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Marachkov type stability conditions for non-autonomous functional differential equations with unbounded right-hand sides

Dedicated to T. A. Burton on his 80th birthday

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Abstract. Sufficient conditions for uniform equi-asymptotic stability and uniform asymptotic stability of the zero solution of the retarded equation

$$x'(t) = f(t, x_t), \qquad (x_t(s) := x(t+s),\ -h \le s \le 0)$$

are given. In the stability theory of non-autonomous differential equations a result is of Marachkov type if it contains some kind of boundedness or growth condition on the right-hand side of the equation with respect to t. Using Lyapunov's direct method and the annulus argument we prove theorems for equations whose right-hand sides may be unbounded with respect to t. The derivative of the Lyapunov function is not supposed to be negative definite, it may be negative semi-definite. The results are applied to the retarded scalar differential equation with distributed delay

$$x'(t) = -a(t)x(t) + b(t)\int_{t-h}^{t} x(s)\,\mathrm{d}s, \qquad (a(t) > 0),$$

where a and b may be unbounded on $[0, \infty)$. The growth conditions do not concern function a, they contain only function b. In addition, the function $t \mapsto a(t) - \int_t^{t+h} |b(u)|\,\mathrm{d}u$, measuring the dominance of the negative instantaneous feedback over the delayed feedback, is not supposed to remain above a positive constant, even it may vanish on long intervals.

Keywords: Lyapunov functional with negative semi-definite derivative; annulus argument; uniform asymptotic stability; uniform equi-asymptotic stability.

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1 Introduction

We consider the system

$$x'(t) = f(t, x_t), \qquad (\cdot)' = \frac{\mathrm{d}}{\mathrm{d}t}(\cdot), \tag{1.1}$$

where $f\colon \mathbb{R}_+ \times C_H \to \mathbb{R}^m$ is continuous and takes bounded sets into bounded sets and $f(t,0) \equiv 0$; $\mathbb{R}_+ := [0,\infty)$, C is the Banach space of continuous functions $\varphi\colon [-h,0] \to \mathbb{R}^m$ with the maximum norm $\|\varphi\| := \max_{-h\le s\le 0} |\varphi(s)|$, $|\cdot|$ denotes an arbitrary norm in $\mathbb{R}^m$, h is a nonnegative constant, C_H is the open ball of radius H in C around $\varphi = 0$. As is usual, if $x\colon [-h,\beta) \to \mathbb{R}^m$ $(\beta > 0)$, then $x_t(s) := x(t+s)$ for $-h \le s \le 0$, $0 \le t < \beta$. Let $x(\cdot;t_0,\varphi)\colon [t_0-h, t_0+\alpha) \to \mathbb{R}^m$ denote a solution of (1.1) satisfying the initial condition $x_{t_0}(\cdot;t_0,\varphi) = \varphi$. It is known [5] that for each $t_0 \in \mathbb{R}_+$ and each $\varphi \in C$ there is at least one solution $x(\cdot;t_0,\varphi)\colon [t_0-h, t_0+\alpha) \to \mathbb{R}^m$ with some $\alpha > 0$, and if this solution remains bounded on every bounded subinterval of $[t_0, t_0+\alpha)$, then $\alpha = \infty$.

We will use Lyapunov's direct method [2, 5]. A continuous functional $V\colon \mathbb{R}_+ \times C \to \mathbb{R}_+$ which is locally Lipschitz in φ is called a *Lyapunov functional* if its right-hand side derivative with respect to system (1.1) is non-positive:

$$V'_{(1.1)}(t,\varphi) = V'(t,\varphi) := \limsup_{\delta\to 0+0} \left(\frac{1}{\delta} \left(V(t+\delta, x_{t+\delta}(\cdot,t,\varphi)) - V(t,\varphi) \right) \right) \le 0.$$

A Lyapunov functional is called *positive definite* if there exists a *wedge* (i.e., a continuous, strictly increasing function $W\colon \mathbb{R}_+ \to \mathbb{R}_+$ with $W(0) = 0$) such that

$$V(t,\varphi) \ge W(|\varphi(0)|) \qquad (V(t,0) \equiv 0).$$

The following stability concepts are standard [2, 5].

