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A PRIORI ESTIMATE FOR DISCONTINUOUS SOLUTIONS OF A SECOND ORDER LINEAR HYPERBOLIC PROBLEM

S. S. Akhiev
(Azerbaijan State Pedagogical University)

Abstract. *In the paper we investigate a non-local contact-boundary value problem for a system of second order hyperbolic equations with discontinuous solutions. Under some conditions on input data a priori estimate is obtained for the solution of this problem.*

In the paper we consider the following hyperbolic system:

$$(Lz)(t,x) \equiv z_{tx}(t,x) + z(t,x)\,A_{0,0}(t,x) +$$

$$+z_t(t,x)\,A_{1,0}(t,x) + z_x(t,x)\,A_{0,1}(t,x) = \varphi(t,x)\,, \tag{1}$$

$$(t,x) \in G = G_0 \cup G_1,\ G_0 = (0,T)\times(0,\alpha)\,,\ G_1 = (0,T)\times(\alpha,l)\,,$$

where $z(t,x) = (z_1(t,x),....,z_n(t,x))$ is the desired vector-function; $A_{i,j}(t,x)$, $i,j = 0,1$ are the given $n\times n$-matrices on G; $\varphi(t,x)$ is the given n-dimensional vector-function on G; α is a fixed point from $(0,l)$.

For the system (1) we give the following non-local contact boundary conditions

$$(L_k z)(t) \equiv z(t,0)\,\beta_{0,k}(t) + z(t,\alpha-0)\,\beta_{1,k}(t) + z(t,\alpha+0)\,\beta_{2,k}(t) +$$

$$+z(t,l)\,\beta_{3,k}(t) + z_t(t,0)\,g_{0,k}(t) + z_t(t,\alpha-0)\,g_{1,k}(t) + z_t(t,\alpha+0)\,g_{2,k}(t) +$$

$$+z_t(t,l)\,g_{3,k}(t) = \varphi_k(t)\ ,\quad t\in(0,T)\ ,\quad k=1,2\,; \tag{2}$$

$$(L_3 z)(x) \equiv z_x(0,x) = \varphi_3(x)\ \ ,\ x\in(0,l)\,; \tag{3}$$

$$L_0 z \equiv z(0,0) = \varphi_0. \tag{4}$$

Here: $\beta_{i,k}(t)$, $g_{i,k}(t)\,,\ i=0,1,2,3;\ k=1,2$ are the given $n\times n$ matrices on $(0,T)$; $\varphi_k(t)\,, k=1,2$ are the given n-dimensional vector-functions on $(0,T)\,;$ $\varphi_3(x)$ is the given n-dimensional vector-function on $(0,l)\,;$ φ_0 is the given constant n-dimensional vector.

We assume that the following conditions are satisfied:

1) The matrices $A_{i,j}(t,x)$ are measurable on G, $A_{0,0} \in \mathcal{L}_{p,n\times n}(G)$; there exit the functions $A^0_{1,0} \in \mathcal{L}_p(0,l)$ and $A^0_{0,1} \in \mathcal{L}_p(0,T)$, such that $\|A_{1,0}(t,x)\| \leq A^0_{1,0}(x)$, $\|A_{0,1}(t,x)\| \leq A^0_{0,1}(t)$ almost everywhere on G, where $\mathcal{L}_{p,n\times n}(G)$, $1 \leq p \leq \infty$ is a Banach space of $n \times n$ matrices $g = (g_{ij})$ with elements $g_{ij} \in \mathcal{L}_p(G)$, wherein the norm is determined by the equality $\|g\|_{\mathcal{L}_{p,n\times n}(G)} = \|g^0\|_{\mathcal{L}_p(G)}$, moreover $g^0 = \|g\| \equiv \sum\limits_{i,j=1}^{n} |g_{ij}|$ is the norm of the matrix g;

