Electronic Journal of Qualitative Theory of Differential Equations
2017, No. **4**, 1–9; doi: 10.14232/ejqtde.2017.1.4 http://www.math.u-szeged.hu/ejqtde/

Isospectral Dirac operators

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Received 30 December 2015, appeared 23 January 2017
Communicated by Miklós Horváth

Abstract. We give the description of self-adjoint regular Dirac operators, on $[0, \pi]$, with the same spectra.

Keywords: inverse spectral theory, Dirac operator, isospectral operators.

2010 Mathematics Subject Classification: 34A55, 34B30, 47E05.

1 Introduction and statement of result

Let p and q are real-valued, summable on $[0, \pi]$ functions, i.e. $p, q \in L^1_{\mathbb{R}}[0, \pi]$. By $L(p, q, \alpha) = L(\Omega, \alpha)$ we denote the boundary-value problem for canonical Dirac system (see [5, 6, 9, 13, 14]):

$$\ell y \equiv \left\{ B\frac{d}{dx} + \Omega(x) \right\} y = \lambda y, \quad x \in (0, \pi), \quad y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \quad \lambda \in \mathbb{C}, \tag{1.1}$$

$$y_1(0)\cos\alpha + y_2(0)\sin\alpha = 0, \quad \alpha \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right], \tag{1.2}$$

$$y_1(\pi) = 0, \tag{1.3}$$

where

$$B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \Omega(x) = \begin{pmatrix} p(x) & q(x) \\ q(x) & -p(x) \end{pmatrix}.$$

By the same $L(p, q, \alpha)$ we also denote a self-adjoint operator generated by differential expression ℓ in Hilbert space of two component vector-function $L^2([0, \pi]; \mathbb{C}^2)$ on the domain

$$D = \left\{ y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}; \, y_k \in AC[0, \pi], \, (\ell y)_k \in L^2[0, \pi], \, k = 1, 2; \right.$$
$$\left. y_1(0)\cos\alpha + y_2(0)\sin\alpha = 0, \quad y_1(\pi) = 0 \right\}$$

where $AC[0, \pi]$ is the set of absolutely continuous functions on $[0, \pi]$ (see, e.g. [13, 16]). It is well known (see [1, 5, 9]) that under these conditions the spectra of the operator $L(p, q, \alpha)$ is purely discrete and consists of simple, real eigenvalues, which we denote by $\lambda_n = \lambda_n(p, q, \alpha) =$

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$\lambda_n(\Omega,\alpha)$, $n \in \mathbb{Z}$, to emphasize the dependence of λ_n on quantities p, q and α. It is also well known (see, e.g. [1,5,9]) that the eigenvalues form a sequence, unbounded below as well as above. So we will enumerate it as $\lambda_k < \lambda_{k+1}, k \in \mathbb{Z}$, $\lambda_k > 0$, when $k > 0$ and $\lambda_k < 0$, when $k < 0$, and the nearest to zero eigenvalue we will denote by λ_0. If there are two nearest to zero eigenvalue, then by λ_0 we will denote the negative one. With this enumeration it is proved (see [1,5,9]), that the eigenvalues have the asymptotics:

$$\lambda_n(\Omega,\alpha) = n - \frac{\alpha}{\pi} + r_n, \quad r_n = o(1), \quad n \to \pm\infty. \tag{1.4}$$

In what follows, writing $\Omega \in A$ will mean $p, q \in A$. If $\Omega \in L^2_{\mathbb{R}}[0,\pi]$, then we know, (see, e.g. [9]), that instead of $r_n = o(1)$ we have:

$$\sum_{n=-\infty}^{\infty} r_n^2 < \infty. \tag{1.5}$$

Let $\varphi(x,\lambda) = \varphi(x,\lambda,\alpha,\Omega)$ be the solution of the Cauchy problem

$$\ell\varphi = \lambda\varphi, \qquad \varphi(0,\lambda) = \begin{pmatrix} \sin\alpha \\ -\cos\alpha \end{pmatrix}. \tag{1.6}$$

