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POSITIVE SYMMETRIC SOLUTIONS OF SINGULAR SEMIPOSITONE BOUNDARY VALUE PROBLEMS

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Honoring the Career of John Graef on the Occasion of His Sixty-Seventh Birthday

Abstract

Using the method of upper and lower solutions, we prove that the singular boundary value problem,

$$-u'' = f(u)\, u^{-\alpha} \quad \text{in} \quad (0,1), \quad u'(0) = 0 = u(1)\,,$$

has a positive solution when $0 < \alpha < 1$ and $f : \mathbb{R} \to \mathbb{R}$ is an appropriate nonlinearity that is bounded below; in particular, we allow f to satisfy the semipositone condition $f(0) < 0$. The main difficulty of this approach is obtaining a positive subsolution, which we accomplish by piecing together solutions of two auxiliary problems. Interestingly, one of these auxiliary problems relies on a novel fixed–point formulation that allows a direct application of Schauder's fixed–point theorem.

Key words and phrases: Singular boundary value problems, semipositone boundary value problems, upper and lower solutions, radial solutions.

AMS (MOS) Subject Classifications: 34B, 35J

1 Introduction

We are interested here in the existence of positive solutions to the singular boundary value problem (BVP)

$$-u'' = f(u)\, u^{-\alpha} \quad \text{in} \quad (0,1), \quad u'(0) = 0 = u(1)\,, \tag{1}$$

where $\alpha > 0$ is a given exponent and $f : \mathbb{R} \to \mathbb{R}$ is a continuous function that is bounded below. By a positive solution, we mean a suitable function u that satisfies $u > 0$ on $(0, 1)$.

Since the lower bound on f may be negative, we allow f to satisfy the *semipositone* condition

$$f(0) < 0\,. \tag{2}$$

By reflection across the origin, a positive solution u of (1) yields a positive symmetric solution w of the BVP

$$-w'' = f(w)\, w^{-\alpha} \quad \text{in} \quad (-1,1), \quad w(-1) = 0 = w(1)\,.$$

Our interest in problem (1) stems from its relation to elliptic partial differential equations of the form

$$-\Delta_p u = f(u)u^{-\alpha} \quad \text{in} \quad \Omega, \quad u = 0 \quad \text{on} \quad \partial\Omega\,, \tag{3}$$

where $p > 1$, $\Omega \subset \mathbb{R}^N$ is a bounded region with smooth boundary $\partial\Omega$, and Δ_p is the p-Laplacian. More precisely, problem (1) is the one–dimensional analogue of problem (3) when $p = 2$, Ω is the unit ball, and one seeks radial solutions. Progress on this simplified case therefore provides some insight into what can happen in the more complicated higher–dimensional problem (3). Note that [8] establishes the existence of positive radial solutions of problem (3) for various exponents p and dimensions N when f is nonsingular; the present paper employs different methods to extend the results of [8] to the singular case.

Furthermore, the analysis of positive solutions to BVPs is of significance due to the fact that when natural phenomena are modelled via BVPs, only positive solutions to the problem will make physical sense.

Since we rely on the well–known method of upper and lower solutions to obtain positive solutions of (1), we briefly recall the relevant facts for the reader's convenience. See Chapter 2 of [6], for example, for a much more general development of such techniques for two-point BVPs. For our purposes, a function $\underline{u} \in C^2[0,1]$ is called a *lower solution* of (1) if

$$\underline{u}'' + f(\,\underline{u}\,)\,\underline{u}^{-\alpha} \;\geq\; 0 \quad \text{in} \quad (0,1), \quad \underline{u}'(0) = 0 \geq \underline{u}(1)\,, \tag{4}$$

while a function $\overline{u} \in C^2[0,1]$ that satisfies the reversed inequalities is termed an *upper solution* of (1). The following result will be fundamental (cf. Theorem 1.3, p. 77 of [6]).

Theorem 1.1 *Let $\underline{u}$ and $\overline{u}$ be lower and upper solutions, respectively, of (1) such that $\underline{u} \leq \overline{u}$. Then there exists a solution u of (1) such that $\underline{u} \leq u \leq \overline{u}$.*

To find a positive solution of a semipositone problem via the method of upper and lower solutions, it is well–known that the principal difficulty is identifying a positive lower solution. Doing so is the objective of the following section, while Section 3 uses its results to establish the existence of a positive solution of (1) for certain nonlinearities f.

