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Ground state solutions for a quasilinear Kirchhoff type equation

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Abstract. We study the ground state solutions of the following quasilinear Kirchhoff type equation

$$-\left(1+b\int_{\mathbb{R}^3}|\nabla u|^2dx\right)\Delta u+V(x)u-[\Delta(u^2)]u=|u|^{10}u+\mu|u|^{p-1}u,\qquad x\in\mathbb{R}^3,$$

where $b\geq 0$ and μ is a positive parameter. Under some suitable conditions on $V(x)$, we obtain the existence of ground state solutions of the above equation with $1<p<11$.

Keywords: Kirchhoff type equations, ground state solution, quasilinear, variational methods.

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1 Introduction and main results

Consider the following Kirchhoff type equation

$$-\left(1+b\int_{\mathbb{R}^3}|\nabla u|^2dx\right)\Delta u+V(x)u-[\Delta(u^2)]u=|u|^{10}u+\mu|u|^{p-1}u,\qquad x\in\mathbb{R}^3,\tag{1.1}$$

where $b\geq 0$, $1<p<11$, $\mu>0$ is a parameter and the potential $V(x)$ satisfies the following condition:

(V) $V\in C(\mathbb{R}^3,\mathbb{R})$ satisfies $\inf V(x)=V_0>0$ and for each $M>0$, $\operatorname{meas}\{x\in\mathbb{R}^3 : V(x)\leq M\}<+\infty$, where meas denotes the Lebesgue measure in $\mathbb{R}^3$.

Problem (1.1) arises in an interesting physical context. In fact, if $V(x)=0$ and replacing $\mathbb{R}^3$ by a bounded domain $\Omega\subset\mathbb{R}^3$ in (1.1), problem (1.1) without the term $[\Delta(u^2)]u$ reduces to the following Dirichlet problem of Kirchhoff type

$$\begin{cases}-\left(1+b\int_{\mathbb{R}^3}|\nabla u|^2dx\right)\Delta u=f(u), & x\in\Omega,\\ u=0, & x\in\partial\Omega,\end{cases}\tag{1.2}$$

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which is related to the stationary analogue of the equation

$$\rho \frac{\partial^2 u}{\partial t^2} - \left(\frac{P_0}{h} + \frac{E}{2L} \int_0^L \left| \frac{\partial u}{\partial x} \right|^2 dx \right) \frac{\partial^2 u}{\partial x^2} = 0 \tag{1.3}$$

presented by Kirchhoff in [7], where ρ, ρ_0, h, E and L are constants. Problem (1.3) extends the classical D'Alembert's wave equation, by considering the effects of the changes in the length of the strings during the vibrations. After Lions [10] proposed an abstract framework to the problem (1.2), many papers devoted to the existence of multiple nontrivial solutions, infinitely many solutions and ground solution to the semiliner (without the term $[\Delta(u^2)]u$) Kirchhoff type problems by applying the modern variational methods. See for instance, Liu and He [15], Wu [23], Sun and Wu [21], Chen and Li [3], Li and Ye [8], He and Zou [6], Zhang et al. [26], Zhang and Zhang [27], Liu and Guo [12–14], Liu and Chen [4, 11] and the references therein.

On the other hand, many papers concerned with the following quasilinear Schrödinger equation

$$-\Delta u + V(x)u - [\Delta(u^2)]u = h(u), \qquad x \in \mathbb{R}^N. \tag{1.4}$$

Such equations arise in various branches of mathematical physics and have been extensively studied in recent years. For example, the problem (1.4) was transformed to be a semilinear one by a change of variables and the existence of positive solutions of problem (1.4) in [18] was obtained on an Orlicz space by using the mountain pass theorem. The same method was also used in [5], but the usual Sobolev space $H^1(\mathbb{R}^N)$ framework was used as the working space. Liu, Wang and Wang [19] obtained the existence of both one sign and nodal ground state type solutions of problem (1.4) by the Nehari method. In [17], the authors presented an approach to study problem (1.4) and proved that the solutions of problem (1.4) can be obtained as limits of 4-Laplacian perturbations.

However, to the best of our knowledge, very few papers deal with problem (1.1) in the literature. More precisely, Liang and Shi [9] studied the problem (1.1) and obtained infinitely many solutions which tend to zero via a concentration-compactness principle and the minimax methods. In [24], the authors got the infinitely many small energy solutions of problem (1.1) by applying Clark's theorem.

Motivated by the reasons above, the aim of this paper is to show the existence of ground state solutions of problem (1.1). Different from the semilinear problems, the feature of the quasilinear problem is the appearance of the term $[\Delta(u^2)]u$. This makes the problem more challenging and interesting because in general there is no suitable space in which the energy functional enjoys both smoothness and compactness. Therefore, the variational methods can not be applied directly. As we shall see in the present paper, problem (1.1) can be viewed as an elliptic equation coupled with a non-local term. The competing effect of the non-local term with the critical nonlinearity and the lack of compactness of the embedding of $H^1(\mathbb{R}^3)$ into the space $L^p(\mathbb{R}^3)$, prevent us from using the variational methods in a standard way. Following the idea of [5, 18], we transform the problem to a semilinear one by a change of variables. Note that the problem (1.1) becomes problem (1.4) when $b = 0$. It is worth pointing out that although the idea was used to solve the problem (1.4) above, the adaptation to the procedure to our problem is not trivial at all since the appearance of non-local term. To obtain the ground state solution of problem (1.1), however, some more delicate estimates are needed in the present paper.

Before stating our main results we need to introduce some notations and definitions.

Notation 1.1. Throughout this paper, we denote by $\|\cdot\|_r$ the L^r-norm, $1 \le r \le \infty$, and we use the notation $\to$ ($\rightharpoonup$) to denote strong (weak) convergence. Also, if we take a subsequence of a sequence $\{u_n\}$ we shall denote it again $\{u_n\}$. We use $o(1)$ to denote any quantity which tends to zero when $n \to \infty$. C and C_i express distinct positive constants which may vary from line to line.

