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Positive evanescent solutions of singular elliptic problems in exterior domains

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Abstract. We investigate the existence of positive solutions for the following class of nonlinear elliptic problems

$$\operatorname{div}(a(\|x\|)\nabla u(x)) + f(x,u(x)) - (u(x))^{-\alpha}\|\nabla u(x)\|^{\beta} + g(\|x\|)x \cdot \nabla u(x) = 0,$$

where $x \in \mathbb{R}^n$ and $\|x\| > R$, with the condition $\lim_{\|x\|\to\infty} u(x) = 0$. We present the approach based on the subsolution and supersolution method for bounded subdomains and a certain convergence procedure. Our results cover both sublinear and superlinear cases of f. The speed of decaying of solutions will be also characterized more precisely.

Keywords: singular elliptic problems, positive evanescent solutions, subsolution and supersolution method, exterior domain.

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1 Introduction

We consider the solvability of the following singular elliptic equation

$$\operatorname{div}(a(\|x\|)\nabla u(x)) + f(x,u(x)) - (u(x))^{-\alpha}\|\nabla u(x)\|^{\beta} + g(\|x\|)x \cdot \nabla u(x) = 0, \tag{1.1}$$

for $x \in \Omega_R$, in the case when we look for solutions satisfying the condition

$$\lim_{\|x\|\to\infty} u(x) = 0, \tag{1.2}$$

where $n > 2$, $R > 1$, $0 < 2\alpha \le \beta \le 2$, for all $x, y \in \mathbb{R}^n$, $\|x\| := \sqrt{\sum_{i=1}^n x_i^2}$, and $x \cdot y := \sum_{i=1}^n x_i y_i$, $\Omega_R = \{x \in \mathbb{R}^n, \|x\| > R\}$. Precisely, we ask about sufficient conditions which guarantee the existence of function u of $C^{2+\alpha}_{\mathrm{loc}}$ class, which satisfies (1.1) at each point x from a certain neighborhood of infinity and we require the solution vanishes when the Euclidian norm of arguments tends to infinity. Our next aim is to describe more precisely how quickly solutions decay.

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Similar problems without any singular part were widely discussed, among others in [4–6, 10–15]. On the other hand, there are many papers devoted to the singular elliptic problems with Laplace operator, similar singularity at zero and subquadratic growth with respect to the gradient. Here we have to mention paper [8] due to D. P. Covei, who looks for positive solutions of $C^{2+\alpha}_{\mathrm{loc}}$ class for the following problem

$$\begin{gathered} -\Delta u + c(x)u^{-1}\|\nabla u\|^2 = a(x) \quad \text{for } x \in R^N,\ u > 0, \\ \lim_{\|x\|\to\infty} u(x) = 0. \end{gathered} \tag{1.3}$$

Such problems, by making suitable transformation, are associated with widely discussed equation of the form

$$-\Delta u = a(x)h(u) \quad \text{in } \Omega, u > 0 \tag{1.4}$$

when we look for a solution which blows up in a neighborhood of $\partial\Omega$, (see e.g. [8, Remarks 1 and 2]). Precisely, let us consider (1.4) with $h(u) = e^u$. When we apply the transformation $w = e^{-u}$ we get $\Delta u = \frac{1}{w^2}\|\nabla w\|^2 - \frac{1}{w}\Delta w$. Therefore problem (1.4) leads to the following equation

$$-\Delta w + w^{-1}\|\nabla w\|^2 = a(x)$$

which is a special case of (1.3) and of (1.1), where we consider the singularity $(u(x))^{-\alpha}\|\nabla u(x)\|^{\beta}$, with $0 < 2\alpha \le \beta \le 2$. It appears that this assumption plays the special role. First of all, the inequality $2\alpha \le \beta$ allows us to obtain the subsolution of our problem on a bounded domain with the help of an eigenfunction of a certain linear problem. On the other hand, the condition $\beta \le 2$ is necessary to apply the technical tools described in [9]. For the reader's convenience we describe the paper [9], where the existence and nonexistence results are discussed for PDE with singular nonlinearities on a bounded domain $\Omega \subset R^N$ with sufficiently smooth boundary. The author applied his general results, among others, to the problem

$$\begin{aligned} \Delta u - a(x)u^{-q}\|\nabla u\|^2 + b(x)u^{2-p} = 0 \quad &\text{for } x \in \Omega,\ u > 0, \\ u = 0 \quad &\text{on } \partial\Omega \end{aligned}$$

which comes from stochastic process theory and leads (for $q = 1$, $a \equiv 2$, $b \equiv 1$), via substitution $u = 1/v$, to the problem

$$\begin{aligned} \Delta v - v^p = 0, \quad &v > 0 \text{ in } \Omega \\ v(x) \to \infty \quad &\text{as } x \to \partial\Omega. \end{aligned}$$

There are also many results concerning weak solutions. Here it is worth mentioning the paper [20] written by Wen-Shu Zhou who considers the existence and multiplicity of positive weak solutions for the following singular PDE

$$-\Delta u + \lambda u^{-m}\|\nabla u\|^2 = f(x) \quad \text{for } x \in \Omega,\ u = 0 \text{ on } \partial\Omega,$$

where $\Omega \subset R^N$ is bounded, $N \ge 2$, $m > 1$ and $\lambda \ne 0$ and f is a nonnegative measurable function. The results are also based on the subsolution and supersolution method. We can meet such problems in fluid mechanics (see e.g. [17] and references therein). Further, D. Arcoya, S. Barile, P. J. Martínez-Aparicio (in [2]) investigate the problem of the form

$$-\Delta u + g(x,u)\|\nabla u\|^2 = a(x) \quad \text{for } x \in \Omega,\ u \in H^1_0. \tag{1.5}$$

where $\Omega \subset R^N$ is a bounded domain, with $N \geq 3$, $a \in L^q$, with $q > N/2$ and g is a Carathéodory function in $\Omega \times (0, +\infty)$ which can have a singularity at zero. The authors consider a sequence of approximated problems to (1.5) and show the existence of a sequence (w_n) of their solutions which tends to a positive solution of (1.5) in $H^1_{\text{loc}}(\Omega)$.