Since the differential expression ℓ self-adjoint, then the components $\varphi_1(x,\lambda)$ and $\varphi_2(x,\lambda)$ of the vector-function $\varphi(x,\lambda)$ we can choose real-valued for real λ. By $a_n = a_n(\Omega,\alpha)$ we denote the squares of the L^2-norm of the eigenfunctions $\varphi_n(x,\Omega) = \varphi(x,\lambda_n(\Omega,\alpha),\alpha,\Omega)$:

$$a_n = \|\varphi_n\|^2 = \int_0^\pi |\varphi_n(x,\Omega)|^2 dx, \qquad n \in \mathbb{Z}.$$

The numbers a_n are called norming constants. And by $h_n(x,\Omega)$ we will denote normalized eigenfunctions (i.e. $\|h_n(x)\| = 1$):

$$h_n(x,\Omega) = h_n(x) = \frac{\varphi_n(x,\Omega)}{\sqrt{a_n(\Omega,\alpha)}}. \tag{1.7}$$

It is known (see [5,9]) that in the case of $\Omega \in L^2_{\mathbb{R}}[0,\pi]$ the norming constants have an asymptotic form:

$$a_n(\Omega) = \pi + c_n, \qquad \sum_{n=-\infty}^{\infty} c_n^2 < \infty. \tag{1.8}$$

Definition 1.1. Two Dirac operators $L(\Omega,\alpha)$ and $L(\tilde{\Omega},\tilde{\alpha})$ are said to be isospectral, if $\lambda_n(\Omega,\alpha) = \lambda_n(\tilde{\Omega},\tilde{\alpha})$, for every $n \in \mathbb{Z}$.

Lemma 1.2. *Let $\Omega, \tilde{\Omega} \in L^1_{\mathbb{R}}[0,\pi]$ and the operators $L(\Omega,\alpha)$ and $L(\tilde{\Omega},\tilde{\alpha})$ are isospectral. Then $\tilde{\alpha} = \alpha$.*

Proof. The proof follows from the asymptotics (1.4):

$$\frac{\alpha}{\pi} = \lim_{n\to\infty}(n - \lambda_n(\Omega,\alpha)) = \lim_{n\to\infty}(n - \lambda_n(\tilde{\Omega},\tilde{\alpha})) = \frac{\tilde{\alpha}}{\pi}.$$

□

So, instead of isospectral operators $L(\Omega,\alpha)$ and $L(\tilde{\Omega},\tilde{\alpha})$, we can talk about "isospectral potentials" Ω and $\tilde{\Omega}$.

Theorem 1.3 (Uniqueness theorem)**.** *The map*

$$(\Omega, \alpha) \in L^2_{\mathbb{R}}[0,\pi] \times \left(-\frac{\pi}{2}, \frac{\pi}{2}\right] \longleftrightarrow \{\lambda_n(\Omega,\alpha),\, a_n(\Omega,\alpha);\, n \in \mathbb{Z}\}$$

is one-to-one.

Remark 1.4. It is natural to call this a Marchenko theorem, since it is an analogue of the famous theorem of V. A. Marchenko [15], in the case for Sturm–Liouville problem. The proof of this theorem for the case $p, q \in AC[0,\pi]$ there is in the paper [18]. The detailed proof for the case $p, q \in L^2_{\mathbb{R}}[0,\pi]$ there is in [7] (see also [4–6, 8, 10, 19]).