Definition 1.2. A nontrivial solution of problem (1.1) is called a ground state solution if its energy is minimal among the energy of all nontrivial solutions.

Now, we give our main results.

Theorem 1.3. *Suppose that condition (V) holds. Then problem (1.1) has a ground state solution for all $\mu > 0$ when $9 < p < 11$.*

Theorem 1.4. *Suppose that condition (V) holds. Then there exists $\mu^* > 0$ such that problem (1.1) has a ground state solution for all $\mu \in (\mu^*, +\infty)$ when $1 < p \le 9$.*

Remark 1.5. It should be mentioned that the authors in [16] have proved the problem (1.1) with $b = 0$ has no nontrivial solution if $x \cdot \nabla V(x) \ge 0$ and $p \ge 11$. This is the reason why we just consider the problem for $1 < p < 11$.

Remark 1.6. Compared to the previous results (see e.g. [9]), the main novelty in this paper is that we are able to obtain the existence of the ground state solution of problem (1.1) with $1 < p < 11$. Moreover, since we consider the critical case, our main results are also different from [24] in which the authors studied the nontrivial solutions.

The remainder of this paper is organized as follows. In Section 2, we present some preliminaries while the proofs of our main results is given in Section 3.

2 Preliminaries

Define the Hilbert space

$$X = \left\{ u \in H^1(\mathbb{R}^3) : \int_{\mathbb{R}^3} V(x) v^2 dx < \infty \right\}$$

with the inner product

$$\langle u, v \rangle = \int_{\mathbb{R}^3} [\nabla u \nabla v + V(x) uv] dx$$

and the norm

$$\|u\| = \langle u, u \rangle^{\frac{1}{2}}.$$

It is well known that the embedding $X \hookrightarrow L^s(\mathbb{R}^3)$ for $2 \le s < 6$ is compact under the condition (V). Problem (1.1) is the Euler–Lagrange equation associated with the natural energy functional

$$\begin{aligned} I(v) = \frac{1}{2} \int_{\mathbb{R}^3} (1 + 2v^2) |\nabla v|^2 dx + \frac{1}{2} \int_{\mathbb{R}^3} V(x) v^2 dx + \frac{b}{4} \left(\int_{\mathbb{R}^3} |\nabla v|^2 dx \right)^2 \\ - \frac{1}{12} \int_{\mathbb{R}^3} |v|^{12} dx - \frac{\mu}{p+1} \int_{\mathbb{R}^3} |v|^{p+1} dx. \end{aligned} \tag{2.1}$$

Unfortunately, I in general is not well defined on X because $\nabla(v^2)$ is not always in $L^1(\mathbb{R}^3)$. To overcome this difficulty, based on the strategy developed in [5], we introduce a C^∞-function f defined by

$$f'(t) = \frac{1}{\sqrt{1+2|f(t)|^2}}, \quad \text{for } t \in [0,+\infty)$$

and

$$f(-t) = -f(t), \quad \text{for } t \in (-\infty,0].$$

Some properties of the function f are necessary in our arguments which we list below. The corresponding proofs can be found in [5,18,25]. We omit them here.

Lemma 2.1. *The function f enjoys the following properties:*

(1) f is uniquely defined and invertible C^∞-function;

(2) $0 < f'(t) \le 1, \forall\, t \in \mathbb{R}$;

(3) $|f(t)| \le |t|, \forall\, t \in \mathbb{R}$;

(4) $f^2(t) \le \sqrt{2}|t|, \forall\, t \in \mathbb{R}$;

(5) $\frac{f(t)}{t}$ is decreasing for $t > 0$;

(6) There exists a positive constant C such that

$$|f(t)| \ge \begin{cases} C|t|, & |t| \le 1, \\ C|t|^{\frac{1}{2}}, & |t| \ge 1. \end{cases}$$

By making the change of variables $v = f(u)$, the functional I can be rewritten as

$$\begin{aligned} J(u) := I(f(u)) = \frac{1}{2}\int_{\mathbb{R}^3}|\nabla u|^2dx + \frac{1}{2}\int_{\mathbb{R}^3}V(x)f^2(u)dx + \frac{b}{4}\left(\int_{\mathbb{R}^3}(f'(u)|\nabla u|)^2dx\right)^2 \\ - \frac{1}{12}\int_{\mathbb{R}^3}|f(u)|^{12}dx - \frac{\mu}{p+1}\int_{\mathbb{R}^3}|f(u)|^{p+1}dx, \end{aligned} \tag{2.2}$$

which is well defined on X. Moreover, by Lemma 2.1, standard arguments (see e.g. Proposition 1.12 in [22]) show that $J \in C^1(X,\mathbb{R})$ and

$$\begin{aligned} \langle J'(u),\varphi\rangle = &\int_{\mathbb{R}^3}\nabla u\nabla\varphi dx + \int_{\mathbb{R}^3}V(x)f(u)f'(u)\varphi dx - \int_{\mathbb{R}^3}|f(u)|^{10}f(u)f'(u)\varphi dx \\ &- \mu\int_{\mathbb{R}^3}|f(u)|^{p-1}f(u)f'(u)\varphi dx + b\left(\int_{\mathbb{R}^3}\frac{|\nabla u|^2}{1+2f^2(u)}dx\right) \\ &\times\left(\int_{\mathbb{R}^3}\frac{\nabla u\nabla\varphi(1+2f^2(u)) - 2|\nabla u|^2f(u)f'(u)\varphi}{[1+2f^2(u)]^2}dx\right), \end{aligned} \tag{2.3}$$

for all u, $\varphi \in X$. As in [9], if u is a nontrivial critical point of J, then u is nontrivial solution of problem

$$-\Delta u + V(x)f(u)f'(u) - b\int_{\mathbb{R}^3}|f'(u)|^2|\nabla u|^2dx\cdot\left(f'(u)f''(u)|\nabla u|^2 + |f'(u)|^2\Delta u\right) = g(x,u), \tag{2.4}$$

where

$$g(x,u) = f'(u)\left[\mu|f(u)|^{p-1}f(u) + |f(u)|^{10}f(u)\right].$$

Let

$$B(\rho) = \left\{u \in X : \int_{\mathbb{R}^3}[|\nabla u|^2 + V(x)f^2(u)]dx \leq \rho^2\right\}$$

and

$$S(\rho) = \partial B(\rho) = \left\{u \in X : \int_{\mathbb{R}^3}[|\nabla u|^2 + V(x)f^2(u)]dx = \rho^2\right\}.$$

The following two lemmas show that the functional J has a mountain pass geometric structure.