In the end we recall the results presented by D. Arcoya et al. in [3], where we can find the more general problem

$$-\operatorname{div}(M(x,u)\nabla u) + g(x,u)\|\nabla u\|^2 = f(x) \quad \text{for } x \in \Omega \subset R^N,\ u = 0 \text{ on } \partial\Omega \tag{1.6}$$

with f being strictly positive on every compact subset of Ω and a Carathéodory function $g : \Omega \times (0, +\infty) \to R$, which can be singular at 0. The authors also prove that for the following special case of (1.6)

$$-\Delta u + u^{-\gamma}\|\nabla u\|^2 = f(x) \quad \text{for } x \in \Omega,\ u = 0 \text{ on } \partial\Omega \tag{1.7}$$

with $\gamma > 0$, the condition $\gamma < 2$ is necessary and sufficient for the existence of distributional solution of (1.7).

We also want to join in this discussion and deal with positive solutions for (1.1)–(1.2) and their asymptotic behavior. We start with the definitions of solution of our problem. We have to emphasize that we use standard definitions based on the ideas from the seventies and the eighties (described e.g. by Amann or Noussair and Swanson in [1] and [18]) which are met also in papers mentioned above.

Definition 1.1. As a solution of our problem we understand a function $u \in C^{2+\alpha}_{\text{loc}}(\Omega_R)$ which satisfies (1.1) at every point $x \in \Omega_R$ and condition (1.2).

Our results are based on the following assumptions

(A_a) *$a : [1, +\infty) \to (0, +\infty)$ belongs to $C^{1+\alpha}([1, +\infty))$, $\int_1^\infty \frac{l^{1-n}}{a(l)} dl < +\infty$ and $\lim_{l \to +\infty} a(l) \in (0, +\infty)$;*

(A_f) *$f : \Omega_1 \times \mathbb{R} \to \mathbb{R}$, $\Omega_1 = \{x \in \mathbb{R}^n, \|x\| > 1\}$, is locally Hölder continuous, there exist $d > 0$ and continuous function $M : [1, +\infty) \to (0, +\infty)$ such that $\sup_{\|x\|=r} \sup_{u \in [0,d]} |f(x, u)| \leq M(r)$ in $[1, +\infty)$ and*

$$\int_1^\infty r^{n-1} M(r) dr < (n-2)\frac{d}{c}, \tag{1.8}$$

where $c := (n-2)\int_1^\infty \frac{l^{1-n}}{a(l)} dl$ and for each bounded domain $\widetilde{\Omega} \subset \Omega_1$, $f(x, u) \geq f_{\min} > 0$, for all $x \in \widetilde{\Omega}$ and $u \in [0, d]$;

(A_g) *$g : [1, +\infty) \to \mathbf{R}$ is continuously differentiable and there exists $r_0 \geq 1$ such that $g(r) \geq 0$ for all $r \geq r_0$.*

2 Supersolution on exterior domain

Our task is now to obtain the existence of function v of the class $C^2(\Omega_R)$, such that

$$\operatorname{div}(a(\|x\|)\nabla v(x)) + f(x, v(x)) - (v(x))^{-\alpha}\|\nabla v(x)\|^\beta + g(\|x\|)x \cdot \nabla v(x)) \leq 0,$$

for $x \in \Omega_R$, and $\lim_{\|x\| \to \infty} v(x) = 0$. In the sequel we call such function v a supersolution of (1.1)–(1.2). To this effect we use the ideas presented in the paper [19] and consider the auxiliary linear elliptic problem

$$\left\{ \begin{array}{c} -\operatorname{div}(a(\|x\|)\nabla v(x)) = M(\|x\|) \text{ on } \Omega_1 \\ v(x) = 0, \quad \text{for } \|x\| = 1, \\ \lim_{\|x\|\to\infty} v(x) = 0. \end{array} \right\} \tag{2.1}$$

for function M given in (A_f). We show that there exists a radial positive solution of (2.1) which is a supersolution of (1.1)–(1.2) in a certain neighborhood Ω_R of infinity. To prove its existence we employ the standard reasoning applying suitable transformation. Then the problem of the existence of radial solutions for (2.1) leads to the existence of positive solutions of the following singular Dirichlet problem

$$\begin{cases} -(\widetilde{a}(t)z'(t))' = h(t) \quad \text{in } (0,1), \\ z(0) = z(1) = 0, \end{cases} \tag{2.2}$$

where

$$h(t) = \frac{1}{(n-2)^2}(1-t)^{\frac{2n-2}{2-n}} M\left((1-t)^{\frac{1}{2-n}}\right) \quad \text{and} \quad \widetilde{a}(t) = a\left((1-t)^{\frac{1}{2-n}}\right).$$

Precisely, we use the transformation $\|x\| = (1-t)^{\frac{1}{2-n}}$ and the well-known fact that if z is a solution of (2.2) then $v(x) = z(1-\|x\|^{2-n})$ is a radial solution of (2.1), and conversely, if we have a radial solution $v(x) = \widetilde{z}(\|x\|)$ of (2.1), with $\widetilde{z} : [1, +\infty) \to \mathbb{R}$, then $z(t) = \widetilde{z}\big((1-t)^{\frac{1}{2-n}}\big)$ satisfies (2.2).