Definition 1.1. The zero solution of (1.1) is:

(a) *stable* if for every $\varepsilon > 0$ and $t_0 \ge 0$ there is a $\delta(\varepsilon, t_0) > 0$ such that $\|\varphi\| < \delta$, $t \ge t_0$ imply that $|x(t;t_0,\varphi)| < \varepsilon$;

(b) *uniformly stable* if for every $\varepsilon > 0$ there is a $\delta(\varepsilon) > 0$ such that $\|\varphi\| < \delta, t_0 \ge 0, t \ge t_0$ imply that $|x(t;t_0,\varphi)| < \varepsilon$;

(c) *asymptotically stable* if it is stable and for every $t_0 \ge 0$ there is a $\sigma(t_0) > 0$ such that $\|\varphi\| < \sigma$ implies $\lim_{t\to\infty} x(t;t_0,\varphi) = 0$;

(d) *uniformly equi-asymptotically stable* (UEAS) if it is uniformly stable and there is a $D > 0$ and for each $\mu > 0$, $t_0 \ge 0$ there is a $T(\mu, t_0)$ such that $\|\varphi\| < D$, $t \ge t_0 + T$ imply that $|x(t;t_0,\varphi| < \mu$;

(e) *uniformly asymptotically stable* (UAS) if it is uniformly stable and there is a $D > 0$ and for each $\mu > 0$ there is a $T(\mu)$ such that $t_0 \in \mathbb{R}_+$, $\|\varphi\| < D$, $t \ge t_0 + T$ imply that $|x(t;t_0,\varphi)| < \mu$.

In stability theory of non-autonomous differential equations a result is of Marachkov's type if it contains some kind of boundedness or growth condition on the right-hand side of the equation with respect of t [9]. One of the most classical results in stability theory of functional differential equations is the following theorem.

Theorem A ([2, 5])**.** *Suppose that there are a Lyapunov functional V, wedges W_1, W_2, and constants $M, H > 0$ such that the following conditions are satisfied:*

(i) $W_1(|\varphi(0)|) \le V(t,\varphi)$,

(ii) $V'(t,\varphi) \le -W_2(|\varphi(0)|)$,

(iii) $|f(t,\varphi)| \le M$, *provided* $\|\varphi\| \le H$.

Then the zero solution of (1.1) *is asymptotically stable.*

Condition (iii) is very restrictive, it often raises difficulties in applications of the theorem. T. A. Burton and G. Makay [4] have taken an important step to overcome these difficulties.

Theorem B (T. A. Burton and G. Makay)**.** *Suppose there are $H > 0$, $V\colon \mathbb{R}_+ \times C_H \to \mathbb{R}_+$, wedges W_1, W_2, W_3, and a continuous increasing function $F\colon \mathbb{R}_+ \to [1,\infty)$ such that*

(i) $W_1(|\varphi(0)|) \le V(t,\varphi) \le W_2(\|\varphi\|)$,

(ii) $V'(t,\varphi) \le -W_3(|\varphi(0)|)$,

(iii) $|f(t,\varphi)| \le F(t)$ *on* $\mathbb{R}_+ \times C_H$,

(iv) $\int_1^\infty (1/F(t))\,\mathrm{d}t = \infty$.

Then the zero solution of (1.1) *is uniformly equi-asymptotically stable.*

Throughout this paper we will illustrate abstract results with applications to the retarded scalar differential equation with distributed delay

$$x'(t) = -a(t)x(t) + b(t)\int_{t-h}^{t} x(s)\,\mathrm{d}s, \tag{1.2}$$

where $a, b\colon \mathbb{R}_+ \to \mathbb{R}$ are continuous and $a(t) \ge 0$ $(t \in \mathbb{R}_+)$. This is an important model equation: it describes a process in which there are an instantaneous and a delayed feedback. Define the Lyapunov functional

$$\begin{aligned} V(t,\varphi) &:= |\varphi(0)| + \int_{-h}^{0}\int_{s}^{0} |b(t+\tau-s)||\varphi(\tau)|\,\mathrm{d}\tau\,\mathrm{d}s \\ &= |\varphi(0)| + \int_{-h}^{0} |\varphi(\tau)| \left(\int_{-h}^{\tau} |b(t+\tau-s)|\,\mathrm{d}s\right)\mathrm{d}\tau \\ &\le |\varphi(0)| + \left(\int_{t}^{t+h} |b(u)|\,\mathrm{d}u\right)\int_{-h}^{0}|\varphi(s)|\,\mathrm{d}s. \end{aligned} \tag{1.3}$$

If x is a solution of (1.2), then

$$V'(t,x_t) = \frac{\mathrm{d}}{\mathrm{d}t}V(t,x_t) = \frac{\mathrm{d}}{\mathrm{d}t}\left(|x(t)| + \int_{-h}^{0}\int_{t+s}^{t} |b(u-s)||x(u)|\,\mathrm{d}u\,\mathrm{d}s\right).$$

It can be seen that

$$V'(t,x_t) \le -\left(a(t) - \int_{t}^{t+h}|b(u)|\,\mathrm{d}u\right)|x(t)|,$$

therefore

$$\begin{aligned} &V'(t,\varphi) \le -\eta(t)W_3(|\varphi(0)|); \\ &\eta(t) := a(t) - \int_{t}^{t+h}|b(u)|\,\mathrm{d}u, \qquad W_3(r) := r, \end{aligned} \tag{1.4}$$

so we can apply the Burton–Makay Theorem B to equation (1.2).