Let us fix some $\Omega \in L^2_{\mathbb{R}}[0,\pi]$ and consider the set of all canonical potentials $\tilde{\Omega} = \begin{pmatrix} \tilde{p} & \tilde{q} \\ \tilde{q} & -\tilde{p} \end{pmatrix}$, with the same spectra as Ω:

$$M^2(\Omega) = \{\tilde{\Omega} \in L^2_{\mathbb{R}}[0,\pi] : \lambda_n(\tilde{\Omega}, \tilde{\alpha}) = \lambda_n(\Omega,\alpha),\, n \in \mathbb{Z}\}.$$

Our main goal is to give the description of the set $M^2(\Omega)$ as explicit as it possible.

From the uniqueness theorem the next corollary easily follows.

Corollary 1.5. *The map*

$$\tilde{\Omega} \in M^2(\Omega) \leftrightarrow \{a_n(\tilde{\Omega}),\, n \in \mathbb{Z}\}$$

is one-to-one.

Since $\tilde{\Omega} \in M^2(\Omega)$, then $a_n(\tilde{\Omega})$ have similar to (1.8) asymptotics. Since $a_n(\Omega)$ and $a_n(\tilde{\Omega})$ are positive numbers, there exist real numbers $t_n = t_n(\tilde{\Omega})$, such that $\frac{a_n(\Omega)}{a_n(\tilde{\Omega})} = e^{t_n}$. From the latter equality and from (1.8) follows that

$$e^{t_n} = 1 + d_n, \qquad \sum_{n=-\infty}^{\infty} d_n^2 < \infty. \tag{1.9}$$

It is easy to see, that the sequence $\{t_n;\, n \in \mathbb{Z}\}$ is also from l^2, i.e. $\sum_{n=-\infty}^{\infty} t_n^2 < \infty$. Since all $a_n(\Omega)$ are fixed, then from the corollary 1.5 and the equality $a_n(\tilde{\Omega}) = a_n(\Omega)e^{-t_n}$ we will get the following corollary.

Corollary 1.6. *The map*

$$\tilde{\Omega} \in M^2(\Omega) \leftrightarrow \{t_n(\tilde{\Omega}),\, n \in \mathbb{Z}\} \in l^2$$

is one-to-one.

Thus, each isospectral potential is uniquely determined by a sequence $\{t_n;\, n \in \mathbb{Z}\}$. Note, that the problem of description of isospectral Sturm–Liouville operators was solved in [3, 11, 12, 17].

For Dirac operators the description of $M^2(\Omega)$ is given in [8]. This description has a "recurrent" form, i.e. at the first in [8] is given the description of a family of isospectral potentials $\Omega(x,t)$, $t \in \mathbb{R}$, for which only one norming constant $a_m(\Omega(\cdot,t))$ different from $a_m(\Omega)$ (namely, $a_m(\Omega(\cdot,t)) = a_m(\Omega)e^{-t}$), while the others are equal, i.e. $a_m(\Omega(\cdot,t)) = a_m(\Omega)$, when $n \neq m$.

Theorem 1.7 ([8]). *Let $t \in \mathbb{R}$, $\alpha \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and*

$$\Omega(x,t) = \Omega(x) + \frac{e^t - 1}{\theta_m(x,t,\Omega)}\{Bh_m(x,\Omega)h_m^*(x,\Omega) - h_m(x,\Omega)h_m^*(x,\Omega)B\},$$

where $\theta_n(x,t,\Omega) = 1 + (e^t - 1)\int_0^x |h_n(s,\Omega)|^2 ds$, and $*$ is a sign of transponation, e.g. $h_m^* = \begin{pmatrix} h_{m_1} \\ h_{m_2} \end{pmatrix}^* = (h_{m_1}, h_{m_2})$. Then, for arbitrary $t \in \mathbb{R}$, $\lambda_n(\Omega,t) = \lambda_n(\Omega)$ for all $n \in \mathbb{Z}$, $a_n(\Omega,t) = a_n(\Omega)$ for all $n \in \mathbb{Z}\backslash\{m\}$ and $a_m(\Omega,t) = a_m(\Omega)e^{-t}$. The normalized eigenfunctions of the problem $L(\Omega(\cdot,t),\alpha)$ are given by the formulae:

$$h_n(x,\Omega(\cdot,t)) = \begin{cases} \dfrac{e^{-t/2}}{\theta_m(x,t,\Omega)} h_m(x,\Omega), & \text{if } n = m, \\ h_n(x,\Omega) - \dfrac{(e^t-1)\int_0^x h_m^*(s,\Omega)h_n(s,\Omega)ds}{\theta_m(x,t,\Omega)} h_m(x,\Omega), & \text{if } n \neq m. \end{cases}$$