Taking into account the properties of functions M and a, one can infer that h and $\widetilde{a}$ satisfy conditions:

(A_$\widetilde{a}$) *$\widetilde{a} \in C^1([0,1))$ is positive, $\lim_{t\to 1^-} \widetilde{a}(t) := \widetilde{a}_1 \in (0, +\infty)$, $c = \int_0^1 \frac{1}{\widetilde{a}(s)} ds$.*

(A_h) *$h : (0,1) \to (0, +\infty)$ is continuous and for all $t \in (0,1)$ and*

$$\int_0^1 h(s) ds \le 4dc\widetilde{a}^2_{\min}, \tag{2.3}$$

where $\widetilde{a}_{\min} := \inf_{t\in(0,1)} \widetilde{a}(t)$.

Applying the approach described in [12] and [19] we prove existence of a positive radial solution v of (2.1) having the special properties, which allow us to show that v is a supersolution of our problem on each bounded domain $\Omega \subset \Omega_R$. We start with the singular ODE.

Lemma 2.1. *If conditions (A_h) and (A_$\widetilde{a}$) are satisfied then we state the existence of at least one positive classical solution z of (2.2) such that*

1. *there exists $t_0 \in (0,1)$ for which $z'(t) \le 0$ for all $t \in (t_0, 1)$,*
2. *for all $t \in (0,1)$,*
$$z(t) \le d, \tag{2.4}$$
3. $$z(t) = O(1-t) \quad \text{for } t \to 1^-, \tag{2.5}$$
4. $$z(t) = o(\phi(t)) \quad \text{for } t \to 1^-, \tag{2.6}$$

where ϕ is any function $\phi \in C^1(0,1)$ such that $\lim_{t\to 1^-} \phi(t) = 0$ and $\lim_{t\to 1^-} \phi'(t) = +\infty$.

Proof. Firstly, we note that the function z given by the formula

$$z(t) = \int_0^1 \mathbf{G}(s,t)h(s)ds, \tag{2.7}$$

is a solution of (2.2), when $\mathbf{G}$ is the Green's function

$$\mathbf{G}(s,t) := \frac{1}{c}\begin{cases} \int_0^s \frac{1}{\widetilde{a}(r)}dr \int_t^1 \frac{1}{\widetilde{a}(r)}dr & \text{for } 0 \le s \le t \\ \int_0^t \frac{1}{\widetilde{a}(r)}dr \int_s^1 \frac{1}{\widetilde{a}(r)}dr & \text{for } t < s \le 1. \end{cases}$$

It is clear that $z \in C([0,1]) \cap C^2(0,1)$, $z(0) = z(1) = 0$, z satisfies (2.2) *and*

$$0 \le z(t) \le \frac{1}{c}\frac{1}{4\widetilde{a}^2_{\min}} \int_0^1 h(s)ds \le d.$$

Our task is now to show the existence of t_0 such that $z'(t) \le 0$ for all $t \in (t_0,1)$ and the positivity of z in $(0,1)$. To show the first assertion we state the existence of $t_0 \in (0,1)$ such that $z'(t_0) = 0$ what is a simply consequence of Rolle's theorem. It is clear that $k(t) := \widetilde{a}(t)z'(t)$, for all $t \in (0,1)$, is nonincreasing in $(0,1)$, and consequently $k(t) \le k(t_0) = 0$ for all $t \in (t_0,1)$ which gives $z'(t) \le 0$ for all $t \in (t_0,1)$.

Now we prove that $z > 0$ in $(0,1)$. By (2.7), we know that z is nonnegative. Suppose that there exists at least one argument $\widetilde{t} \in (0,1)$ at which $z(\widetilde{t}) = 0$. Here we can use Rolle's theorem again which leads to the existence of numbers $\underline{t}_1 \in (0,\widetilde{t})$ and $\overline{t}_1 \in (\widetilde{t},1)$ such that $z'(\underline{t}_1) = z'(\overline{t}_1) = 0$, which implies, by the properties of k, that for all $t \in [\underline{t}_1,\overline{t}_1]$, $z'(t) = 0$ in $[\underline{t}_1,\overline{t}_1]$, and further $z(t) = z(\widetilde{t}) = 0$ in $[\underline{t}_1,\overline{t}_1]$. Now the iteration process gives us two sequences: $(\underline{t}_m)_{m\in N} \subset (0,1)$, which is decreasing, and $(\overline{t}_m)_{m\in N} \subset (0,1)$, which is increasing, and such that $z \equiv 0$ in $[\underline{t}_m,\overline{t}_m]$. The properties of both sequences lead to the existence of their limits. Let $\underline{t} := \lim_{m\to\infty} \underline{t}_m$ and $\overline{t} := \lim_{m\to\infty} \overline{t}_m$. Since z is continuous in $[0,1]$, $z(t) = 0$ in $[\underline{t},\overline{t}]$. It is easy to show that $\underline{t} = 0$ and $\overline{t} = 1$, which means that $z \equiv 0$ in $[0,1]$. We get a contradiction to (A_h). Thus $z > 0$ in $(0,1)$.