Corollary C. *Suppose that there are constants $c_1, c_2 > 0$ such that*

(i) $\int_t^{t+h} |b(s)| \, \mathrm{d}s \le c_1$,

(ii) $a(t) - \int_t^{t+h} |b(s)| \, \mathrm{d}s \ge c_2 > 0$ *for all* $t \in \mathbb{R}_+$,

and

(iii) $$\int_1^\infty \frac{\mathrm{d}s}{\max_{0 \le s \le t}\{a(s) + h|b(s)|\}} = \infty.$$

Then the zero solution of (1.2) *is uniformly equi-asymptotically stable.*

If a, b are constants, i.e., $a(t) \equiv a_0 > 0$, $b(t) \equiv b_0$, then Corollary C says that the zero solution of (1.2) is UEAS, provided that $a_0 > h|b_0|$. In other words, the dominance of the negative instantaneous feedback over the delayed one suffices UEAS. Conditions (i) and (ii) are in accordance with this experience in the case of the more general nonautonomous equation (1.2), but condition (iii) contradicts "the larger $a(t)$ is the better" principle. The following problem arises: is the zero solution UEAS if $a(t)$ is large enough and $|b(t)|$ is bounded (e.g., $a(t) = t^2$, $b(t) \equiv \sin t$), which is excluded by the growth condition (iii)? We can also ask a question regarding condition (ii) in Theorem B (and in Corollary C). One can expect that the dominance of a over b is not necessarily as uniform as condition (ii) requires. For example, can the zero solution of (1.2) be UEAS if η vanishes on intervals of the same length infinitely many times in $\mathbb{R}_+$?

In this paper we develop further Theorem B essentially weakening both conditions (ii) and (iii). For example, the corollary of the main result for equation (1.2) will imply that the answers to both of the questions above are affirmative.

The paper is organized as follows. Section 2 contains the main theorem and its corollaries. Section 3 is the proof of the main Theorem 3.1 based upon an annulus argument [6]. In Section 4 we formulate some applications to equation (1.2).

2 Main results

To weaken the uniformity of conditions (ii) in Theorem B and Corollary C we need the following concepts, which have played an important role in the stability theory of non-autonomous systems [7, 3, 8] for a long time.

Definition 2.1. A locally integrable function $\eta \colon \mathbb{R}_+ \to \mathbb{R}_+$ is called

(a) *integrally positive* (IP) if for every $\delta > 0$ the inequality

$$\liminf_{t\to\infty} \int_t^{t+\delta} \eta(u) \, \mathrm{d}u > 0 \tag{2.1}$$

holds.

(b) *weakly integrally positive* (WIP) if for any sequences $\{t_i'\}_{i=1}^\infty$, $\{t_i''\}_{i=1}^\infty$ satisfying conditions

$$t_i' + \delta \le t_i'' < t_{i+1}' \le t_i'' + \Delta \qquad (i = 1, 2, \dots) \tag{2.2}$$

with some $\delta > 0$, $\Delta > 0$, we have

$$\sum_{i=1}^\infty \int_{t_i'}^{t_i''} \eta(t) \, \mathrm{d}t = \infty. \tag{2.3}$$

For example, $t \mapsto |\cos t| - \cos^2 t$ is IP; $t \mapsto |\cos t| - \cos t$ is WIP but it is not IP.

To control the growth of an integral function we introduce a notation. For a locally integrable function $M\colon \mathbb{R}_+ \to \mathbb{R}_+$ and numbers $t \in \mathbb{R}_+$, $\varepsilon > 0$ define

$$\Gamma_M(t,\varepsilon) := \sup\left\{\tau > 0 : \int_{t-\tau}^{t} M(u)\,\mathrm{d}u \le \varepsilon\right\} \tag{2.4}$$

(see [6]).