2) $\beta_{i,k} \in \mathcal{L}_{p,n\times n}(0,T)$ and $g_{i,k} \in \mathcal{L}_{\infty,n\times n}(0,T)$;

3) $\varphi \in \mathcal{L}_{p,n}(G)$, $\varphi_k \in \mathcal{L}_{p,n}(0,T)$, $\varphi_3 \in \mathcal{L}_{p,n}(0,l)$, where $\mathcal{L}_{p,n}(G)$, $1 \leq p \leq \infty$, is a space of n-dimensional vector-functions $\varphi = (\varphi_1, ..., \varphi_n)$ with elements from $\mathcal{L}_p(G)$; norm of $\varphi \in \mathcal{L}_{p,n}(G)$ is defined as $\|\varphi\|_{\mathcal{L}_{p,n}(G)} = \|\varphi^0\|_{\mathcal{L}_p(G)}$ and $\varphi^0(t,x) = \|\varphi(t,x)\| = \sum\limits_{i=1}^{n} |\varphi_i(t,x)|$ is norm of n-vectors $\varphi(t,x) \in R^n$ for fixed $(t,x) \in G$. R^n is the space of all vectors $\rho = (\rho_1, \ldots, \rho_n)$ with norm $\|\rho\| = \sum\limits_{i=1}^{n} |\rho_i|$.

Non-local boundary value problems for integro-differential equations with continuous coefficients were studied in the paper [2].

We'll consider the solution of problem (1)-(4) in the space $\widehat{W}_{p,n}(G), 1 \leq p \leq \infty$, [4] (p. 52) of all n-dimensional vector-functions $z(t,x)$, which on each domain G_k $(k = 0,1)$ belong to $W_{p,n}(G_k)$ and are continuous at the point $(0,\alpha)$. Here $W_{p,n}(G_k)$ is a space of all n-dimensional vector-functions $z \in \mathcal{L}_{p,n}(G_k)$, possessing generalized in S.L.Sobolev's sense derivatives z_t, z_x and z_{tx} from $\mathcal{L}_{p,n}(G_k)$, $k = 0,1$. We'll define the norm in the space $\widehat{W}_{p,n}(G)$ by the equality [4] (p. 54)

$$\|z\|_{\widehat{W}_{p,n}(G)} = \sum_{k=0}^{1} \|z\|_{W_{p,n}(G_k)},$$

where

$$\|z\|_{W_{p,n}(G_k)} = \|z\|_{\mathcal{L}_{p,n}(G_k)} + \|z_t\|_{\mathcal{L}_{p,n}(G_k)} + \|z_x\|_{\mathcal{L}_{p,n}(G_k)} + \|z_{tx}\|_{\mathcal{L}_{p,n}(G_k)}.$$

Since the operator $Nz = (z(0,0), z_t(t,0), z_t(t,\alpha+0), z_x(0,x), z_{tx}(t,x))$, brings about isomorphism from $\widehat{W}_{p,n}(G)$ to $\widehat{Q}_{p,n} = R^n \times \mathcal{L}_{p,n}(0,T) \times \mathcal{L}_{p,n}(0,T) \times \mathcal{L}_{p,n}(0,l) \times \mathcal{L}_{p,n}(G)$, [1], we can reduce problem (1) - (4) to the following operator equation

$$\widehat{L}z = \widehat{\varphi},$$

where $\widehat{L} = (L_0, L_1, L_2, L_3, L)$, $z \in \widehat{W}_{p,n}(G)$ is desired solution and $\widehat{\varphi} = (\varphi_0, \varphi_1, \varphi_2, \varphi_3, \varphi) \in \widehat{Q}_{p,n}$ is the given element. This equation is equivalent to the system of integro-algebraic equations with respect to elements of five components

$$\widehat{b} = (b_0, b_1(t), b_2(t), b_3(x), b(t,x)) \equiv$$
$$\equiv (z(0,0), z_t(t,0), z_t(t,a+0), z_x(0,x), z_{tx}(t,x))$$