We start the proof of parts 3 and 4 with the observation that using (2.7) and (A_$\widetilde{a}$) we get for all $t \in (0,1)$,

$$z'(t) = \frac{1}{c}\frac{1}{\widetilde{a}(t)}\left[-\int_0^1 \left(\int_0^s \frac{1}{\widetilde{a}(r)}dr\right) h(s)ds + c\int_t^1 h(s)ds\right]$$

and further

$$\lim_{t\to 1^-} z'(t) = -\frac{1}{c}\frac{1}{\widetilde{a}_1}\int_0^1 \int_0^s \frac{1}{\widetilde{a}(r)}drh(s)ds.$$

We also have $\lim_{t\to 1^-} z(t) = 0$. Now, applying (as in [11] and [12]) L'Hospital's rule and the above equalities, we obtain

$$\lim_{t\to 1^-}\frac{z(t)}{(1-t)} = \lim_{t\to 1^-}\frac{z'(t)}{-1} = \frac{1}{c}\frac{1}{\widetilde{a}_1}\int_0^1 \int_0^s \frac{1}{\widetilde{a}(r)}drh(s)ds \in (0,+\infty).$$

Therefore $z(t) = O((1-t))$ for $t \to 1^-$. If we take any function $\phi \in C^1(0,1)$ satisfying $\lim_{t\to 1^-}\phi(t) = 0$ and $\lim_{t\to 1^-}\phi'(t) = +\infty$, we can apply again L'Hospital's rule and get $\lim_{t\to 1^-}\frac{z(t)}{\phi(t)} = \lim_{t\to 1^-}\frac{z'(t)}{\phi'(t)} = 0$. In consequence, we have $z(t) = o(\phi(t))$ as $t \to 1^-$. □

As a consequence of the above lemma we get the existence of supersolutions for (1.1)–(1.2).

Corollary 2.2. *If we assume (A_f) and (A_a) then there exists a positive supersolution v of our singular problem in Ω_R, for a certain $R > 1$. Moreover the following estimates hold*

$$v \leq d \quad in\ \overline{\Omega}_R, \tag{2.8}$$

$$v(x) = O\left(\frac{1}{\|x\|^{n-2}}\right) \quad as\ \|x\| \to +\infty, \tag{2.9}$$

and

$$v(x) = o\left(\widetilde{\phi}(\|x\|)\right) \qquad as\ \|x\| \to +\infty \tag{2.10}$$

for any $\widetilde{\phi} \in C^1(1, +\infty)$ satisfying conditions $\lim_{r\to+\infty} \widetilde{\phi}(r) = 0$ and $\lim_{r\to+\infty} \widetilde{\phi}'(r) r^{n-1} = +\infty$.

Proof. Applying Lemma 2.1 we state that there exists at least one positive radial solution $v(x) = z(1 - \|x\|^{2-n}) > 0$ for $x \in \Omega_1$, of (2.1), where z is a positive solution of (2.2). The first part of Lemma 2.1 guarantees the existence of $t_0 \in (0,1)$ such that $z'(t) \leq 0$ for all $t \in (t_0, 1)$. Let us put $R_0 := (1 - t_0)^{\frac{1}{2-n}} > 1$. Then for all $x \in \mathbb{R}^n$ such that $\|x\| \geq R_0$, we have the following estimate

$$\begin{aligned}
x \cdot \nabla v(x) &= \sum_{j=1}^{n} x_j \frac{\partial v(x)}{\partial x_j} \\
&= \sum_{j=1}^{n} \left[x_j z'(1 - \|x\|^{2-n}) \left(-(2-n)\|x\|^{1-n} \frac{x_j}{\|x\|} \right) \right] \\
&= z'(1 - \|x\|^{2-n})(n-2)\|x\|^{2-n} \leq 0.
\end{aligned}$$

Moreover for all $\|x\| \geq r_0$, $g(\|x\|) \geq 0$. Finally, we have for all $x \in \mathbb{R}^n$ such that $\|x\| \geq R$, where $R := \max\{r_0, R_0\}$

$$\begin{aligned}
&\operatorname{div}(a(\|x\|)\nabla v(x)) + f(x, v(x)) - (v(x))^{-\alpha} \|\nabla v(x)\|^{\beta} + g(\|x\|) x \cdot \nabla v(x)) \\
&\quad \leq \operatorname{div}(a(\|x\|)\nabla v(x)) + M(\|x\|) = 0,
\end{aligned}$$

namely v is a supersolution of our singular problem in Ω_R.

Applying assertions (2.5) and (2.6) and the definition of v we obtain (2.9) and (2.10). □

3 Solutions on bounded domain

Let $\Omega \subset R^n$ be a bounded domain with $C^{2+\alpha}$ boundary such that $\Omega \subset \Omega_R$. Our task is now to prove the existence of a positive solution of the elliptic singular PDE (1.1) in Ω. To this end we use the ideas presented by S. Cui in [9] and formulate the lemma which gives us the solvability of our problem in Ω. For the reader's convenient we recall subsolution and supersolution results from [9]. We start with the following operator

$$Lu \equiv \sum_{i,j=1}^{n} a_{i,j}(x) \frac{\partial^2}{\partial x_i \partial x_j} u + \sum_{i=1}^{n} b_i(x) \frac{\partial}{\partial x_i} u,$$

where $a_{i,j}, b_i \in C^{\alpha}\left(\overline{\Omega}\right)$, for some $\alpha \in (0,1)$, $a_{i,j}(x) = a_{i,j}(x)$ in Ω, and there exists a constant $\lambda_0 > 0$ such that for all $x \in \Omega$ and $\zeta \in R^n$, $\sum_{i,j=1}^{n} a_{i,j}(x)\zeta_i\zeta_j \geq \lambda_0|\zeta|^2$. Let us consider the function F satisfying the following assumptions