Theorem 2.2. *Suppose that for system (1.1) there are a Lyapunov function $V\colon \mathbb{R}_+ \times C \to \mathbb{R}_+$ and wedges W_1, W_2, W_3 such that the following conditions are satisfied:*

(i) $W_1(|\varphi(0)|) \le V(t,\varphi) \le W_2(\|\varphi\|)$;

(ii) there is a locally integrable function $\eta\colon \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$V'(t,\varphi) \le -\eta(t)W_3(|\varphi(0)|);$$

(iii) $\lim_{T\to\infty} \int_t^{t+T} \eta(u)\,\mathrm{d}u = \infty$ *uniformly with respect to $t \in \mathbb{R}_+$;*

(iv) there are $H \in \mathbb{R}_+$ and a locally integrable function $G\colon \mathbb{R}_+ \to \mathbb{R}_+$ such that if a solution is bounded by H, then $|x(t)|' \le G(t)$ $(t \in \mathbb{R}_+)$; in addition, for every $\varepsilon > 0$ there is L such that for every $\{t_i\}_{i=1}^{\infty}$ satisfying the inequalities

$$t_i > ih, \qquad t_{i+1} < t_i + L \tag{2.5}$$

we have

$$\sum_{i=1}^{\infty} \int_{t_i - \Gamma_G(t_i,\varepsilon)}^{t_i} \eta(t)\,\mathrm{d}t = \infty. \tag{2.6}$$

Then the zero solution of (1.1) is uniformly equi-asymptotically stable.

The Burton–Makay theorem is a special case of Theorem 2.2.

Proposition 2.3. *Theorem B is a corollary of Theorem 2.2.*

Proof. Suppose that conditions (i)–(iv) in Theorem B are fulfilled, and set $\eta(t) \equiv 1$. Then conditions (ii) and (iii) in Theorem 2.2 are satisfied. We show that condition (iv) is also satisfied.

Since $|x(t)|' \le |x'(t)| \le |f(t,x_t)| \le F(t)$, we can choose $G(t) := F(t)$. This function is increasing, therefore $\Gamma_G(t,\varepsilon) > \varepsilon/G(t)$ $(t \in \mathbb{R}_+,\ \varepsilon > 0)$, and for every $\varepsilon > 0$, $\{t_i\}_{i=1}^{\infty}$ with property (2.5) we have

$$\sum_{i=1}^{\infty} \int_{t_i-\Gamma_G(t_i,\varepsilon)}^{t_i} 1\,\mathrm{d}t \ge \varepsilon \sum_{i=1}^{\infty} \frac{1}{G(t_i)} \ge \frac{\varepsilon}{L} \sum_{i=1}^{\infty} \int_{t_i}^{t_{i+1}} \frac{\mathrm{d}t}{G(t)} = \infty,$$

i.e., (2.6) is satisfied. □

Condition (iv) in Theorem 2.2 has a simple form also in the case, when the integral function of G is uniformly continuous.

Corollary 2.4. *Assume that conditions (i)–(iii) in Theorem 2.2 are satisfied. Suppose, in addition, that*

(iv′) function G in condition (iv) of Theorem 2.2 has the additional property that $t \mapsto \int_0^t G(u)\,\mathrm{d}u$ is uniformly continuous and, instead of (2.5)–(2.6), function η is weakly integrally positive.

Then the zero solution of (1.1) is uniformly equi-asymptotically stable.

Proof. We have to prove that (iv) in Theorem 2.2 is satisfied. $\int_0^t G$ is uniformly continuous, so for every $\varepsilon > 0$ there is a $\delta(\varepsilon) > 0$ such that $t - \delta(\varepsilon) < r < t$ implies $\int_r^t G < \varepsilon$. Therefore $\Gamma_G(t,\varepsilon) \geq \delta(\varepsilon)$. If $\varepsilon > 0$, $\{t_i\}_{i=1}^{\infty}$ are arbitrary with

$$t_i + \delta(\varepsilon) < t_{i+1} < t_i + L \qquad (i = 1, 2, \dots),$$

then

$$\sum_{i=1}^{\infty} \int_{t_i - \Gamma_G(t_i,\varepsilon)}^{t_i} \eta(t)\,\mathrm{d}t \geq \sum_{i=1}^{\infty} \int_{t_i - \delta(\varepsilon)}^{t_i} \eta(t)\,\mathrm{d}t = \infty$$

because η is weakly integrally positive. □

Burton and Makay gave a sophisticated counterexample showing that it is impossible to strengthen the conclusion of Theorem B to uniform asymptotic stability. The following theorem says that if we strengthen condition (iv$'$) in Corollary 2.4 to "integral positivity", then we get UAS. Therefore, if F is bounded in Theorem B, then the conclusion of the theorem can already be strengthened to UAS, and we get Theorem 5.2.1 in [5]. So the following theorem can be considered as a generalization of this theorem.

Theorem 2.5. *Assume that conditions (i)–(ii) in Corollary 2.4 (i.e., conditions (i)–(ii) in Theorem 2.2) are satisfied. Suppose, in addition, that*

(iv$''$) function η in condition (iv$'$) is not only weakly integrally positive but integrally positive.