Theorem 1.7 shows that it is possible to change exactly one norming constant, keeping the others. As examples of isospectral potentials Ω and $\tilde{\Omega}$ we can present $\Omega(x) \equiv 0 = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and

$$\tilde{\Omega}(x) = \Omega_{m,t}(x) = \frac{\pi(e^t-1)}{\pi + (e^t-1)x} \begin{pmatrix} -\sin 2mx & \cos 2mx \\ \cos 2mx & \sin 2mx \end{pmatrix},$$

where $t \in \mathbb{R}$ is an arbitrary real number and $m \in \mathbb{Z}$ is an arbitrary integer.

Changing successively each $a_m(\Omega)$ by $a_m(\Omega)e^{-t_m}$, we can obtain any isospectral potential, corresponding to the sequence $\{t_m; m \in \mathbb{Z}\} \in l^2$. It follows from the uniqueness Theorem 1.3 that the sequence, in which we change the norming constants, is not important.

In [8] were used the following designations:

$$\begin{aligned} T_{-1} &= \{\ldots, 0, \ldots\}, \\ T_0 &= \{\ldots, 0, \ldots, 0, t_0, 0, \ldots, 0, \ldots\}, \\ T_1 &= \{\ldots, 0, \ldots, 0, 0, t_0, t_1, 0, \ldots, 0, \ldots\}, \\ T_2 &= \{\ldots, 0, \ldots, 0, t_{-1}, t_0, t_1, 0, \ldots, 0, \ldots\}, \\ &\vdots \\ T_{2n} &= \{\ldots, 0, 0, t_{-n}, \ldots, t_{-1}, t_0, t_1, \ldots, t_{n-1}, t_n, 0, \ldots\}, \\ T_{2n+1} &= \{\ldots, 0, t_{-n}, t_{-n+1}, \ldots, t_{-1}, t_0, t_1, \ldots, t_n, t_{n+1}, 0, \ldots\}, \\ &\vdots \end{aligned}$$

Let $\Omega(x, T_{-1}) \equiv \Omega(x)$ and

$$\Omega(x, T_m) = \Omega(x, T_{m-1}) + \triangle\Omega(x, T_m), \qquad m = 0, 1, 2, \ldots,$$

where

$$\triangle\Omega(x, T_m) = \frac{e^{t_{\tilde{m}}} - 1}{\theta_m(x, t_{\tilde{m}}, \Omega(\cdot, T_{m-1}))}[Bh_{\tilde{m}}(x, \Omega(\cdot, T_{m-1}))h_{\tilde{m}}^*(\cdot) - h_{\tilde{m}}(\cdot)h_{\tilde{m}}^*(\cdot)B],$$

where $\tilde{m} = \frac{m+1}{2}$, if m is odd and $\tilde{m} = -\frac{m}{2}$, if m is even. The arguments in others $h_{\tilde{m}}(\cdot)$ and $h_{\tilde{m}}^*(\cdot)$ are the same as in the first. And after that in [8] the following theorem was proved.

Theorem 1.8 ([8]). *Let $T = \{t_n, n \in \mathbb{Z}\} \in l^2$ and $\Omega \in L^2_{\mathbb{R}}[0,\pi]$. Then*

$$\Omega(x,T) \equiv \Omega(x) + \sum_{m=0}^{\infty} \triangle\Omega(x,T_m) \in M^2(\Omega). \tag{1.10}$$

We see, that each potential matrix $\triangle\Omega(x,T_m)$ defined by normalized eigenfunctions $h_{\tilde{m}}(x,\Omega(x,T_{m-1}))$ of the previous operator $L(\Omega(\cdot,T_{m-1}),\alpha)$. This approach we call "recurrent" description.