(D1) F is locally Hölder continuous in $\Omega \times (0, +\infty) \times R^n$ and continuously differentiable with respect to the variables u and ξ;

(D2) for bounded domain $Q \subset\subset \Omega$ and any $a, b \in (0, +\infty)$, $a < b$, there exists a corresponding constant $C = C(Q, a, b) > 0$ such that for all $x \in \overline{Q}$, $u \in [a, b]$, $\xi \in R^n$,
$$|F(x, u, \xi)| \le C(1 + |\xi|^2).$$

By a solution of the problem
$$Lu + F(x, u, \nabla u) = 0, \quad u > 0 \text{ in } \Omega \tag{3.1}$$
and
$$u = \psi \quad \text{on } \partial\Omega, \tag{3.2}$$
S. Cui understands function $u \in C^{2+\alpha}(\Omega) \cap C(\overline{\Omega})$ which satisfies (3.1) at every point $x \in \Omega$ and (3.2). Subsolutions of (3.1)–(3.2), i.e. functions w satisfying $Lw + F(x, w, \nabla w) \ge 0$ and (3.2), and supersolutions, i.e. functions v satisfying $Lv + F(x, v, \nabla v) \le 0$ and (3.2), are described analogously.

We base ourselves on the below results proved by Cui (see [9, Lemma 3]).

Lemma 3.1. *Suppose that the function F satisfies conditions $(D1)$ and $(D2)$. Suppose furthermore that problem (3.1)–(3.2) has a pair of subsolution $\underline{u}$ and supersolutions $\overline{u}$ satisfying the conditions*

(1) $\overline{u}, \underline{u} \in C^2(\Omega) \cap C(\overline{\Omega})$;

(2) $0 < \underline{u}(x) \le \overline{u}(x)$ *for all* $x \in \Omega$;

(3) $\underline{u}(x) = \overline{u}(x) = \psi(x)$; *for all* $x \in \partial\Omega$.

Then problem (3.1)–(3.2) has a solution $u \in C^{2+\alpha}(\Omega) \cap C(\overline{\Omega})$ satisfying $\underline{u}(x) \le u(x) \le \overline{u}(x)$ for all $x \in \Omega$.

In spite of the fact that in our case assumptions (D1) and (D2) are satisfied, we have to emphasize that we cannot apply directly the above result. As we see in the next lemma we will construct a subsolution which is equal to zero on the boundary of Ω. On the other hand the supersolution v of our problem will be positive on $\partial\Omega$. Thus condition (3) in Lemma 3.1 does not hold. But it appears that a small modification of the proof of Lemma 3.1 gives us the required assertion.

It is clear that (1.1) is a particular case of (3.1) with $F(x, u, z) = f(x, u) - (u)^{-\alpha}\|z\|^\beta - \nabla(a(\|x\|))z - g(\|x\|)x \cdot z$. Now we consider the equation
$$\operatorname{div}(a(\|x\|)\nabla u(x)) + f(x, u(x)) - (u(x))^{-\alpha}\|\nabla u(x)\|^\beta \\ + g(\|x\|)x \cdot \nabla u(x)) = 0 \quad \text{for all } x \in \Omega, \tag{3.3}$$
where Ω is a bounded domain. We say that $w \in C^2(\Omega) \cap C(\overline{\Omega})$ is a subsolution of (3.3) in Ω if, at each point of Ω, w satisfies
$$\operatorname{div}(a(\|x\|)\nabla w(x)) + f(x, w(x)) - (w(x))^{-\alpha}\|\nabla w(x)\|^\beta + g(\|x\|)x \cdot \nabla w(x)) \ge 0.$$
Analogously, we say that $v \in C^2(\Omega) \cap C(\overline{\Omega})$ is a supersolution of (3.3) in Ω if, at each point of Ω, v satisfies
$$\operatorname{div}(a(\|x\|)\nabla v(x)) + f(x, v(x)) - (v(x))^{-\alpha}\|\nabla v(x)\|^\beta + g(\|x\|)x \cdot \nabla v(x)) \le 0.$$
Applying the steps of the reasoning described in the proof of the Lemma 3.1 (see [9, Lemma 3]) we can prove the below result.

Lemma 3.2. *Assume that equation* (3.3) *has a pair of subsolution and supersolutions $\underline{u}$ and $\overline{u}$ such that $\overline{u}, \underline{u} \in C^2(\Omega) \cap C(\overline{\Omega})$, $0 < \underline{u}(x) \leq \overline{u}(x)$ for all $x \in \Omega$ and $\underline{u}(x) \leq \overline{u}(x)$ for all $x \in \partial\Omega$. Then the equation* (3.3) *has a solution u_0 belonging to $C^{2+\alpha}(\Omega)$ and satisfying $\underline{u}(x) \leq u_0(x) \leq \overline{u}(x)$ for all $x \in \Omega$.*

Corollary 2.2 gives the existence of the supersolution v of (3.3) on Ω. We have to emphasize that v is independent of the set Ω, namely for each bounded domain $\Omega \subset \Omega_R$, v is the supersolution of (3.3). Our task is now to find a positive subsolution for (3.3) in Ω.