For recent works on positive symmetric solutions of two–point boundary value problems involving singular and/or semipositone nonlinearities, we refer the reader to [1] – [4], [7], [10] – [14] and the references therein. Contrary to these works, we adopt a different fixed–point formulation of our auxiliary problem to which the Schauder Fixed Point Theorem applies directly (cf. Lemma 2.1 below). Finally, we remark that much less is known about such boundary value problems in higher dimensions, but a recent preprint by Chhetri and Robinson [5] provides some interesting results in this direction.

2 Constructing Positive Lower Solutions

In this section, we consider the auxiliary BVP

$$-u'' = \sigma(u)u^{-\alpha} \quad \text{on} \quad (0,1), \quad u'(0) = 0 = u(1)\,, \tag{5}$$

where the exponent $\alpha \in (0,1)$ is given and the function $\sigma : \mathbb{R} \to \mathbb{R}$ is defined by

$$\sigma(t) := \begin{cases} K, & \text{for} \quad t > 1\,, \\ L, & \text{for} \quad t < 1\,, \end{cases}$$

for constants $K > 0$ and $L < 0$. We will see in Section 3 that a positive solution of (5) yields a positive subsolution of (1); we will solve (5) by concatenating solutions of two related problems. First, given any constant $L < 0$, we show in Section 2.1 that there exist $\rho \in (0,1)$ and $v > 0$ such that

$$-v'' = Lv^{-\alpha} \quad \text{on} \quad (\rho,1), \quad v(\rho) = 1, \quad v(1) = 0\,. \tag{6}$$

Having found ρ, Section 2.2 determines a corresponding $K > 0$ such that

$$-w'' = Kw^{-\alpha} \quad \text{on} \quad (0,\rho), \quad w'(0) = 0, \quad w(\rho) = 1 \tag{7}$$

has a solution $w \geq 1$ with $w'(\rho) = v'(\rho)$. With v and w in hand, the function u defined by

$$u(r) := \begin{cases} w(r), & \text{for} \quad 0 \leq r \leq \rho\,, \\ v(r), & \text{for} \quad \rho \leq r \leq 1\,, \end{cases} \tag{8}$$

is a solution of (5). To complement these results, Section 2.5 shows that problem (6) cannot have a positive solution if $\alpha \geq 1$.

Before solving (6), we note that a natural ansatz for its solution is

$$v(r) := c\,(1-r)^{\beta}\,. \tag{9}$$

One easily finds that this ansatz yields a solution of the differential equation in (6) when

$$\beta := \frac{2}{1+\alpha} \quad \text{and} \quad c := \left(\frac{-L}{\beta(\beta-1)}\right)^{1/(1+\alpha)}.$$

It follows that $v(\rho) = 1$ when

$$\rho := 1 - \left(\frac{1}{c}\right)^{1/\beta},$$

and this value of ρ belongs to $(0, 1)$ if and only if $c > 1$, i.e.,

$$-L > \beta(\beta - 1)\,. \tag{10}$$

Thus, problem (6) only has a solution of the form (9) if $|L|$ is sufficiently large. In contrast, we show that (6) has a solution for any $L < 0$; even when (10) holds, the solution of (6) produced below cannot be of the form (9). As a result, one expects to find multiple positive solutions of both problems (6) and (5), and it would be interesting to determine precisely how many positive solutions these problems have.

2.1 Existence for Problem (6)

Suppose that $v > 0$ is a decreasing solution of (6); since

$$v'' = (-L)v^{-\alpha}\,,$$

multiplying both sides of this equation by $2v'$ and integrating yields

$$(v')^2 = \frac{-2L}{1-\alpha}v^{1-\alpha} + d\,,$$

for an integration constant d to be specified. Since $v' < 0$, we have

$$v' = -\left(\frac{-2L}{1-\alpha}v^{1-\alpha} + d\right)^{1/2}, \tag{11}$$

and integrating from r to 1 gives

$$v(r) = \int_r^1 \left(\frac{-2L}{1-\alpha}\,(v(s))^{1-\alpha} + d\right)^{1/2} ds\,.$$

This preliminary calculation motivates the following definition: for a given constant $d > 1$, let $T_d : C[0, 1] \to C[0, 1]$ be the operator

$$(T_d v)\,(r) := \int_r^1 \left(\frac{-2L}{1-\alpha}\,(v(s))^{1-\alpha} + d\right)^{1/2} ds \quad \text{for} \quad v \in C[0, 1]\,. \tag{12}$$