Lemma 2.2. *There exist $\rho, \alpha > 0$ such that $J(u) \geq \alpha$ for all $u \in S(\rho)$.*

Proof. Since $1 < p < 11$, for any $\varepsilon > 0$, there exists a constant $C(\varepsilon) > 0$ such that

$$|t|^{p+1} \leq \varepsilon|t|^2 + C(\varepsilon)|t|^{12}, \qquad \forall\, t \in \mathbb{R}. \tag{2.5}$$

By (2.5), condition (V), Lemma 2.1 (4) and the Sobolev inequality, for $u \in S(\rho)$, it deduces that

$$\begin{aligned}
J(u) &\geq \frac{1}{2}\int_{\mathbb{R}^3}|\nabla u|^2dx + \frac{1}{2}\int_{\mathbb{R}^3}V(x)f^2(u)dx - \frac{1}{12}\int_{\mathbb{R}^3}|f(u)|^{12}dx - \frac{\mu}{p+1}\int_{\mathbb{R}^3}|f(u)|^{p+1}dx \\
&\geq \frac{1}{4}\int_{\mathbb{R}^3}|\nabla u|^2dx + \frac{1}{4}\int_{\mathbb{R}^3}V(x)f^2(u)dx + \frac{1}{4}\int_{\mathbb{R}^3}V_0f^2(u)dx \\
&\quad - \frac{\mu\varepsilon}{p+1}\int_{\mathbb{R}^3}|f(u)|^2dx - \left(\frac{1}{12} + \frac{\mu C(\varepsilon)}{p+1}\right)\int_{\mathbb{R}^3}|f(u)|^{12}dx \\
&\geq \frac{1}{4}\int_{\mathbb{R}^3}[|\nabla u|^2 + V(x)f^2(u)]dx - \left(\frac{1}{12} + \frac{\mu C(\varepsilon)}{p+1}\right)\int_{\mathbb{R}^3}|f(u)|^{12}dx \\
&\geq \frac{1}{4}\int_{\mathbb{R}^3}[|\nabla u|^2 + V(x)f^2(u)]dx - \left(\frac{1}{12} + \frac{\mu C(\varepsilon)}{p+1}\right)8\int_{\mathbb{R}^3}|u|^6dx \\
&\geq \frac{1}{4}\int_{\mathbb{R}^N}[|\nabla u|^2 + V(x)f^2(u)]dx - C\left(\int_{\mathbb{R}^3}|\nabla u|^2dx\right)^3 \\
&\geq \frac{1}{4}\rho^2 - C\rho^6,
\end{aligned}$$

for $\varepsilon > 0$ small. Choose $\rho > 0$ with $\frac{1}{4}\rho^2 - C\rho^6 = \frac{1}{8}\rho^2 := \alpha > 0$. Then $J(u) \geq \alpha$ for all $u \in S(\rho)$. The proof is completed. □

Lemma 2.3. *There exists a $u \in X$ such that $J(u) < 0$.*

Proof. Choosing $w \in X \cap L^{12}(\mathbb{R}^3)$ with $0 < |w| \leq 1$, it follows from Lemma 2.1 (5) that $f(tw) \geq f(t)w$ for $t > 0$. Hence for $t \geq 1$, by Lemma 2.1 (3) and (6), we have

$$\begin{aligned}
J(tw) &= \frac{t^2}{2}\int_{\mathbb{R}^3}|\nabla w|^2dx + \frac{1}{2}\int_{\mathbb{R}^3}V(x)f^2(tw)dx + \frac{b}{4}\left(\int_{\mathbb{R}^3}(f'(tw)|\nabla tw|)^2dx\right)^2 \\
&\quad - \frac{1}{12}\int_{\mathbb{R}^3}|f(tw)|^{12}dx - \frac{\mu}{p+1}\int_{\mathbb{R}^N}|f(tw)|^{p+1}dx \\
&\leq \frac{t^2}{2}\int_{\mathbb{R}^3}|\nabla w|^2dx + \frac{t^2}{2}\int_{\mathbb{R}^3}V(x)w^2dx + \frac{bt^4}{4}\left(\int_{\mathbb{R}^3}|\nabla w|^2dx\right)^2 \\
&\quad - \frac{t^6C}{12}\int_{\mathbb{R}^3}|w|^6dx \to -\infty, \quad \text{as } t \to +\infty,
\end{aligned}$$

which implies that there exists a large $t > 0$ such that $J(tw) < 0$. We complete the proof. □

Define the mountain pass level c of the functional J as

$$c = \inf_{\gamma\in\Gamma} \max_{r\in[0,1]} J(\gamma(t)), \tag{2.6}$$

where $\Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = 0, J(\gamma(1)) < 0\}$. Let

$$\Phi(u) = \int_{\mathbb{R}^3} \left[|\nabla u|^2 + V(x) f^2(u)\right] dx. \tag{2.7}$$

It follows from Lemma 2.2 that $J(u) \geq 0$ for $u \in B(\rho)$. This implies that $\Phi(\gamma(1)) > \rho$ for all $\gamma \in \Gamma$. Hence there exists a $t_\gamma \in (0,1)$ such that $\Phi(\gamma(t_\gamma)) = \rho$ for every $\gamma \in \Gamma$. By the definition of c, we have $c \geq \alpha > 0$, where α is given in Lemma 2.2.