Lemma 3.3. *There exists a positive subsolution w_Ω of the problem* (3.3) *on Ω, such that $w_\Omega \leq v$ in Ω.*

Proof. To this effect we consider φ being the eigenfunction corresponding to the real eigenvalue $\lambda_1 > 0$ of the following operator $\widetilde{L}u := -\operatorname{div}(a(\|x\|)\nabla u(x)) - g(\|x\|)x \cdot \nabla u(x)$, namely

$$\begin{cases} -\operatorname{div}(a(\|x\|)\nabla \varphi(x)) - g(\|x\|)x \cdot \nabla \varphi(x) = \lambda_1 \varphi(x) \text{ on } \Omega \\ \varphi(x) = 0 \text{ on } \partial\Omega. \end{cases}$$

We know that $\varphi \in C^{2+\alpha}(\Omega) \cap C^1(\overline{\Omega})$ is positive in Ω. We show that function $w_\Omega = s\varphi^2$ with s satisfying

$$0 < s \leq \min\left\{1, \left(\frac{f_{\min}}{2\lambda_1\varphi_{\max}^2 + 2^\beta \varphi_{\max}^{\beta-2\alpha}\|\nabla\varphi_{\max}\|^\beta)}\right)^{\frac{1}{\alpha}}\right\}.$$

is a subsolution of (3.3). We start with the proof that $w_\Omega(x) < v(x) < d$ for all $x \in \Omega$, which allows us to use properties of f in $\Omega \times [0, d]$ and, in consequence, we will be able to show that w_Ω is the subsolution of (3.3) in Ω. To this effect we note that the following chain of inequalities holds

$$\begin{aligned} &-\operatorname{div}(a(\|x\|)\nabla(v(x) - w_\Omega(x))) - g(\|x\|)x \cdot \nabla(v(x) - w_\Omega(x)) \\ &\quad \geq -\operatorname{div}(a(\|x\|)\nabla v(x)) + \operatorname{div}(a(\|x\|)\nabla w_\Omega(x)) + g(\|x\|)x \cdot \nabla w_\Omega(x) \\ &\quad = M(\|x\|) + \operatorname{div}(a(\|x\|)\nabla w_\Omega(x)) + g(\|x\|)x \cdot \nabla w_\Omega(x) \\ &\quad > f_{\min} + 2s\varphi(x)[\operatorname{div}(a(\|x\|)\nabla\varphi(x)) + g(\|x\|)x\nabla\varphi(x)] + 2sa(\|x\|)\|\nabla\varphi(x)\|^2 \\ &\quad = f_{\min} - 2s\lambda_1\varphi^2(x) + 2sa(\|x\|)\|\nabla\varphi(x)\|^2 \\ &\quad \geq f_{\min} - 2s\lambda_1\varphi^2(x) \geq 0. \end{aligned}$$

for all $x \in \Omega$. By the maximum principle we get $v(x) \geq w_\Omega(x)$ on $\overline{\Omega}$.

Now we have for all $x \in \Omega$,

$$\begin{aligned} &\operatorname{div}(a(\|x\|)\nabla w_\Omega(x)) + f(x, w_\Omega(x)) - (w_\Omega(x))^{-\alpha}\|\nabla w_\Omega(x)\|^\beta + g(\|x\|)x \cdot \nabla w_\Omega(x)) \\ &\quad = 2s\varphi(x)[\operatorname{div}(a(\|x\|)\nabla\varphi(x)) + g(\|x\|)x\nabla\varphi(x)] + 2sa(\|x\|)\|\nabla\varphi(x)\|^2 + f(x, s\varphi^2(x)) \\ &\qquad - \frac{4s^\beta\varphi^\beta(x)\|\nabla\varphi(x)\|^\beta}{s^\alpha\varphi^{2\alpha}(x)} \\ &\quad = -2s\lambda_1\varphi^2(x) + 2sa(\|x\|)\|\nabla\varphi(x)\|^2 + f(x, s\varphi^2(x)) - 2^\beta s^{\beta-\alpha}\varphi^{\beta-2\alpha}(x)\|\nabla\varphi(x)\|^\beta \\ &\quad \geq -2s\lambda_1\varphi_{\max}^2 + f_{\min} - 2^\beta s^{\beta-\alpha}\varphi_{\max}^{\beta-2\alpha}\|\nabla\varphi_{\max}\|^\beta \\ &\quad \geq -2s^\alpha\lambda_1\varphi_{\max}^2 + f_{\min} - 2^\beta s^\alpha\varphi_{\max}^{\beta-2\alpha}\|\nabla\varphi_{\max}\|^\beta \geq 0. \end{aligned}$$

Finally we have the positive function $w_\Omega = s\varphi^2$, such that w_Ω is a subsolution of (3.3) on Ω. ∎

Summarizing, we proved the existence the subsolution w_Ω (Lemma 3.3) and the supersolution v (Corollary 2.2) of (3.3) on Ω such that $w_\Omega \le v$ in $\overline{\Omega}$. Thus as consequence of Lemma 3.2 we get the below result.

Theorem 3.4. *Let $\Omega \subset R^n$ be a bounded domain with $C^{2+\alpha}$ boundary such that $\Omega \subset \Omega_R$. If we assume (A_f) and (A_a), then there exists a solution $u_\Omega \in C^{2+\alpha}(\Omega)$ of (3.3) such that $w_\Omega \le u_\Omega \le v$ in Ω.*

4 Solutions on exterior domain

Theorem 4.1. *If we assume (A_f) and (A_a), then there exists a positive solution $\overline{\overline{u}} \in C^{2+\alpha}(\Omega_R)$ of the problem (1.1)–(1.2) such that*

$$0 < \overline{\overline{u}}(x) \le v(x) \le d \quad \text{for all } x \in \Omega_R, \tag{4.1}$$

$$\overline{\overline{u}}(x) = O\left(\frac{1}{\|x\|^{n-2}}\right) \quad \text{as } \|x\| \to +\infty, \tag{4.2}$$

and

$$\overline{\overline{u}}(x) = o\left(\widetilde{\varphi}(\|x\|)\right) \quad \text{as } \|x\| \to +\infty, \tag{4.3}$$

where $\widetilde{\varphi}$ is any function $\widetilde{\varphi} \in C^1(1,+\infty)$ such that $\lim_{r\to+\infty} \widetilde{\varphi}(r) = 0$ and $\lim_{r\to+\infty} \widetilde{\varphi}'(r) r^{n-1} = +\infty$.