of the space $\widehat{Q}_{p,n}$:

$$b(t,x) + \int_0^t \int_0^x b(\tau,\zeta)\, q_1(\zeta,x)\, A_{0,0}(t,x)\, d\tau d\zeta +$$
$$+ \int_0^x b(t,\zeta)\, q_1(\zeta,x)\, A_{1,0}(t,x)\, d\zeta + \int_0^t b(\tau,x)\, A_{0,1}(t,x)\, d\tau +$$
$$+ \left(\int_0^t b_1(\tau)\, \theta(\alpha - x)\, d\tau + \int_0^t b_2(\tau)\, \theta(x-\alpha)\, d\tau \right) A_{0,0}(t,x) +$$
$$+ (b_1(t)\, \theta(\alpha - x) + b_2(t)\, \theta(x-\alpha))\, A_{1,0}(t,x) + b_3(x)\, A_{0,1}(t,x) +$$
$$+ \int_0^x b_3(\zeta)\, A_{0,0}(t,x)\, d\zeta + b_0 A_{0,0}(t,x) = \varphi(t,x), \quad (t,x) \in G; \tag{5}$$

$$b_1(t)\, (g_{0,k}(t) + g_{1,k}(t)) + b_2(t)\, (g_{2,k}(t) + g_{3,k}(t)) +$$
$$+ \int_0^t b_1(\tau)\, (\beta_{0,k}(t) + \beta_{1,k}(t))\, d\tau +$$
$$+ \int_0^t b_2(\tau)\, (\beta_{2,k}(t) + \beta_{3,k}(t))\, d\tau = \varphi_k^{\,0}(t), \quad t \in (0,T), \quad k = 1,2; \tag{6}$$

where

$$\varphi_k^0(t) = \varphi_k(t) - b_0(\beta_{0,k}(t) + \beta_{1,k}(t) + \beta_{2,k}(t) + \beta_{3,k}(t)) -$$
$$- \int_0^\alpha b_3(\zeta)(\beta_{1,k}(t) + \beta_{2,k}(t)) d\zeta - \int_0^l b_3(\zeta)\beta_{3,k}(t) d\zeta - \varphi_{k,b}(t);$$
$$\varphi_{k,b}(t) = \int_0^t \int_0^\alpha b(\tau,\zeta)\beta_{1,k}(t) d\tau d\zeta + \int_0^t \int_\alpha^l b(\tau,\zeta)\beta_{3,k}(t) d\tau d\zeta +$$
$$+ \int_0^\alpha b(t,\zeta) g_{1,k}(t) d\zeta + \int_\alpha^l b(t,\zeta) g_{3,k}(t) d\zeta, t \in (0,T); \tag{6*}$$

$$b_3(x) = \varphi_3(x), \ x \in (0,l); \tag{7}$$

$$b_0 = \varphi_0, \tag{8}$$

where $\theta(y)$ is one –dimensional Heaviside function on $R = R^1$ and $q_1(\zeta,x) = \theta(\zeta - \alpha)\, \theta(x-\alpha) + \theta(\alpha - x)$.

If we succeed to estimate the components $b_0, b_1(t), b_2(t), b_3(x), b(t,x)$ of the vector $\widehat{b}$, on the basis of [1] we get a priori estimate for the solution $z \in \widehat{W}_{p,n}(G)$ of problem (1)-(4)

$$z(t,x) = b_0 + \theta(\alpha - x)\int_0^T b_1(\tau)\theta(t-\tau)\,d\tau +$$

$$+\theta(x-\alpha)\int_0^T b_2(\tau)\theta(t-\tau)\,d\tau + \int_0^l b_3(\zeta)\theta(x-\zeta)\,d\zeta +$$

$$+\int_G\int \theta(t-\tau)\,\theta(x-\zeta)\,q_1(\zeta,x)\,b(\tau,\zeta)d\tau d\zeta\,,\ (t,x)\in G. \tag{9}$$

The components $b_1(t), b_2(t), b(t,x)$ are determined from the system of equations (5),(6), since the components $b_0, b_3(x)$ are explicitly given by conditions (7), (8), therefore, it remains to estimate only $b_1(t), b_2(t), b(t,x)$.