3 Proof of main results

Now, we are in the position to verify the main results. To this end, a further estimate of the mountain pass level value c is necessary. We recall that the best constant S for the Sobolev embedding $D^{1,2}(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N)$ is given by

$$S = \inf_{v\in D^{1,2}(\mathbb{R}^N), \|v\|_{2^*}=1} \|\nabla v\|_2^2. \tag{3.1}$$

Consider the function w_ε defined by

$$w_\varepsilon = \frac{[N(N-2)]^{\frac{N-2}{8}}}{[\varepsilon + |x|^2]^{\frac{N-2}{4}}}, \qquad \forall\, \varepsilon > 0. \tag{3.2}$$

Let $0 < R < 1$ and $u_\varepsilon = \phi w_\varepsilon$, where ϕ is a smooth cut-off function satisfying $\phi(x) = 1$ for $|x| \leq R$ and $\phi(x) = 0$ for $|x| \geq 2R$. For any $\varepsilon > 0$, it is known that

$$-\Delta(w_\varepsilon^2) = w_\varepsilon^{\frac{2(N+2)}{N-2}}$$

and the infimum in (3.1) is achieved by the function w_ε^2. Moreover, followed by [2], a direct computation yields that

$$\int_{\mathbb{R}^N} |\nabla(u_\varepsilon^2)|^2 dx = S^{\frac{N}{2}} + O\left(\varepsilon^{\frac{N-2}{2}}\right), \tag{3.3}$$

$$\int_{\mathbb{R}^N} u_\varepsilon^{22^*} dx = S^{\frac{N}{2}} + O\left(\varepsilon^{\frac{N}{2}}\right), \tag{3.4}$$

$$\int_{\mathbb{R}^N} |\nabla u_\varepsilon|^2 dx \leq O\left(\varepsilon^{\frac{N-2}{2}} |\ln \varepsilon|\right), \tag{3.5}$$

$$\int_{\mathbb{R}^N} u_\varepsilon^2 dx = O\left(\varepsilon^{\frac{N-2}{4}}\right), \tag{3.6}$$

and

$$\int_{\mathbb{R}^N} u_\varepsilon^q dx = O\left(\varepsilon^{\frac{N}{2} - \frac{1}{8}q(N-2)}\right), \quad \forall\, 2^* < q < 22^*. \tag{3.7}$$

Taking u_ε as a test function, we have the following estimates for the level value c given in (2.6).

Lemma 3.1.

(*i*) *If* $9 < p < 11$, *then* $c < \frac{1}{6}S^{\frac{3}{2}}$ *for any* $\mu > 0$.

(*ii*) *If* $1 < p \leq 9$, *then there exists a constant* $\mu^* > 0$ *such that* $c < \frac{1}{6}S^{\frac{3}{2}}$ *for any* $\mu > \mu^*$.

Proof. Let I be the functional defined in (2.1). Then $I(u_\varepsilon)$ is well defined since $u_\varepsilon \in X \cap L^\infty(\mathbb{R}^3)$.

Firstly, we consider the case of $9 < p < 11$. From $I(u)$ and u_ε, we can define $t_\varepsilon > 0$ satisfying

$$I(t_\varepsilon u_\varepsilon) = \sup_{t \geq 0} I(tu_\varepsilon).$$

Here, we claim that there exist positive constants t_1, t_2 and ε_0 such that $t_1 \leq t_\varepsilon \leq t_2$ for all $\varepsilon \in (0, \varepsilon_0)$. Indeed, by (3.3), (3.4), (3.5) and (3.6), there exists a small $\varepsilon_2 > 0$ such that

$$\begin{aligned}
0 \leq I(t_\varepsilon u_\varepsilon) &= \sup_{t>0} I(tu_\varepsilon) \\
&= \frac{1}{2}\int_{\mathbb{R}^3} (1 + 2t^2u_\varepsilon^2)|\nabla tu_\varepsilon|^2 dx + \frac{1}{2}\int_{\mathbb{R}^3} V(x)t^2u_\varepsilon^2 dx + \frac{b}{4}\left(\int_{\mathbb{R}^3} |\nabla tu_\varepsilon|^2 dx\right)^2 \\
&\quad - \frac{1}{12}\int_{\mathbb{R}^3} |tu_\varepsilon|^{12} dx - \frac{\mu}{p+1}\int_{\mathbb{R}^3} |tu_\varepsilon|^{p+1} dx \\
&= \frac{t^2}{2}\int_{\mathbb{R}^3} |\nabla u_\varepsilon|^2 dx + \frac{t^2}{2}\int_{\mathbb{R}^3} V(x)u_\varepsilon^2 dx + \frac{t^4}{4}\int_{\mathbb{R}^3} |\nabla(u_\varepsilon^2)|^2 dx \\
&\quad + \frac{bt^4}{4}\left(\int_{\mathbb{R}^3} |\nabla u_\varepsilon|^2 dx\right)^2 - \frac{t^{12}}{12}\int_{\mathbb{R}^3} |u_\varepsilon|^{12} dx - \frac{\mu t^{p+1}}{p+1}\int_{\mathbb{R}^3} |u_\varepsilon|^{p+1} dx \\
&\leq \frac{t^2}{2}\int_{\mathbb{R}^3} |\nabla u_\varepsilon|^2 dx + \frac{t^2}{2}\int_{\mathbb{R}^3} V(x)u_\varepsilon^2 dx + \frac{t^4}{4}\int_{\mathbb{R}^3} |\nabla(u_\varepsilon^2)|^2 dx \\
&\quad + \frac{bt^4}{4}\left(\int_{\mathbb{R}^3} |\nabla u_\varepsilon|^2 dx\right)^2 - \frac{t^{12}}{12}\int_{\mathbb{R}^3} |u_\varepsilon|^{12} dx \\
&\leq t^2 + \frac{t^4}{2}(S^{\frac{3}{2}} + 1) + \frac{bt^4}{4} - \frac{t^{12}}{12}S^{\frac{3}{2}},
\end{aligned} \tag{3.8}$$

which means that $t^2 + \frac{t^4}{2}(S^{\frac{3}{2}} + 1) + \frac{bt^4}{4} \geq \frac{t^{12}}{12}S^{\frac{3}{2}}$. Thus, there exists $t_2 > 0$ small such that $t_\varepsilon \leq t_2 < 1$ for all $\varepsilon \in (0, \varepsilon_2)$. It follows from (V), (3.3), (3.4) and (3.7) that there exists $\varepsilon_1 \in (0, \varepsilon_2)$ such that