This operator is clearly completely continuous. Moreover, if $v \in C[0, 1]$ satisfies $0 \le v(r) \le M$ for all $r \in [0, 1]$, then it follows that

$$\sqrt{d} \le \left(\frac{-2L}{1-\alpha}\,(v(r))^{1-\alpha} + d\right)^{1/2} \le \left(\frac{-2L}{1-\alpha}M^{1-\alpha} + d\right)^{1/2}$$

and hence that

$$\sqrt{d}\,(1-r) \;\le\; (T_d v)(r) \;\le\; \left(\frac{-2L}{1-\alpha}M^{1-\alpha}+d\right)^{1/2}\,(1-r) \tag{13}$$

for any $r \in [0,1]$. Since $L<0$ and $0<1-\alpha<1$, it is possible to choose an $M_d>0$ such that

$$\left(\frac{-2L}{1-\alpha}M_d^{1-\alpha}+d\right)^{1/2} \le M_d\,. \tag{14}$$

For such an M_d, the bounds in (13) guarantee that the set $D \subset C[0,1]$ defined by

$$D := \{\, v \in C[0,1] \,:\, 0 \le v \le M_d \,\} \tag{15}$$

is invariant under T_d, i.e., $T(D) \subset D$. As D is clearly bounded, closed and convex, the Schauder Fixed Point Theorem [9] pp.25–26 applies and yields the following result.

Lemma 2.1 *Let $d>1$ be given, and suppose that $M_d>0$ satisfies (14). Then there is a function $v \in C[0,1]$ such that $0 \le v \le M_d$ and $T_d v = v$.*

A fixed point $v \in D$ of T_d is automatically decreasing, vanishes at $r=1$, and is therefore positive on $(0,1)$. Since $v \ge 0$ and $d>1$,

$$v(0) = \int_0^1 \left(\frac{-2L}{1-\alpha}v^{1-\alpha}+d\right)^{1/2} ds \ge \sqrt{d} > 1\,,$$

and we conclude that there must be a point $\rho \in (0,1)$ such that

$$v(\rho) = \int_\rho^1 \left(\frac{-2L}{1-\alpha}v^{1-\alpha}+d\right)^{1/2} ds = 1\,.$$

We thus obtain the following result.

Corollary 2.1 *Let $L<0$ and $\alpha \in (0,1)$ be given. There is a point $\rho \in (0,1)$ such that the singular boundary value problem*

$$-v'' = Lv^{-\alpha} \quad \textit{on} \quad (\rho,1), \quad v(\rho)=1, \quad v(1)=0$$

has a positive solution v.

2.2 Existence for Problem (7)

Lemma 2.2 *Let $\alpha \in (0,1)$, $\rho \in (0,1)$ and $K>0$ be given. Problem (7),*

$$-w'' = Kw^{-\alpha} \quad \textit{on} \quad (0,\rho), \quad w'(0)=0, \quad w(\rho)=1\,,$$

has a solution w satisfying $w \ge 1$.

Proof. Direct calculations show that $\underline{w} \equiv 1$ is a lower solution of (7), while the unique solution $\overline{w}_K$ of

$$-\overline{w}_K'' = K \quad \text{on} \quad (0,\rho), \quad \overline{w}_K'(0) = 0, \quad \overline{w}_K(\rho) = 1 \tag{16}$$

is an upper solution of (7); note that

$$\overline{w}_K(r) = \frac{K}{2}\left(\rho^2 - r^2\right) + 1\,. \tag{17}$$

It follows from Theorem 1.1 that there is a solution w of the problem such that $1 \le w \le \overline{w}_K$. ∎

The main result of this paper, Theorem 2.1, relies on finding a solution w of problem (7) with a prescribed slope at ρ. The following sequence of lemmas shows that this can be done.

