}^{t} [f(s, x_n(t_i^n)) + u_i^n] ds$$

whenever $t \in]t_i^n, t_{i+1}^n],\ 0 \leq i \leq n-1$, with $x_n(0) = x_0,\ u_i^n \in F(x_n(t_i^n))$.
Now let us define the step functions from $[0, T]$ to $[0, T]$ by

$$\theta_n(t) = t_i^n,\ u_n(t) = u_i^n\ \forall t \in [t_i^n, t_{i+1}^n[,\ 1 \leq i \leq n-1.$$

the last interval, for $i = n-1$, is taken closed ($[t_{n-1}^n, T]$). Then, $\forall n \in \mathbb{N}^*, \forall t \in [0, T]$, we have the following properties

$$0 \leq t - \theta_n(t) \leq \frac{T}{n} \tag{3.23}$$

$$x_n(t) = x_0 + \int_0^t [f(s, x_n(\theta_n(s))) + u_n(s)] ds \tag{3.24}$$

$$u_n(t) \in F(x_n(\theta_n(t))) \tag{3.25}$$

Observe that $x_n(t) \in \bar{\mathbb{B}}(x_0, r)$ for all $n \in \mathbb{N}^*$and all $t \in [0, T]$. Indeed it is obvious that

$$\|x_n(t) - x_0\| = \| \int_0^t [f(s, x_n(\theta_n(s))) + u_n(s)] ds\| \leq (m_1 + m)T < r$$

then

$$\|x_n(t) - x_n(t')\| \leq (m_1 + m)|t' - t|$$

whenever $0 \leq t \leq t' \leq T$ and $n \in \mathbb{N}^*$. Hence $(x_n)_{n\in\mathbb{N}^*}$ is an equi-Lipschitz subset of $C([0, T], \mathbb{H})$. The set $\{x_n(t) : n \in \mathbb{N}^*\}$ is relatively compact in $\mathbb{H}$ for every $t \in [0, T]$; indeed we have for all $n \in \mathbb{N}^*$ and all $t \in [0, T]$

$$x_n(t) \in x_0 + T\{K_1 + (1 + \|x_0\| + r)K\} := K_2$$

which is compact. Then by Ascoli's theorem, $(x_n)_{n\in\mathbb{N}^*}$ is relatively compact in the Banach space $C([0, T], \mathbb{H})$. Further, the sequence $(u_n)_{n\in\mathbb{N}}$ is relatively $\sigma(L^1([0, T], \mathbb{H}); L^\infty([0, T], \mathbb{H}))$-compact since we have a.e.

$$\forall n \in \mathbb{N}^*,\ u_n(t) \in (1 + \|x_0\| + r)K$$

Therefore by extracting subsequences if necessary, we can suppose that there exists $x \in C([0, T], \mathbb{H}),\quad u \in L^1([0, T], \mathbb{H})$ such that $x_n \to x$ in $C([0, T], \mathbb{H}), u_n \to u$ for $\sigma(L^1([0, T], \mathbb{H}); L^\infty([0, T], \mathbb{H}))$-topology.
We have also $f(., x_n(\theta_n(.))) \to f(., x(.))$ for the norm of $L^1([0, T], \mathbb{H})$. Consequently, one has for all $t \in [0, T]$

$$x(t) = x_0 + \lim_{n\to\infty} \int_0^t [f(s, x_n(\theta_n(s))) + u_n(s)] ds = x_0 + \int_0^t [f(s, x(s)) + u(s)] ds$$

which gives the equality

$$\dot{x}(t) = f(s, x(s)) + u(s) \text{ for almost every } t \in [0, T]. \tag{3.26}$$

Now we claim that $\dot{x}_n$ converges strongly to x. Since u_n converges weakly to u, $\dot{x}_n$ converges weakly to x. By construction, we have for a.e $t \in [0, T]$

$$\dot{x}_n(t) - f(t, x_n(\theta_n(t))) = u_n(t) \in F(x_n(\theta_n(t)))$$

and by hypothesys $(H_2^{'})$,

$$\begin{aligned} u_n(t) \in F(x_n(\theta_n(t))) &\subset (1 + \|x_n(\theta_n(t))\|)K \\ &\subset (2 + \|x(t)\|)K \end{aligned}$$

Then u_n is in the fixed compact set $(2+\|x(t)\|)K$, consequently it converges strongly to u which gives the strong convergence of $\dot{x}_n$. Since the graph of F is closed, we get

$$\dot{x}(t) \in F(x(t)) + f(t, x(t)) \text{ a.e. on } [0, T].$$

Therefore, the differential inclusion (1.1) admits a solution. □

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