$$\begin{aligned}
I(tu_\varepsilon) &= \frac{t^2}{2}\int_{\mathbb{R}^3} |\nabla u_\varepsilon|^2 dx + \frac{t^2}{2}\int_{\mathbb{R}^3} V(x)u_\varepsilon^2 dx + \frac{t^4}{4}\int_{\mathbb{R}^3} |\nabla(u_\varepsilon^2)|^2 dx \\
&\quad + \frac{bt^4}{4}\left(\int_{\mathbb{R}^3} |\nabla u_\varepsilon|^2 dx\right)^2 - \frac{t^{12}}{12}\int_{\mathbb{R}^3} |u_\varepsilon|^{12} dx - \frac{\mu t^{p+1}}{p+1}\int_{\mathbb{R}^3} |u_\varepsilon|^{p+1} dx \\
&\geq \frac{t^4}{4}\int_{\mathbb{R}^3} |\nabla(u_\varepsilon^2)|^2 dx - \frac{t^{12}}{12}\int_{\mathbb{R}^3} |u_\varepsilon|^{12} dx - \frac{\mu t^{p+1}}{p+1}\int_{\mathbb{R}^3} |u_\varepsilon|^{p+1} dx \\
&\geq \frac{t^4}{4}S^{\frac{3}{2}} - \frac{\mu}{p+1}t^{p+1}\varepsilon^{\frac{12-p}{8}} - \frac{t^{12}}{12}S^{\frac{3}{2}}.
\end{aligned}$$

Let $k := \max_{0 \leq t \leq 1}\left(\frac{1}{4}t^4 - \frac{1}{12}t^{12}\right)S^{\frac{3}{2}}$. Then $k > 0$. We can find a small $\varepsilon_0 < \varepsilon_1$ with $\frac{\mu}{11}\varepsilon^{\frac{12-p}{8}} \leq \frac{k}{2}$ for all $\varepsilon \in (0, \varepsilon_0)$. Therefore,

$$I(t_\varepsilon u_\varepsilon) \geq \max_{0 \leq t \leq 1}\left\{\frac{1}{4}S^{\frac{3}{2}}t^4 - \frac{\mu}{p+1}\varepsilon^{\frac{12-p}{8}}t^{p+1} - \frac{1}{12}t^{12}S^{\frac{3}{2}}\right\} \geq \frac{k}{2}. \tag{3.9}$$

Combining (3.9) with (3.8) yields that

$$0 < \frac{k}{2} \le I(t_\varepsilon u_\varepsilon) \le t_\varepsilon^2 + \left(\frac{1}{2}(S^{\frac{3}{2}}+1) + \frac{b}{4}\right) t_\varepsilon^4 - \frac{1}{12} S^{\frac{3}{2}} t_\varepsilon^{12},$$

which implies that there exists a $t_1 > 0$ such that $t_\varepsilon \ge t_1$ for all $\varepsilon \in (0, \varepsilon_0)$. Hence our claim is true.

For any $\varepsilon \in (0, \varepsilon_0)$, applying (3.3)–(3.7) again, we have

$$\begin{aligned}
I(t_\varepsilon u_\varepsilon) &= \frac{t_\varepsilon^2}{2}\int_{\mathbb{R}^3} |\nabla u_\varepsilon|^2 dx + \frac{t_\varepsilon^2}{2}\int_{\mathbb{R}^3} V(x) u_\varepsilon^2 dx + \frac{t_\varepsilon^4}{4}\int_{\mathbb{R}^3} |\nabla(u_\varepsilon^2)|^2 dx \\
&\quad + \frac{bt_\varepsilon^4}{4}\left(\int_{\mathbb{R}^3} |\nabla u_\varepsilon|^2 dx\right)^2 - \frac{t_\varepsilon^{12}}{12}\int_{\mathbb{R}^3} |u_\varepsilon|^{12} dx - \frac{\mu t_\varepsilon^{p+1}}{p+1}\int_{\mathbb{R}^3} |u_\varepsilon|^{p+1} dx \\
&\le \frac{t_2^2}{2}\int_{\mathbb{R}^3} |\nabla u_\varepsilon|^2 dx + \frac{t_2^2}{2}\int_{\mathbb{R}^3} V(x) u_\varepsilon^2 dx + \frac{t_\varepsilon^4}{4}\int_{\mathbb{R}^3} |\nabla(u_\varepsilon^2)|^2 dx \\
&\quad + \frac{bt_\varepsilon^4}{4}\left(\int_{\mathbb{R}^3} |\nabla u_\varepsilon|^2 dx\right)^2 - \frac{t_\varepsilon^{12}}{12}\int_{\mathbb{R}^3} |u_\varepsilon|^{12} dx - \frac{\mu t_1^{p+1}}{p+1}\int_{\mathbb{R}^3} |u_\varepsilon|^{p+1} dx \\
&\le \frac{t_2^2}{2} O(\varepsilon^{\frac{1}{4}}|\ln\varepsilon|) + \frac{t_2^2}{2}|V|_\infty O(\varepsilon^{\frac{1}{4}}) + \frac{t_\varepsilon^4}{4}\left[S^{\frac{3}{2}} + O(\varepsilon^{\frac{3}{2}})\right] \\
&\quad + \frac{bt_\varepsilon^4}{4}\left[O(\varepsilon^{\frac{1}{4}}|\ln\varepsilon|)\right]^2 - \frac{t_\varepsilon^{12}}{12}\left[S^{\frac{3}{2}} + O(\varepsilon^{\frac{3}{2}})\right] - CO(\varepsilon^{\frac{12-p}{8}}) \\
&\le \frac{1}{6} S^{\frac{3}{2}} + C\left[O(\varepsilon^{\frac{1}{4}}|\ln\varepsilon|) - O(\varepsilon^{\frac{12-p}{8}})\right] \\
&\le \frac{1}{6} S^{\frac{3}{2}}.
\end{aligned}$$