Then the zero solution of (1.1) is uniformly asymptotically stable.

3 Proofs of the theorems

The proof of Theorem 2.2 is based upon the annulus argument [1, 6]. This is a method of the proof for the existence of a limit, which can detect that a trajectory $x\colon \mathbb{R}_+ \to \mathbb{R}^m$ crosses the annulus $\varepsilon_1 \leq |x| \leq \varepsilon_2$ infinitely many times.

3.1. Proof of Theorem 2.2

Suppose that conditions (i)–(iv) in Theorem 2.2 are satisfied. (i) and (ii) guarantee uniform stability for the zero solution of (1.1) [2, 5]; take $\delta(\varepsilon)$ from the definition of this property. Define $D := \delta(H)$, where H is from condition (iv). We always suppose throughout this proof that initial functions φ satisfy $\|\varphi\| < D$, i.e., we have $|x(t)| < H$ for all solutions x and for all t_0, t with $t_0 \leq t$. Since $t \mapsto v(t) := V(t, x_t)$ is nonincreasing, we also have

$$v(t) \leq v(t_0) \leq V(t_0, x_{t_0}) \leq W_2(D) \qquad (t \geq t_0).$$

In the first step we prove that (ii) and (iii) imply the following property: for every $\varepsilon > 0$ there exists a $\Delta(\varepsilon) > 0$ such that if a solution x satisfies $|x(u)| \geq \varepsilon$ on $[t, t+T]$, then $T \leq \Delta(\varepsilon)$. In fact, define $\Delta(\varepsilon) > 0$ so large that

$$\int_t^{t+\Delta(\varepsilon)} \eta(u)\,\mathrm{d}u > \frac{W_2(D)}{W_3(\varepsilon)} \qquad (t \in \mathbb{R}_+)$$

(the existence of such $\Delta(\varepsilon)$ is a consequence of (iii)). If $|x(u)| \geq \varepsilon$ on $[t, t+T]$, then

$$\begin{aligned} 0 \leq v(t+T) &\leq v(t) - W_3(\varepsilon) \int_t^{t+T} \eta(u)\,\mathrm{d}u \\ &\leq W_2(D) - W_3(\varepsilon) \int_t^{t+T} \eta(u)\,\mathrm{d}u, \end{aligned} \tag{3.1}$$

i.e.,

$$\int_t^{t+T} \eta(u)\,\mathrm{d}u \le \frac{W_2(D)}{W_3(\varepsilon)}.$$

If $T > \Delta(\varepsilon)$ were possible, then by the choice of $\Delta(\varepsilon)$ the reversed strict inequality would also hold, what is impossible.

To prove UEAS we have to show the existence of $T(\mu, t_0)$ in the definition of the property. Thanks to the US, it is enough to guarantee the existence of $\widetilde{T}(\mu, t_0)$ such that for every φ with $\|\varphi\| < D$ there is a $t \in [t_0, t_0 + \widetilde{T}(\mu, t_0)]$ such that $\|x_t(\cdot; t_0, \varphi)\| < \delta(\mu)$. Suppose the contrary, i.e., there are $\overline{\mu}, \bar{t}_0$ such that for each $T > 0$ there exists $\varphi = \varphi(\cdot; T)$ with $\|\varphi\| < D$ such that

$$\|x_t(\cdot; \bar{t}_0, \varphi)\| \ge \delta(\overline{\mu}) =: 3\varepsilon \qquad (\bar{t}_0 \le t \le \bar{t}_0 + T).$$

Let us fix T arbitrarily, take a corresponding φ and denote $x(t) = x(t; \bar{t}_0, \varphi)$. Then there are sequences $\{t'_i\}_{i=N_0}^{N_0+N(T)}$, $\{t''_i\}_{i=N_0}^{N_0+N(T)}$ such that

$$\begin{aligned} & i\,(\Delta(\varepsilon) + h) \le t'_i < t''_i \le (i+1)\,(\Delta(\varepsilon) + h)\,, \\ & |x(t'_i)| = \varepsilon, \quad |x(t''_i)| = 3\varepsilon; \quad \varepsilon \le |x(t)| \le 3\varepsilon \quad \text{for } t \in [t'_i, t''_i]. \end{aligned} \tag{3.2}$$

Here N_0 and $N(T)$ are defined by

$$N_0 := \left[\frac{\bar{t}_0}{\Delta(\varepsilon) + h}\right] + 1, \qquad N(T) := \left[\frac{T}{\Delta(\varepsilon) + h}\right] - 2, \tag{3.3}$$

where $[\alpha]$ denotes the integer part of a real number α. That is, N_0 is independent of T, but $N(T)$ does depend on T and $\lim_{T\to\infty} N(T) = \infty$.