Lemma 2.3 *Let $\alpha \in (0,1)$, $\rho \in (0,1)$, and $m < 0$ be given. There exists a constant $K > 0$ such that problem (7) has a solution w with $m < w'(\rho) < 0$.*

Proof. Lemma 2.2 guarantees that, for any $K > 0$, problem (7) has a decreasing solution w satisfying $1 \le w \le \overline{w}_K$. Consequently,

$$1 \le w(0) \le \overline{w}_K(0) = 1 + \frac{K}{2}\rho^2\,,$$

and the calculations used earlier to define the operator T_d show that

$$\left(w'(\rho)\right)^2 = \frac{-2K}{1-\alpha} + \frac{2K}{1-\alpha}\left(w(0)\right)^{1-\alpha}\,. \tag{18}$$

It follows that decreasing K yields a solution w whose slope at ρ is as small (in absolute value) as desired. ∎

Lemma 2.4 *Let $\alpha \in (0,1)$, $\rho \in (0,1)$, and $m < 0$ be given. There exists a constant $K > 0$ such that problem (7) has a solution w with $w'(\rho) < m$.*

Proof. Let $\overline{w}_{K_1}$ denote the function defined by (17); we know that

$$1 \le \overline{w}_{K_1} \le 1 + \frac{K_1}{2}\rho^2\,,$$

from which we obtain

$$\left(\overline{w}_{K_1}\right)^{-\alpha} \ge \left(1 + \frac{K_1}{2}\rho^2\right)^{-\alpha}\,.$$

It then follows by direct calculation that $\overline{w}_{K_1}$ will be a lower solution of (7) if $K > 0$ satisfies

$$K \geq K_1 \left(1 + \frac{K_1}{2}\rho^2\right)^{\alpha} . \tag{19}$$

Since $\overline{w}_K$ provides an upper solution as in the proof of Lemma 2.2, there exists a solution w of (7) such that $\overline{w}_{K_1} \leq w \leq \overline{w}_K$. In particular,

$$w(0) \geq \overline{w}_{K_1}(0) = 1 + \frac{K_1}{2}\rho^2 .$$

By taking K_1 sufficiently large, choosing a K that satisfies (19), and then using identity (18), we see that there exists a K such that (7) has a solution with arbitrarily large slope at ρ. ∎

Lemma 2.5 *Let $\alpha \in (0,1)$, $\rho \in (0,1)$, and $m < 0$ be given. There exists a constant $K > 0$ such that problem (7) has a solution w satisfying $w'(\rho) = m$.*

Proof. Using the two previous lemmas, there exist positive constants $\underline{K} < \overline{K}$ and corresponding solutions $\underline{w} \leq \overline{w}$ of problem (7) such that $\overline{w}'(\rho) < m < \underline{w}'(\rho)$. Let K be any constant between $\underline{K}$ and $\overline{K}$; since $\underline{w}$ and $\overline{w}$ are distinct lower and upper solutions, respectively, of problem (7) with coefficient K, there must exist a solution w_K of this problem that lies between $\underline{w}$ and $\overline{w}$. Since we can use w_K as an upper or lower solution for problem (7) with other coefficients, we thus obtain a family of solutions $\mathcal{F} := \left\{\, w_K \,:\, \underline{K} \leq K \leq \overline{K} \,\right\}$ with $w_{K_1} \leq w_{K_2}$ if $K_1 \leq K_2$.

To see that $w'_K(\rho)$ varies continuously with K, fix a constant K^* between $\underline{K}$ and $\overline{K}$, let K_n be an increasing sequence that converges to K^*, let $w_n \in \mathcal{F}$ denote the solution corresponding to K_n, and let $w^* \in \mathcal{F}$ be the solution corresponding to K^*. The set of functions $\{\, w_n \,\}$ is clearly uniformly bounded (by $w_{K^*}(0)$), and the calculations leading to (11) and (18) show that these functions are equicontinuous. The Arzelà-Ascoli Theorem therefore guarantees that some subsequence converges uniformly; relabeling as necessary, we find that the functions w_n converge uniformly to some function w. Combining these convergence results with a fixed point characterization of problem (7) (obtained by, e.g., proceeding as in Section 2.1) shows that w solves problem (7) with coefficient K^*. w and w^* are therefore the unique positive solutions of the initial value problem

$$-v'' = K^* v^{-\alpha} , \quad v(0) = v_0 , \quad v'(0) = 0 ,$$

with $v_0 = w(0)$ and $v_0 = w^*(0)$, respectively. The uniqueness of trajectories and the fact that $w(\rho) = w^*(\rho) = 1$ force $w(0) = w^*(0)$, and it then follows from (18) that both $w_K(0)$ and $w'_K(\rho)$ depend continuously on the parameter K. Having verified this continuous dependence on K, the remainder of the proof is a direct application of the Intermediate Value Theorem. ∎

2.3 Existence for Problem (5)

Combining the preceding results provides a proof of existence for problem (5):

Theorem 2.1 *Let $L < 0$ and $\alpha \in (0,1)$ be given. There exists a corresponding $K > 0$ such that the singular boundary value problem*

$$-u'' = \sigma(u)u^{-\alpha} \quad on \quad (0,1), \quad u'(0) = 0 = u(1)\,,$$

has a positive solution u.