In this paper, we want to give a description of the set $M^2(\Omega)$ only in terms of eigenfunctions $h_n(x,\Omega)$ of the initial operator $L(\Omega,\alpha)$ and sequence $T \in l^2$. With this aim, let us denote by $N(T_m)$ the set of the positions of the numbers in T_m, which are not necessary zero, i.e.

$$\begin{aligned}
N(T_0) &= \{0\},\\
N(T_1) &= \{0,1\},\\
N(T_2) &= \{-1,0,1\},\\
&\vdots\\
N(T_{2n}) &= \{-n,-(n-1),\dots,0,\dots,n-1,n\},\\
N(T_{2n+1}) &= \{-n,-(n-1),\dots,0,\dots,n,n+1\},\\
&\vdots
\end{aligned}$$

in particular $N(T) \equiv \mathbb{Z}$. By $S(x,T_m)$ we denote the $(m+1)\times(m+1)$ square matrix

$$S(x,T_m) = \left(\delta_{ij} + (e^{t_j}-1)\int_0^x h_i^*(s)h_j(s)ds\right)_{i,j\in N(T_m)} \tag{1.11}$$

where δ_{ij} is a Kronecker symbol. By $S_p^{(k)}(x,T_m)$ we denote a matrix which is obtained from the matrix $S(x,T_m)$ by replacing the kth column of $S(x,T_m)$ by $H_p(x,T_m) = \{-(e^{t_k}-1)h_{k_p}(x)\}_{k\in N(T_m)}$ column, $p=1,2$, Now we can formulate our result as follows.

Theorem 1.9. *Let $T = \{t_k\}_{k\in\mathbb{Z}} \in l^2$ and $\Omega \in L^2_{\mathbb{R}}[0,\pi]$. Then the isospectral potential from $M^2(\Omega)$, corresponding to T, is given by the formula*

$$\Omega(x,T) = \Omega(x) + G(x,x,T)B - BG(x,x,T) = \begin{pmatrix} p(x,T) & q(x,T)\\ q(x,T) & -p(x,T)\end{pmatrix}, \tag{1.12}$$

where

$$G(x,x,T) = \frac{1}{\det S(x,T)}\sum_{k\in\mathbb{Z}}\begin{pmatrix}\det S_1^{(k)}(x,T)\\ \det S_2^{(k)}(x,T)\end{pmatrix} h_k^*(x),$$

and $\det S(x,T) = \lim_{m\to\infty}\det S(x,T_m)$ *(the same for* $\det S_p^k(x,T),\ p=1,2$*).*

In addition, for $p(x,T)$ and $q(x,T)$ we get explicit representations:

$$p(x,T) = p(x) - \frac{1}{\det S(x,T)}\sum_{k\in\mathbb{Z}}\sum_{p=1}^{2}\det S_p^{(k)}(x,T)h_{k_{(3-p)}}(x),$$

$$q(x,T) = q(x) + \frac{1}{\det S(x,T)}\sum_{k\in\mathbb{Z}}\sum_{p=1}^{2}(-1)^{p-1}S_p^{(k)}(x,T)h_{k_p}(x).$$

2 Proof of Theorem 1.9

The spectral function of an operator $L(\Omega, \alpha)$ defined as

$$\rho(\lambda) = \begin{cases} \sum\limits_{0<\lambda_n\le\lambda} \dfrac{1}{a_n(\Omega)}, & \lambda > 0, \\ -\sum\limits_{\lambda<\lambda_n\le 0} \dfrac{1}{a_n(\Omega)}, & \lambda < 0, \end{cases}$$

i.e. $\rho(\lambda)$ is left-continuous, step function with jumps in points $\lambda = \lambda_n$ equals $\frac{1}{a_n}$ and $\rho(0) = 0$.