Let us observe that

$$\varepsilon < 2\varepsilon = \int_{t'_i}^{t''_i} |x(t)|'\,\mathrm{d}t \le \int_{t'_i}^{t''_i} G(t)\,\mathrm{d}t,$$

consequently, $t'_i < t''_i - \Gamma_G(t''_i, \varepsilon)$.

For the function $t \mapsto v(t) := V(t, x_t(\cdot; \bar{t}_0, \varphi))$ we have

$$\begin{aligned} 0 \le v(t''_{N_0+N(T)}) &\le v(\bar{t}_0) - W_3(\varepsilon) \sum_{i=N_0}^{N_0+N(T)} \int_{t'_i}^{t''_i} \eta(t)\,\mathrm{d}t \\ &\le W_2(D) - W_3(\varepsilon) \sum_{i=N_0}^{N_0+N(T)} \int_{t''_i - \Gamma_G(t''_i,\varepsilon)}^{t''_i} \eta(t)\,\mathrm{d}t. \end{aligned} \tag{3.4}$$

To get a contradiction we want to apply condition (iv) taking $T \to \infty$. However, the problem is that the initial function φ and, consequently, sequences $\{t'_i\}$, $\{t''_i\}$ depend on T. We overcome this difficulty by the use of a universal sequence $\{t_i\}_{i=N_0}^{\infty}$. Since G is locally integrable, the integral of G is absolute continuous [10] and $\Gamma_G(u, \varepsilon)$ is continuous in u, so we can define $\{t_i\}$ by

$$\int_{t_i - \Gamma_G(t_i,\varepsilon)}^{t_i} \eta(t)\,\mathrm{d}t = \min\left\{\int_{u-\Gamma_G(u,\varepsilon)}^{u} \eta(t)\,\mathrm{d}t : i(\Delta + h) \le u \le (i+1)(\Delta + h)\right\}. \tag{3.5}$$

Then $t_i > ih$ and $t_{i+1} < t_i + 2(\Delta + h)$ for all $i \in \mathbb{N}$, so (2.5) is satisfied with $L := 2(\Delta + h)$, consequently condition (iv) implies

$$\sum_{i=1}^{\infty} \int_{t_i - \Gamma_G(t_i,\varepsilon)}^{t_i} \eta(t)\,\mathrm{d}t = \infty. \tag{3.6}$$

On the other hand, by the definition (3.5) of $\{t_i\}$, from (3.4) we obtain

$$0 \le v(t''_{N_0+N(T)}) \le v(\bar{t}_0) - W_3(\varepsilon) \sum_{i=N_0}^{N_0+N(T)} \int_{t_i-\Gamma_G(t_i,\varepsilon)}^{t_i} \eta(t)\,\mathrm{d}t$$

for all $T > 0$. Now we can already take the limit $T \to \infty$ and write

$$\sum_{i=N_0}^{\infty} \int_{t_i-\Gamma_G(t_i,\varepsilon)}^{t_i} \eta(t)\,\mathrm{d}t < \frac{v(\bar{t}_0)}{W_3(\varepsilon)} < \infty,$$

which contradicts (3.6).

3.2. Proof of Theorem 2.5

We will construct an upper bound for T of properties (3.2)–(3.3) independent of φ and also of $\bar{t}_0$.

Step 1. At first we prove: the integral positivity of η implies that condition (iii) in Theorem 2.2 is satisfied. In fact, for $\alpha > 0$ introduce the notation

$$\beta(\alpha) := \frac{1}{2} \liminf_{t\to\infty} \int_t^{t+\alpha} \eta(u)\,\mathrm{d}u > 0.$$

Then for every $\alpha > 0$ there exists an $L_1(\alpha)$ such that $t > L_1(\alpha)$ implies

$$\int_t^{t+\alpha} \eta(u)\,\mathrm{d}u > \beta(\alpha).$$

Given $K > 0$ arbitrarily, define $L(K) := L_1(1) + K/\beta(1)$. If $t \in \mathbb{R}_+$ and $T > L(K)$, then

$$\int_t^{t+T} \eta(u)\,\mathrm{d}u \ge \int_t^{t+L_1(1)} \eta(u)\,\mathrm{d}u + \int_{t+L_1(1)}^{t+L_1(1)+K/\beta(1)} \eta(u)\,\mathrm{d}u \ge \frac{K}{\beta(1)}\beta(1) = K,$$

which means that condition (iii) in Theorem 2.2 is satisfied.