Hence we can find a small $\bar{\varepsilon} > 0$ such that

$$\sup_{t\ge 0} J(f^{-1}(tu_{\bar{\varepsilon}})) = \sup_{t\ge 0} I(tu_{\bar{\varepsilon}}) = I(t_{\bar{\varepsilon}} u_{\bar{\varepsilon}}) \le \frac{1}{6} S^{\frac{3}{2}}.$$

Moreover, we conclude from (3.8) that $J(f^{-1}(tu_{\bar{\varepsilon}})) = I(tu_{\bar{\varepsilon}}) \to -\infty$ as $t \to +\infty$, which shows that there exists a $\bar{t} > 0$ such that $J(f^{-1}(\bar{t}u_{\bar{\varepsilon}})) < 0$. Let $\bar{\gamma}(t) = f^{-1}(t\bar{t}u_{\bar{\varepsilon}})$. Then $\bar{\gamma} \in \Gamma$ and for any $\mu > 0$,

$$c \le \max_{0\le t\le 1} J(\bar{\gamma}(t)) < \frac{1}{6} S^{\frac{3}{2}}.$$

Secondly, we consider the other case $1 < p \le 9$. Now, we rewrite the functional I as I_μ. Let $u_0 \in C_0^\infty(\mathbb{R}^N)$ with $u_0 \ne 0$ and define $t_\mu > 0$ such that $I_\mu(t_\mu u_0) = \sup_{t\ge 0} I_\mu(tu_0)$. We claim that $t_\mu \to 0$ as $\mu \to +\infty$. Indeed, if the assertion does not hold, then there exists a constant $t_0 > 0$ and a sequence $\{\mu_n\}$ such that $\mu_n \to +\infty$ and $t_{\mu_n} \ge t_0$ for all n. Without loss of generality, we assume that $\mu_n \ge 1$ for all n. Let $t_n = t_{\mu_n}$ and $I_1 = I_\mu|_{\mu=1}$. Then $0 \le I_{\mu_n}(t_n u_0) \le I_1(t_n u_0)$ for all n, which implies that t_n is bounded from above. Moreover, we have

$$\begin{aligned}
I_{\mu_n}(t_n u_0) &= \frac{t_n^2}{2}\int_{\mathbb{R}^3} |\nabla u_0|^2 dx + \frac{t_n^2}{2}\int_{\mathbb{R}^N} V(x) u_0^2 dx + \frac{t_n^4}{4}\int_{\mathbb{R}^N} |\nabla(u_0^2)|^2 dx \\
&\quad + \frac{bt_n^4}{4}\left(\int_{\mathbb{R}^3} |\nabla u_0|^2 dx\right)^2 - \frac{t_n^{12}}{12}\int_{\mathbb{R}^3} |u_0|^{12} dx - \frac{\mu_n t_n^{p+1}}{p+1}\int_{\mathbb{R}^3} |u_0|^{p+1} dx
\end{aligned}$$

$$\leq \frac{t_n^2}{2}\int_{\mathbb{R}^3}|\nabla u_0|^2dx+\frac{t_n^2}{2}\int_{\mathbb{R}^3}V(x)u_0^2dx+\frac{t_n^4}{4}\int_{\mathbb{R}^3}|\nabla(u_0^2)|^2dx$$
$$+\frac{bt_n^4}{4}\left(\int_{\mathbb{R}^3}|\nabla u_0|^2dx\right)^2-\frac{\mu_n t_n^{p+1}}{p+1}\int_{\mathbb{R}^3}|u_0|^{p+1}dx$$
$$\leq C-\frac{\mu_n t_n^{p+1}}{p+1}\int_{\mathbb{R}^3}|u_0|^{p+1}dx\to-\infty,\quad \text{as } n\to\infty,$$

which contradicts $I_{\mu_n}(t_n u_0)\geq 0$. Hence the claim holds. Since $t_\mu\to 0$ as $\mu\to+\infty$ and

$$I_\mu(t_\mu u_0)\leq\frac{t_\mu^2}{2}\int_{\mathbb{R}^3}|\nabla u_0|^2dx+\frac{t_\mu^2}{2}\int_{\mathbb{R}^3}V(x)u_0^2dx+\frac{t_\mu^4}{4}\int_{\mathbb{R}^3}|\nabla(u_0^2)|^2dx,$$

we have $I_\mu(t_\mu u_0)\to 0$ as $\mu\to+\infty$ and hence there exists a $\mu^*>0$ such that $\sup_{t\geq 0}I_\mu(tu_0)<\frac{1}{6}S^{\frac{3}{2}}$ for all $\mu>\mu^*$. This implies that $c<\frac{1}{6}S^{\frac{3}{2}}$ for all $\mu>\mu^*$. The proof is completed. □

Recall that, for any $c\in\mathbb{R}$, we say $\{u_n\}$ is a $(C)_c$ sequence of J if $J(u_n)\to c$ and $(1+\|u_n\|)J'(u_n)\to 0$ as $n\to\infty$. In order to obtain the existence of ground state solutions, we need to study some behaviors of a $(C)_c$ sequence of J carefully.

Lemma 3.2. *Let $c\in\mathbb{R}$ and $\{u_n\}\subset X$ be a $(C)_c$ sequence of J. Then $\{\Phi(u_n)\}$ is bounded, where Φ is defined in (2.7). In particular, $\{u_n\}$ is bounded in $H^1(\mathbb{R}^3)$.*

Proof. Let $w_n=\frac{f(u_n)}{f'(u_n)}$. Then Lemma 2.1 (3) and (4) imply that

$$\nabla w_n=\left(1+\frac{2f^2(u_n)}{1+2f^2(u_n)}\right)\nabla u_n.$$

Hence, $\langle J'(u_n),w_n\rangle\to 0$ as $n\to\infty$.