It is obvious that by means of the matrix

$$\Delta(t) = \begin{pmatrix} g_{0,1}(t)+g_{1,1}(t) & g_{0,2}(t)+g_{1,2}(t) \\ \\ g_{2,1}(t)+g_{3,1}(t) & g_{2,2}(t)+g_{3,2}(t) \end{pmatrix}$$

we can write the equality (6) in the compact form

$$(b_1(t),b_2(t))\Delta(t) + \int_0^t (b_1(\tau),b_2(\tau))B(t)d\tau = (\varphi_1^0(t),\varphi_2^0(t)), t\in(0,T), \tag{10}$$

where

$$B(t) = \begin{pmatrix} \beta_{0,1}(t)+\beta_{1,1}(t) & \beta_{0,2}(t)+\beta_{1,2}(t) \\ \\ \beta_{2,1}(t)+\beta_{3,1}(t) & \beta_{2,2}(t)+\beta_{3,2}(t) \end{pmatrix}$$

Assume that almost for all $t \in (0,T)$ the matrix $\Delta(t)$ is invertible and it holds

$$\|\Delta(t)\| \le M_1, \left\|\Delta^{-1}(t)\right\| \le M_1 \tag{11}$$

in the sense of almost everywhere on $(0,T)$. Then, from (10) we have

$$(b_1(t),b_2(t)) + \int_0^t (b_1(\tau),b_2(\tau))B_1(t)d\tau = (\varphi_1^0(t),\varphi_2^0(t))\Delta^{-1}(t), t\in(0,T), \tag{12}$$

where

$$B_1(t) = B(t)\Delta^{-1}(t).$$

Passing in (12) to the vector norm we have

$$\alpha(t) \le \int_0^t \alpha(\tau)l(t)d\tau + S^0(t), t\in(0,T), \tag{13}$$

where

$$\alpha(t) = \|b_1(t)\| + \|b_2(t)\|\,,$$

$$l(t) = \|B_1(t)\| \leq M_1\,\|B(t)\|\,,$$

$$S^0(t) = \left\|(\varphi_1^0(t), \varphi_2^0(t))\Delta^{-1}(t)\right\| \leq M_1(\left\|\varphi_1^0(t)\right\| + \left\|\varphi_2^0(t)\right\|);$$

here and below M_i are constants independent on $\hat{\varphi} = (\varphi_0, \varphi_1, \varphi_2, \varphi_3, \varphi)$.

Let the point $\tau \in (0, T)$ be fixed and $t \in (0, \tau)$. Then integrating (13) with respect to t on $(0, \tau)$ we get

$$R(\tau) \leq \int_0^\tau R(t)l(t)dt + S^1(\tau), \tau \in (0, T), \tag{14}$$

where

$$S^1(\tau) = \int_0^\tau S^0(t)dt,$$

$$R(\tau) = \int_0^\tau \alpha(t)dt.$$

We write the inequality (14) in the form

$$R(\tau) \leq R^1(\tau) + S^1(\tau) + \varepsilon, \tag{15}$$

where $\varepsilon > 0$ is an arbitrary number and

$$R^1(\tau) = \int_0^\tau R(t)l(t)dt.$$

Hence

$$\frac{R\,(\tau)}{R^1\,(\tau) + S^1\,(\tau) + \varepsilon} \leq 1, \tau \in (0, T).$$

Therefore

$$\frac{R\,(\tau)\,l\,(\tau)}{R^1\,(\tau) + S^1\,(\tau) + \varepsilon} \leq l(\tau), \tau \in (0, T),$$

or

$$\frac{\dot{R}^1\,(\tau)}{R^1\,(\tau) + S^1\,(\tau) + \varepsilon} \leq l(\tau), \tau \in (0, T), \tag{16}$$

where the sign of point over some function of one argument means its first derivative.