Step 2. Let us estimate $\Gamma_G(t,\varepsilon)$. Since $\int_0^t G$ is uniformly continuous in $\mathbb{R}_+$, for every $\varepsilon > 0$ there is a $\kappa(\varepsilon)$ such that $0 \le t - s < \kappa(\varepsilon)$ implies $\int_s^t G < \varepsilon$. By the definition of $\Gamma_G(t,\varepsilon)$ this means that $\Gamma_G(t,\varepsilon) \ge \kappa(\varepsilon)$.

For arbitrary $\varepsilon > 0$ define the number

$$N_1 = N_1(\varepsilon) := \left[\frac{L_1(\kappa(\varepsilon))}{\Delta(\varepsilon) + h}\right] + 1,$$

where $L_1(\cdot)$ was defined in Step 1. Then $t \ge (\Delta(\varepsilon) + h)N_1(\varepsilon)$ implies

$$\int_{t-\kappa(\varepsilon)}^{t} \eta(s)\,\mathrm{d}s \ge \beta(\kappa(\varepsilon)). \tag{3.7}$$

Step 3. We prove the existence of $T = T(\mu)$ in the definition of UAS. Similarly to the proof of UEAS, it is enough to prove the existence of $\widetilde{T}(\mu)$ such that for every t_0, φ ($t_0 \in \mathbb{R}_+$, $\|\varphi\| < D$) there is a $t \in [t_0, t_0 + \widetilde{T}(\mu)]$ such that $\|x_t(\cdot; t_0, \varphi)\| < \delta(\mu)$. (We use the notation system introduced in the proof of Theorem 2.2.) Suppose the contrary, i.e., there is $\overline{\mu} > 0$ such that for each $T > 0$ there exist t_0, φ such that

$$\|x_t(\cdot; t_0, \varphi)\| \ge \delta(\overline{\mu}) =: 3\varepsilon \qquad (t_0 \le t \le t_0 + T).$$

Let us fix $T > (\Delta(\epsilon) + h)N_1(\varepsilon)$ arbitrarily large and take the corresponding $\bar{t}_0$, $\overline{\varphi}$ with this property. We will show that T cannot be arbitrarily large, which will be a contradiction.

Now, instead of (3.3), we define

$$N_0 := \max\left\{N_1(\varepsilon); \left[\frac{\overline{t_0}}{\Delta(\varepsilon)+h}\right]+1\right\} \geq N_1(\varepsilon), \qquad N(T) := \left[\frac{T}{\Delta(\varepsilon)+h}\right]-2, \tag{3.8}$$

and consider the sequences $\{t_i'\}_{i=N_0}^{N_0+N(T)}$, $\{t_i''\}_{i=N_0}^{N_0+N(T)}$ with properties (3.2). Estimating the sum in (3.4), using also (3.7), we obtain

$$\sum_{i=N_0}^{N_0+N(T)} \int_{t_i''-\Gamma_G(t_i'',\varepsilon)}^{t_i''} \eta(t)\,\mathrm{d}t \geq \sum_{i=N_0}^{N_0+N(T)} \int_{t_i''-\kappa(\varepsilon)}^{t_i''} \eta(t)\,\mathrm{d}t \geq N(T)\beta(\kappa(\varepsilon)).$$

Inequality (3.4) with this estimate has the form

$$0 \leq W_2(D) - W_3(\varepsilon)N(T)\beta(\kappa(\varepsilon)),$$

whence we get

$$N(T) < \frac{W_2(D)}{W_3(\varepsilon)\beta(\kappa(\varepsilon))} \qquad \left(\varepsilon = \frac{\delta(\overline{\mu})}{3}\right).$$

According to the definition (3.8) of $N(T)$ this makes it possible to obtain an upper bound for T independent of $\bar{t}_0$ and $\overline{\varphi}$, which is a contradiction.

4 Application to equation (1.2)

Consider equation (1.2) and Lyapunov functional (1.3), whose derivative admits estimate (1.4). We always suppose that

$$\eta(t) := a(t) - \int_t^{t+h} |b(u)|\,\mathrm{d}u \geq 0. \tag{4.1}$$

To apply Theorem 2.2 let us observe that if x is a solution of (1.2), then

$$|x(t)|' \leq |b(t)| \left|\int_{t-h}^t x(s)\,\mathrm{d}s\right| \leq |b(t)|h\|x_t\| \leq |b(t)|hH =: G(t).$$

Corollary 4.1. *Suppose that*

(i) function $t \mapsto \int_t^{t+h} |b(u)|\,\mathrm{d}u$ is bounded in $\mathbb{R}_+$;