Define two real functions $g(t)=|t|^{10}t+\mu|t|^{p-1}t$ and $G(t)=\int_0^t g(s)ds$. Then there exists a constant $\lambda\in(4,12)$ such that

$$\lim_{|t|\to 0}\frac{tg(t)-\lambda G(t)}{t^2}=0\quad\text{and}\quad\lim_{|t|\to\infty}\frac{tg(t)-\lambda G(t)}{t^\lambda}=+\infty.$$

Therefore, there exists $r>0$ such that

$$tg(t)-\lambda G(t)\geq 0,\qquad\forall\,|t|\geq r.\tag{3.10}$$

Moreover, for any $\varepsilon>0$, there exists a positive constant $C(\varepsilon)$ such that

$$|tg(t)-\lambda G(t)|\leq\varepsilon|t|^2+C(\varepsilon)|t|^{12},\qquad\forall\,t\in\mathbb{R}.\tag{3.11}$$

Then it can be deduced from (3.10) that

$$c+o(1)=J(u_n)-\frac{1}{\lambda}\langle J'(u_n),w_n\rangle$$
$$=\frac{1}{2}\int_{\mathbb{R}^3}|\nabla u_n|^2dx-\frac{1}{\lambda}\int_{\mathbb{R}^3}\left(1+\frac{2f^2(u_n)}{1+2f^2(u_n)}\right)|\nabla u_n|^2dx+\left(\frac{1}{2}-\frac{1}{\lambda}\right)\int_{\mathbb{R}^3}V(x)f^2(u_n)dx$$
$$+\left(\frac{b}{4}-\frac{b}{\lambda}\right)\left(\int_{\mathbb{R}^3}(f'(u_n)|\nabla u_n|)^2dx\right)^2+\int_{|f(u_n)|>r}\left[\frac{1}{\lambda}g(f(u_n))f(u_n)-G(f(u_n))\right]dx$$
$$+\int_{|f(u_n)|\leq r}\left[\frac{1}{\lambda}g(f(u_n))f(u_n)-G(f(u_n))\right]dx$$

$$\geq \left(\frac{1}{2}-\frac{2}{\lambda}\right)\int_{\mathbb{R}^3}|\nabla u_n|^2dx + 2\left(\frac{1}{4}-\frac{1}{2\lambda}\right)\int_{\mathbb{R}^3}V(x)f^2(u_n)dx$$
$$+\int_{|f(u_n)|\leq r}\left[\frac{1}{\lambda}g(f(u_n))f(u_n)-G(f(u_n))\right]dx. \tag{3.12}$$

From (3.11), there exists a constant $M > V_0$ such that

$$\left|\frac{1}{\lambda}tg(t)-G(t)\right| \leq \left(\frac{1}{4}-\frac{1}{2\lambda}\right)M|t|^2, \qquad \forall\,|t|\leq r, \tag{3.13}$$

where V_0 is the number given in the condition (V). Let $\mathcal{A} = \{x \in \mathbb{R}^3 : V(x) \leq M\}$. Then it follows from condition (V) that meas$(\mathcal{A}) < \infty$. By (3.13) and condition (V), we have

$$\begin{aligned}
&\left(\frac{1}{4}-\frac{1}{2\lambda}\right)\int_{\mathbb{R}^3}V(x)f^2(u_n)dx + \int_{|f(u_n)|\leq r}\left[\frac{1}{\lambda}g(f(u_n))f(u_n)-G(f(u_n))\right]dx\\
&\quad\geq \left(\frac{1}{4}-\frac{1}{2\lambda}\right)\int_{\{|f(u_n)|\leq r\}}(V(x)-M)f^2(u_n)dx\\
&\quad\geq \left(\frac{1}{4}-\frac{1}{2\lambda}\right)\int_{\{|f(u_n)|\leq r, V(x)\leq M\}}(V(x)-M)r^2dx\\
&\quad\geq \left(\frac{1}{4}-\frac{1}{2\lambda}\right)\cdot \mathrm{meas}\left(\mathcal{A}\cap\{x\in\mathbb{R}^N : |f(u_n)|\leq r\}\right)\cdot(V_0-M)r^2\\
&\quad\geq \left(\frac{1}{4}-\frac{1}{2\lambda}\right)\cdot \mathrm{meas}\,(\mathcal{A})\cdot(V_0-M)r^2.
\end{aligned}$$

This combining with (3.12) implies that

$$\begin{aligned}
&\left(\frac{1}{2}-\frac{1}{\lambda}\right)\int_{\mathbb{R}^3}|\nabla u_n|^2dx + \left(\frac{1}{4}-\frac{1}{2\lambda}\right)\int_{\mathbb{R}^3}V(x)f^2(u_n)dx\\
&\quad\leq \left(\frac{1}{4}-\frac{1}{2\lambda}\right)\cdot \mathrm{meas}\,(\mathcal{A})\cdot(M-V_0)r^2 + c + o(1).
\end{aligned}$$

which means that

$$\int_{\mathbb{R}^N}\left[|\nabla u_n|^2 + V(x)f^2(x)\right]dx < +\infty. \tag{3.14}$$

In particular, by Lemma 2.1 (6) and (3.14), we have

$$\begin{aligned}
\int_{\mathbb{R}^3}|u_n|^2dx &= \int_{\{|u_n|\leq 1\}}|u_n|^2dx + \int_{\{|u_n|>1\}}|u_n|^2dx\\
&\leq C_1\int_{\mathbb{R}^3}V(x)f^2(u_n)dx + \int_{\mathbb{R}^3}|u_n|^6dx\\
&\leq C_1\int_{\mathbb{R}^3}V(x)f^2(u_n)dx + C_2\left(\int_{\mathbb{R}^3}|\nabla u_n|^2dx\right)^3\\
&< +\infty,
\end{aligned}$$

which together with (3.14) implies that $\{u_n\}$ is bounded in $H^1(\mathbb{R}^N)$. The proof is completed. □

Lemma 3.3. *Let $\{u_n\} \subset X$ be a $(C)_c$ sequence of J. If $c < \frac{1}{6}S^{\frac{3}{2}}$, then there exist $R, \xi > 0$ and a sequence $\{y_n\} \subset \mathbb{R}^3$ such that*

$$\limsup_{n\to\infty}\int_{B_R(y_n)}f^2(u_n)dx \geq \xi.$$

Proof. Arguing by contradiction, we suppose that the conclusion is not true, i.e.,

$$\limsup_{n\to\infty}\int_{B_R(y_n)} f^2(u_n)dx = 0.$$

Then Lion's concentration compactness principle (Lemma 1.21 in [22]) implies that

$$f(u_n)\to 0, \quad \text{in } L^s(\mathbb{R}^3) \text{ for all } s\in(2,6). \tag{3.15}$$