Proof. Let us take any bounded domain $\overline{\Omega}' \subset\subset \Omega_R$ with $C^{2+\alpha}$-smooth boundary and sets Ω_1, Ω_2, Ω_3 also with $C^{2+\alpha}$-smooth boundary such that $\Omega' \subset\subset \Omega_1 \subset\subset \Omega_2 \subset\subset \Omega_3 \subset\subset B_{m_0} \cap \Omega_R$, for $B_m := \{x \in \mathbf{R}^n, \|x\| < m\}$ and m_0 sufficiently large. For each $m \in N$, Theorem 3.4 implies the existence of solution $u_m \in C^{2+\alpha}(B_m \cap \Omega_R)$ of (3.3) such that for all $m \ge m_0$,

$$0 < w_{m_0}(x) \le u_m(x) \le v(x) \quad \text{for all } x \in B_{m_0} \cap \Omega_R,$$

where w_m and v are given in Lemma 3.3 and Corollary 2.2, respectively. Let us consider the function

$$h_m(x) := f(x, u_m(x)) - (u_m(x))^{-\alpha} \|\nabla u_m(x)\|^\beta - \nabla(a(\|x\|))\nabla u_m(x) - g(\|x\|)x \cdot \nabla u_m(x))$$

for $x \in \Omega_3$. Since u_m satisfies

$$a(\|x\|)\Delta u(x) = h_m(x), \qquad x \in \Omega_3$$

we state, by the interior gradient estimate theorem of Ladyzenskaya and Ural'tseva [16], that there exists a positive constant C_1 independent of m such that

$$\max_{x\in\overline{\Omega}_2} \|\nabla u_m(x)\| \le C_1 \max_{x\in\overline{\Omega}_3} u_m(x) \le C_1 \max_{x\in\overline{\Omega}_3} v(x).$$

Therefore $(\nabla u_m)_{m=m_0}^\infty$ is uniformly bounded on $\overline{\Omega}_2$, and further, $(h_m)_{m=m_0}^\infty$ is uniformly bounded on $\overline{\Omega}_2$ which implies the boundedness of $(h_m)_{m=m_0}^\infty$ in $L^p(\Omega_2)$ for any $p > 1$. Thus (see e.g. [7, Lemma 2.3]) there exists $C_2 > 0$ independent of m, such that

$$\|u_m\|_{W^{2,p}(\Omega_1)} \le C_2 \left(\|h_m\|_{L^p(\Omega_2)} + \|u_m\|_{L^p(\Omega_2)} \right), \quad \text{for all } m \ge m_0,$$

and consequently, $(u_m)_{m=m_0}^\infty$ is bounded in $W^{2,p}(\Omega_1)$. Let us choose $p > \frac{n}{1-\alpha}$. Then Sobolev's imbedding theorem gives the existence of $C_3 > 0$ such that $\|u_m\|_{C^{1+\alpha}(\overline{\Omega}_1)} < C_3$ for all $m \ge m_0$

(see e.g. [7, Lemma 2.1]). Moreover, we get $h_m \in C^\alpha(\overline{\Omega}_1)$ and there exists $C_4 > 0$ such that $\|h_m\|_{C^\alpha(\overline{\Omega}_1)} < C_4$ for all $m \geq m_0$. Applying the Schauder estimates for solutions of elliptic equations (see e.g. [7, Lemma 2.2]) we have the existence of $C_5 > 0$ independent of m and such that for all $m \geq m_0$

$$\|u_m\|_{C^{2+\alpha}(\overline{\Omega}')} \leq C_5 \left(\|h_m\|_{C^\alpha(\overline{\Omega}_1)} + \sup_{x\in\overline{\Omega}_1} u_m(x) \right) \leq C_5 \left(C_4 + \sup_{x\in\overline{\Omega}_1} v(x) \right) =: C_6.$$

Thus, using the Ascoli–Arzelà theorem we infer the existence of a subsequence (still denoted by u_m) such that $(u_m)_{m=m_0}^\infty$ tends to $\overline{\overline{u}}$ in $C^2(\overline{\Omega}')$. It is clear that $\overline{\overline{u}} \in C^2(\overline{\Omega}')$, $0 < w_{m_0}(x) \leq \overline{\overline{u}}(x) \leq v(x)$ on $\overline{\Omega}'$ and u satisfies

$$\operatorname{div}(a(\|x\|)\nabla\overline{\overline{u}}(x)) + f(x,\overline{\overline{u}}(x)) - (\overline{\overline{u}}(x))^{-\alpha}\|\nabla\overline{\overline{u}}(x)\|^\beta + g(\|x\|)x\cdot\nabla\overline{\overline{u}}(x)) = 0$$

on $\overline{\Omega}'$. Applying the Schauder estimates for solutions of elliptic equations we have $\overline{\overline{u}} \in C^{2+\alpha}(\overline{\Omega}')$. Since Ω' was arbitrary bounded subset of Ω_R, we state that $\overline{\overline{u}} \in C^{2+\alpha}_{\mathrm{loc}}(\Omega_R)$, $0 < \overline{\overline{u}} \leq v$ in Ω_R and

$$\operatorname{div}(a(\|x\|)\nabla\overline{\overline{u}}(x)) + f(x,\overline{\overline{u}}(x)) - (\overline{\overline{u}}(x))^{-\alpha}\|\nabla\overline{\overline{u}}(x)\|^\beta + g(\|x\|)x\cdot\nabla\overline{\overline{u}}(x)) = 0$$

at each point in Ω_R. Since v satisfies (2.8), (2.9), (2.10) and $\overline{\overline{u}} \leq v$ in Ω_R, we state that (4.1), (4.2), (4.3) also hold. □

Now we give an explicit example of (1.1) to illustrate the application of Theorem 4.1.