Let $\Omega, \tilde{\Omega} \in L^2_{\mathbb{R}}[0, \pi]$ and they are isospectral. It is known (see [1,2,6,13]), that there exists a function $G(x, y)$ such that:

$$\varphi(x, \lambda, \alpha, \tilde{\Omega}) = \varphi(x, \lambda, \alpha, \Omega) + \int_0^x G(x, s)\varphi(s, \lambda, \alpha, \Omega)dt. \tag{2.1}$$

It is also known (see, e.g. [1,6,13]), that the function $G(x, y)$ satisfies to the Gelfand–Levitan integral equation:

$$G(x, y) + F(x, y) + \int_0^x G(x, s)F(s, y)ds = 0, \qquad 0 \le y \le x, \tag{2.2}$$

where

$$F(x,y) = \int_{-\infty}^{\infty} \varphi(x, \lambda, \alpha, \Omega)\varphi^*(y, \lambda, \alpha, \Omega)d[\tilde{\rho}(\lambda) - \rho(\lambda)]. \tag{2.3}$$

If the potential $\tilde{\Omega}$ from $M^2(\Omega)$ is such that only finite norming constants of the operator $L(\tilde{\Omega}, \alpha)$ are different from the norming constants of the operator $L(\Omega, \alpha)$, i.e. $a_n(\tilde{\Omega}) = a_n(\Omega)e^{-t_n}$, $n \in N(T_m)$ and the others are equal, then it means, that

$$d\tilde{\rho}(\lambda) - d\rho(\lambda) = \sum_{k\in N(T_m)} \left(\frac{1}{\tilde{a}_k} - \frac{1}{a_k}\right)\delta(\lambda - \lambda_k)d\lambda = \sum_{k\in N(T_m)} \left(\frac{e^{t_k} - 1}{a_k}\right)\delta(\lambda - \lambda_k)d\lambda, \tag{2.4}$$

where δ is Dirac δ-function. In this case the kernel $F(x, y)$ can be written in a form of a finite sum (using notation (1.7)):

$$F(x,y) = F(x, y, T_m) = \sum_{k\in N(T_m)} (e^{t_k} - 1)h_k(x, \Omega)h_k^*(y, \Omega), \tag{2.5}$$

and consequently, the integral equation (2.2) becomes to an integral equation with degenerated kernel, i.e. it becomes to a system of linear equations and we will look for the solution in the following form:

$$G(x, y, T_m) = \sum_{k\in N(T_m)} g_k(x)h_k^*(y), \tag{2.6}$$

where $g_k(x) = \binom{g_{k_1}(x)}{g_{k_2}(x)}$ is an unknown vector-function. Putting the expressions (2.5) and (2.6) into the integral equation (2.2) we will obtain a system of algebraic equations for determining the functions $g_k(x)$:

$$g_k(x) + \sum_{i \in N(T_m)} s_{ik}(x) g_i(x) = -(e^{t_k} - 1) h_k(x), \qquad k \in N(T_m), \tag{2.7}$$

where

$$s_{ik}(x) = (e^{t_k} - 1) \int_0^x h_i^*(s) h_k(s) ds.$$

It would be better if we consider the equations (2.7) for the vectors $g_k = \binom{g_{k_1}}{g_{k_2}}$ by coordinates g_{k_1} and g_{k_2} to be a system of scalar linear equations:

$$g_{k_p}(x) + \sum_{i \in N(T_m)} s_{ik}(x) g_{i_p}(x) = -(e^{t_k} - 1) h_{k_p}(x), \qquad k \in N(T_m), \quad p = 1, 2. \tag{2.8}$$