By Lemma 2.1 (4), Lemma 3.2 and the interpolation, we have

$$\int_{\mathbb{R}^3}|f(u_n)|^s\to 0, \qquad \forall s\in(2,12). \tag{3.16}$$

In view of Lemma 3.2, passing to a subsequence, we may assume that

$$\int_{\mathbb{R}^3}\left(1+\frac{2f^2(u_n)}{1+2f^2(u_n)}\right)|\nabla u_n|^2dx+\int_{\mathbb{R}^3}V(x)f^2(u_n)dx+b\left[\int_{\mathbb{R}^3}(f'(u_n)|\nabla u_n|)^2dx\right]^2\to B \tag{3.17}$$

and

$$\int_{\mathbb{R}^3}|f(u_n)|^{12}dx\to D. \tag{3.18}$$

It follows from the definition of S that

$$\begin{aligned}
S\left(\int_{\mathbb{R}^3}|f(u_n)|^{12}dx\right)^{\frac{1}{3}} &\le \int_{\mathbb{R}^3}|\nabla f^2(u_n)|^2dx=\int_{\mathbb{R}^3}\frac{4f^2(u_n)}{1+2f^2(u_n)}|\nabla u_n|^2dx\\
&\le \int_{\mathbb{R}^3}\left(1+\frac{2f^2(u_n)}{1+2f^2(u_n)}\right)|\nabla u_n|^2dx\\
&\quad+\int_{\mathbb{R}^3}V(x)f^2(u_n)dx+b\left[\int_{\mathbb{R}^3}(f'(u_n)|\nabla u_n|)^2dx\right]^2,
\end{aligned}$$

which combining with (3.17) and (3.18) yields that $SD^{\frac{1}{3}}\le B$. In addition, from (3.16), (3.17) and (3.18) we have

$$0=\lim_{n\to\infty}\langle J'(u_n),w_n\rangle = B-D,$$

where $w_n=\frac{f(u_n)}{f'(u_n)}$. Hence, $B=D\ge S^{\frac{3}{2}}$. Moreover, we deduce that

$$\begin{aligned}
c&=\lim_{n\to\infty}J(u_n)\\
&=\lim_{n\to\infty}\left(\frac{1}{2}\int_{\mathbb{R}^3}|\nabla u_n|^2dx+\frac{1}{2}\int_{\mathbb{R}^3}V(x)f^2(u_n)dx+\frac{b}{4}\left(\int_{\mathbb{R}^3}(f'(u_n)|\nabla u_n|)^2dx\right)^2\right)\\
&\quad+\lim_{n\to\infty}\left(-\frac{1}{12}\int_{\mathbb{R}^3}|f(u_n)|^{12}dx-\frac{\mu}{p+1}\int_{\mathbb{R}^N}|f(u_n)|^{p+1}dx\right)\\
&\ge\lim_{n\to\infty}\left(\frac{1}{4}\int_{\mathbb{R}^3}(1+\frac{2f^2(u_n)}{1+2f^2(u_n)})|\nabla u_n|^2dx+\frac{1}{4}\int_{\mathbb{R}^3}V(x)u_n^2dx\right)\\
&\quad+\lim_{n\to\infty}\left(\frac{b}{4}\left(\int_{\mathbb{R}^3}(f'(u_n)|\nabla u_n|)^2dx\right)^2-\frac{1}{12}\int_{\mathbb{R}^3}|f(u_n)|^{12}dx\right)\\
&=\left(\frac{1}{4}-\frac{1}{12}\right)D\\
&\ge\frac{1}{6}S^{\frac{3}{2}},
\end{aligned}$$

which contradicts $c<\frac{1}{6}S^{\frac{3}{2}}$. The proof is completed. □

In what follows, we shall give the proof of Theorem 1.3 and Theorem 1.4. Since the proofs of them are similar, we just give the details of Theorem 1.3.

Proof of Theorem 1.3. Let c be the mountain pass level given in (2.6). From Lemma 2.2, Lemma 2.3 and the mountain pass theorem (see e.g. Theorem 3 in [20]), the functional J has a $(C)_c$ sequence $\{u_n\} \subset X$. In view of Lemma 3.2, we may assume that $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^3)$ and $f(u_n) \rightharpoonup f(u)$ in X, which implies that $u_n \to u$ in $L^s_{\text{loc}}(\mathbb{R}^3)$ for $2 < s < 6$ and $f(u_n) \to f(u)$ in $L^s_{\text{loc}}(\mathbb{R}^3)$ for $2 < s < 12$. Hence $\langle J'(u_n), \varphi\rangle \to \langle J'(u), \varphi\rangle = 0$ for any $\varphi \in C_0^\infty(\mathbb{R}^3)$, that is, u is a weak solution of (2.4). Moreover, since the embedding $X \hookrightarrow L^2(\mathbb{R}^3)$ is compact for $2 \leq s < 6$, we get $f(u_n) \to f(u)$ in $L^2(\mathbb{R}^3)$ for $2 \leq s < 6$. We conclude from Lemma 3.1 (i) that $c < \frac{1}{6}S^{\frac{3}{2}}$ for $p = 10$, $\mu > 0$. By Lemma 3.3, there exists a constant $\xi > 0$ such that

$$\int_{\mathbb{R}^3} f^2(u) = \lim_{n\to\infty} \int_{\mathbb{R}^3} f^2(u_n) \geq \xi,$$

which shows that u is a nontrivial solution of problem (2.4). Hence $u = f(v)$ is a nontrivial solution of problem (1.1). Finally, letting $d = \inf\{J(u) : u \in X, u \neq 0, J'(u) = 0\}$, we know that d is achieved by the lower semi-continuity. The proof is completed. □

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