Proof. Given $L < 0$ and $\alpha \in (0,1)$, Corollary 2.1 yields $\rho \in (0,1)$ and a positive solution v of

$$-v'' = Lv^{-\alpha} \quad \text{on} \quad (\rho,1), \quad v(\rho) = 1, \quad v(1) = 0\,.$$

The lemmas of the preceding section then provide a constant $K > 0$ and a solution w of

$$-w'' = Kw^{-\alpha} \quad \text{on} \quad (0,\rho), \quad w'(0) = 0, \quad w(\rho) = 1$$

such that $w'(\rho) = v'(\rho)$; defining u by (8) provides the desired solution. ∎

2.4 Solutions for Fixed $L < 0$

A closer inspection of the results just established reveals more about the structure of the positive solutions of problem (5). For a fixed $L < 0$, there exist $\rho_d \in (0,1)$ and a positive solution v_d of (6) (with $\rho = \rho_d$) for every $d > 1$, and the proofs in Section 2.1 show that ρ_d increases as d increases. When one considers problem (7) for $\rho = \rho_1$ and $\rho = \rho_2$, where $\rho_1 < \rho_2$, one finds that the corresponding K_1 and K_2 satisfy $K_1 < K_2$. Thus, once a solution of problem (5) has been found for a particular $K > 0$, the problem will have a solution for any larger value of K.

2.5 Nonexistence for $\alpha \geq 1$

Let $\alpha > 1$, and suppose that (6) has a positive solution v. Calculating as above shows that

$$(v'(r))^2 = \frac{-2L}{1-\alpha}(v(r))^{1-\alpha} + d\,, \tag{20}$$

for some integration constant d. Since $\alpha > 1$ and $L < 0$,

$$\frac{-2L}{1-\alpha} < 0\,.$$

As $r \to 1$, $v(r) \to 0$ and the right-hand side of (20) approaches $-\infty$, while the left-hand side is clearly nonnegative. This inconsistency shows that (6) cannot have a solution if $\alpha > 1$.

If $\alpha = 1$ and v is a positive solution of (6), then

$$v'' = \frac{-L}{v}\,.$$

Multiplying both sides by $2v'$ and integrating now gives

$$(v')^2 = -2L\log(v) + d \tag{21}$$

for some integration constant d. As $r \to 1$, v again approaches 0 and we obtain the same contradiction as in the case $\alpha > 1$.

3 Applications

As indicated earlier, a positive solution of problem (5) will be a positive subsolution of (1) for appropriate nonlinearities f. Under additional assumptions on f, it is easy to find a larger supersolution and thereby obtain a positive solution of (1), as illustrated in our final result.

Theorem 3.1 *Let $L < 0$ be given, and let $K > 0$ be a corresponding value such that (5) has a positive solution ψ. Let M be the maximum of ψ, suppose that the constants a and b satisfy*

$$0 < a < b^{\alpha+1}\,,$$

and let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that

1. *$f(t) > b^{\alpha+1}K \quad$ for $\quad b \le t \le Mb$,*
2. *$f(\gamma) < 0$ at some point $\gamma > Mb$, $\quad$ and*
3. *$f(t) > aL \quad$ for all $\quad t$.*

For a given exponent $\alpha \in (0, 1)$, there is a positive solution u of the singular problem

$$-u'' = f(u)u^{-\alpha} \quad in \quad (0,1), \quad u'(0) = 0 = u(1)\,. \tag{22}$$

Proof. First, define $u_1 := b\psi$. Combining the fact that ψ solves (5) with the hypotheses on f, a direct computation shows that u_1 is a positive subsolution of (22). Next, define $u_2 \equiv \gamma$. Since $\gamma > Mb$, $u_1 < u_2$, and it is easy to see that u_2 is a supersolution of (22). By Theorem 1.1, there is a solution u of (22) such that $u_1 < u < u_2$, completing the proof. ∎

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