The systems (2.8) might be written in matrix form

$$S(x, T_m) g_p(x, T_m) = H_p(x, T_m), \qquad p = 1, 2, \tag{2.9}$$

where the column vectors $g_p(x, T_m) = \{g_{k_p}(x, T_m)\}_{k \in N(T_m)}$, $p = 1, 2$, and the solution can be found in the form (Cramer's rule):

$$g_{k_p}(x, T_m) = \frac{\det S_p^{(k)}(x, T_m)}{\det S(x, T_m)}, \qquad k \in N(T_m), \quad p = 1, 2.$$

Thus we have obtained for $g_k(x)$ the following representation:

$$g_k(x, T_m) = \frac{1}{\det S(x, T_m)} \begin{pmatrix} \det S_1^{(k)}(x, T_m) \\ \det S_2^{(k)}(x, T_m) \end{pmatrix} \tag{2.10}$$

and then by putting (2.10) into (2.6) we find the $G(x, y, T_m)$ function. If the potential Ω is from $L^1_{\mathbb{R}}$, then such is also the kernel $G(x, x, T_m)$ (see [8]), and the relation between them gives as follows:

$$\Omega(x, T_m) = \Omega(x) + G(x, x, T_m) B - B G(x, x, T_m). \tag{2.11}$$

On the other hand we have

$$\Omega(x, T_m) = \Omega(x) + \sum_{k=0}^{m} \triangle \Omega(x, T_k). \tag{2.12}$$

So, using the Theorem 1.8 and the equality (2.12) we can pass to the limit in (2.11), when $m \to \infty$:

$$\Omega(x, T) = \Omega(x) + G(x, x, T) B - B G(x, x, T). \tag{2.13}$$

The potentials $\Omega(x, T)$ in (1.10) and (2.13) have the same spectral data $\{\lambda_n(T), a_n(T)\}_{n \in \mathbb{Z}}$, and therefore they are the same and $\Omega(\cdot, T)$ defined by (2.13) is also from $M^2(\Omega)$.

Using (2.6) and (2.10) we calculate the expression $G(x, x, T_m) B - B G(x, x, T_m)$ and pass to the limit, obtaining for the $p(x, T)$ and $q(x, T)$ the representations:

$$p(x,T) = p(x) - \frac{1}{\det S(x,T)} \sum_{k\in N(T)} \sum_{p=1}^{2} \det S_p^{(k)}(x,T) h_{k_{(3-p)}}(x),$$

$$q(x,T) = q(x) + \frac{1}{\det S(x,T)} \sum_{k\in N(T)} \sum_{p=1}^{2} (-1)^{p-1} S_p^{(k)}(x,T) h_{k_p}(x).$$

Theorem 1.9 is proved.

For example, when we change just one norming constant (e.g. for T_0) we get two independent linear equations:

$$(1 + s_{00}(x))g_{0_1}(x) = -(e^{t_0} - 1)h_{0_1}(x),$$
$$(1 + s_{00}(x))g_{0_2}(x) = -(e^{t_0} - 1)h_{0_2}(x).$$

For the solutions we get:

$$g_{0_1}(x) = -\frac{(e^{t_0} - 1)h_{0_1}(x)}{1 + s_{00}(x)},$$
$$g_{0_2}(x) = -\frac{(e^{t_0} - 1)h_{0_2}(x)}{1 + s_{00}(x)},$$

and for the potentials $p(x,T_0)$ and $q(x,T_0)$:

$$p(x,T_0) = p(x) + \frac{e^{t_0} - 1}{1 + s_{00}(x)}(2h_{0_1}(x)h_{0_2}(x)),$$
$$q(x,T_0) = q(x) + \frac{e^{t_0} - 1}{1 + s_{00}(x)}(h_{0_2}^2(x) - h_{0_1}^2(x)).$$

Acknowledgements

This work was supported by the RA MES State Committee of Science, in the frames of the research project No.15T-1A392.

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