Example 4.2. The following problem

$$\begin{cases} \operatorname{div}\left(\left(\frac{\|x\|^4}{\|x\|^4+1}\right)\nabla u(x)\right) + \frac{(x_1+x_2)^2(u(x)-5)(u(x)-6)(u(x)+1)u(x)}{80\|x\|^8} + \frac{(x_2+x_3)^2}{24\|x\|^6}e^{u(x)} \\ \qquad -(u(x))^{-\alpha}\|\nabla u(x)\|^\beta + (\|x\|^6-1)x\cdot\nabla u(x) = 0, \text{ for } x\in\Omega_R \\ \lim_{\|x\|\to\infty} u(x) = 0, \end{cases}$$

where $\Omega_R := \{x \in \mathbf{R}^3, \|x\| > R\}$, $R > 1$, possesses at least one positive solution $\overline{\overline{u}} \in C^{\alpha+2}_{\mathrm{loc}}(\Omega_R)$. Moreover,

$$0 < \overline{\overline{u}}(x) \leq 1 \quad \text{for all } x \in \Omega_R, \tag{4.4}$$

$$\overline{\overline{u}}(x) = O\left(\frac{1}{\|x\|^{n-2}}\right) \quad \text{as } \|x\| \to +\infty, \tag{4.5}$$

and

$$\overline{\overline{u}}(x) = o\left(\widetilde{\varphi}(\|x\|)\right) \quad \text{as } \|x\| \to +\infty, \tag{4.6}$$

where $\widetilde{\varphi}$ is any function $\widetilde{\varphi} \in C^1(1,+\infty)$ such that $\lim_{r\to+\infty}\widetilde{\varphi}(r) = 0$ and $\lim_{r\to+\infty}\widetilde{\varphi}'(r)r^{n-1} = +\infty$.

Proof. We start with the observation that in our case we have functions

$$a(l) = \frac{l^4}{l^4+1}, \qquad g(r) = r^6 - 1$$

and

$$f(x,u) = \frac{(x_1+x_2)^2\,(u-5)\,(u-6)\,(u+1)u}{80\|x\|^8} + \frac{(x_2+x_3)^2}{24\|x\|^6}e^u$$

which are sufficiently smooth. Moreover, we get $\lim_{l\to+\infty} a(l) = 1$ and $\int_1^\infty \frac{l^{1-n}}{a(l)} dl = \frac{6}{5}$. Thus a satisfies (A_a). It is also clear that $g(r) = r^6 - 1$ is positive for all $r > 1$, thus (A_g) holds. Our task is now to show that f satisfies (A_f). To this effect we estimate f on the product $\Omega_1 \times [0, d]$ with $d = 1$,

$$\begin{aligned} 0 \le f(x,u) &= \frac{(x_1+x_2)^2\,(u-5)\,(u-6)\,(u+1)u}{80\|x\|^8} + \frac{(x_2+x_3)^2}{24\|x\|^6} e^u \\ &\le \frac{(x_1^2+x_2^2)}{\|x\|^8} + \frac{e(x_2^2+x_3^2)}{12\|x\|^6} \le \frac{1}{\|x\|^6} + \frac{1}{4\|x\|^4} =: M(\|x\|). \end{aligned}$$

For the continuous function $M(r) := \frac{1}{r^6} + \frac{1}{4r^4}$ with $r > 1$, we have

$$\int_1^\infty r^{n-1} M(r)dr = \int_1^\infty r^2 \left(\frac{1}{r^6} + \frac{1}{4r^4}\right) dr = \frac{7}{12}.$$

Since $n = 3$, $c := (n-2)\int_1^\infty \frac{l^{1-n}}{a(l)} dl = \frac{6}{5}$ and $d = 1$, we get (1.8).

Finally, all our assumptions are satisfied. Therefore Theorem 4.1 gives the existence of positive solution $\overline{\overline{u}} \in C^{\alpha+2}_{\text{loc}}(\Omega_R)$ for which estimates (4.4), (4.5), (4.6) hold. □

Final remark. The natural question is whether the term $(u(x))^{-\alpha}\|\nabla u(x)\|^\beta$ can be replaced by more general singularity. We answer immediately that it is possible to consider the term $b(x)(u(x))^{-\alpha}\|\nabla u(x)\|^\beta$, where b is a sufficiently smooth, bounded and positive function. On the other hand, it is obvious that the approach presented in this paper can be applied only for the singular function satisfying the assumption described by Cui in [9]. His results allow us to obtain the existence of a smooth solution. It seems that more general singularities could imply less regularity of solution, e.g. in [1] we have a Carathéodory function $g(x, u)$ instead of the term $u^{-\alpha}$, where g may have a singularity at 0. In this case the authors obtain the existence of weak solutions for the similar problem.

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