(ii) $\lim_{T\to\infty} \int_t^{t+T} \eta(u)\,\mathrm{d}u = \infty$ uniformly with respect to $t \in \mathbb{R}_+$;

(iii) for every $\varepsilon > 0$ there is $L = L(\varepsilon)$ such that for every sequence $\{t_i\}_{i=1}^\infty$ with $t_i > ih$, $t_{i+1} < t_i + L$ we have

$$\sum_{i=1}^\infty \int_{t_i-\Gamma_{|b|}(t_i,\varepsilon)}^{t_i} \eta(t)\,\mathrm{d}t = \infty.$$

Then the zero solution of (1.2) is UEAS.

If we want to apply Corollary 2.4, then we have to assume that

(iv) $t \mapsto \int_0^t |b(u)|\,\mathrm{d}u$ is uniformly continuous in $\mathbb{R}_+$ (especially, $|b|$ is bounded in $\mathbb{R}_+$).

It is easy to see that (iv) implies (i), so we obtain the following result.

Corollary 4.2. *Suppose that (ii) in Corollary 4.1 and (iv) are satisfied and, in addition,*

(v) η is weakly integrally positive.

Then the zero solution is UEAS.

If η is integrally positive in Corollary 4.2, then, by Theorem 2.5, we can state UAS.

Corollary 4.3. *Suppose that (iv) is satisfied and*

(vi) η is integrally positive.

Then the zero solution is UAS.

Remark 4.4. Considering equation (1.2), Tingxiu Wang [11, 12] gave sufficient conditions for UAS of the zero solution. He assumed that η was *integrally positive in measure* [3]. This property means that for every $\varepsilon > 0$ there are $T \in \mathbb{R}_+$, $\delta > 0$ such that [$t \geq T$, $Q \subset [t-h,t]$ is open, Lebesgue measure of Q is greater or equal to ε] imply that $\int_Q \eta(t)\,\mathrm{d}t \geq \delta$. Wang proved that if (i) is satisfied and η is integrally positive in measure, then the zero solution is UAS. It can be seen [3] that if η is integrally positive in measure, then it is integrally positive, but the converse is false. So we can say that Corollary 4.3 sharpens Wang's result, provided that condition (iv) is satisfied.

Example 4.5. If $|b|$ is bounded, then $t \mapsto \int_0^t |b(u)|\,\mathrm{d}u$ is uniformly continuous in $\mathbb{R}_+$. The following example shows that the converse statement is not true.

For $k \in \mathbb{N}$ define a function $b_k \colon \mathbb{R}_+ \to \mathbb{R}_+$ so that $b_k(k) = k$,

$$b_k(t) = 0 \quad \text{if } |t-k| \geq \frac{1}{k}, \qquad b_k(t) \leq k \quad \text{if } |t-k| \leq \frac{1}{k}$$

and

$$\int_{k-\frac{1}{k}}^{k+\frac{1}{k}} b_k(u)\,\mathrm{d}u \leq \frac{1}{k}.$$

Such a function exists, and we can suppose that b_k is continuous. Obviously,

$$b(t) := \sum_{k=1}^{\infty} b_k(t)$$

is unbounded. Now we prove, that $t \mapsto \int_0^t b(u)\,\mathrm{d}u$ is uniformly continuous in $\mathbb{R}_+$.

In fact, let $\varepsilon > 0$ be fixed arbitrarily and find a k_0 such that $1/k_0 < \varepsilon$. If $s > k_0$ and $0 < t-s < 1/2$, then

$$\int_s^t b(u)\,\mathrm{d}u \leq \frac{1}{k_0} < \varepsilon.$$

If $s \leq k_0$, and $0 < t-s < \varepsilon/k_0$, then

$$\int_s^t b(u)\,\mathrm{d}u \leq k_0(t-s) < \varepsilon.$$

For $\varepsilon > 0$ we can choose $k_0 = [1/\varepsilon] + 1$. Then

$$\frac{\varepsilon}{k_0} \geq \frac{\varepsilon}{\frac{1}{\varepsilon}+1} = \frac{\varepsilon^2}{\varepsilon+1}.$$

If we define

$$\kappa(\varepsilon) := \min\left\{\frac{1}{2}; \frac{\varepsilon^2}{\varepsilon+1}\right\},$$

then $|t-s| < \kappa(\varepsilon)$ implies $\left|\int_s^t b(u)\,\mathrm{d}u\right| < \varepsilon$, i.e., $t \mapsto \int_0^t b(u)\,\mathrm{d}u$ is uniformly continuous in $\mathbb{R}_+$.

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