The function $S^1(\tau)$ is a monotonically increasing function.Therefore, if t is fixed and $\tau \in (0, t)$, then $S^1(\tau) \leq S^1(t)$. Therefore from (16) we have

$$\frac{\dot{R}^1\,(\tau)}{R^1\,(\tau) + S^1\,(t) + \varepsilon} \leq \frac{\dot{R}^1\,(\tau)}{R^1\,(\tau) + S^1\,(\tau) + \varepsilon} \leq l(\tau), \tau \in (0, t),$$

integrating it with respect to τ on $(0,t)$ we get

$$\ln \frac{R^1(t)+S^1(t)+\varepsilon}{R^1(0)+S^1(t)+\varepsilon} \leq \int_0^t l(\tau)\,d\tau$$

or

$$R^1(t)+S^1(t)+\varepsilon \leq (S^1(t)+\varepsilon)e^{\int_0^t l(\tau)d\tau}.$$

Taking this into account in (15) we get

$$R(t) \leq S^1(t)e^{\int_0^t l(\tau)d\tau}. \tag{17}$$

Writing (13) in the form

$$\alpha(t) \leq R(t)l(t) + S^0(t)$$

and using (17) we get

$$\alpha(t) \leq l(t)e^{\int_0^t l(\tau)d\tau} \int_0^t S^0(\tau)d\tau + S^0(t) \tag{18}$$

Thus, we have proved

Lemma 1. *If, for some non-negative functions $\alpha, l, S^0 \in \mathcal{L}_p(0,T)$, the inequality* (13) *holds, then the function $\alpha(t)$ also satisfies the condition* (18).

Taking into account the expression of the function $l(t)$ in (18) we get

$$\alpha(t) \leq M_2 \|B(t)\| \int_0^t S^0(\tau)d\tau + S^0(t), t \in (0,T), \tag{19}$$

where

$$M_2 = M_1 \exp\left(M_1 \int_0^T \|B(\tau)\|\,d\tau\right).$$

Notice that from the conditions imposed on the matrix functions $\beta_{i,k}(t)$ it follows that $\|B(\cdot)\| \in \mathcal{L}_p(0,T)$. Therefore $M_2 < +\infty$.

Now by means of this lemma for the sum $\alpha(t) = \|b_1(t)\| + \|b_2(t)\|$ we have the estimate

$$\alpha(t) = \|b_1(t)\| + \|b_2(t)\| \leq M_2 \int_0^t S^0(\tau)d\tau \,\|B(t)\| + S^0(t), t \in (0,T).$$

Therefore, using the Hölder inequality, we obtain

$$\|b_k(t)\| \leq S^0(t) + M_2 T^{\frac{1}{q}} \left\|S^0\right\|_{\mathcal{L}_p(0,T)} \|B(t)\|, t \in (0,T), k = 1,2,$$

here and below $q = p/(p-1)$ denotes the number conjugate to p.

Here, passing to the norm, by Minkowsky inequality we get

$$\|b_k\|_{\mathcal{L}_{p,n}(0,T)} \leq \left\|S^0\right\|_{\mathcal{L}_{p,}(0,T)} (1 + M_2 T^{\frac{1}{q}} \|B\|_{\mathcal{L}_{p,2n\times 2n}(0,T)}) =$$

$$= M_3 \left\|S^0\right\|_{\mathcal{L}_p(0,T)}, \quad k = 1,2;$$

$$M_3 = 1 + M_2 T^{\frac{1}{q}} \|B\|_{\mathcal{L}_{p,2n\times 2n}(0,T)}.$$

Obviously

$$\left\|S^0\right\|_{\mathcal{L}_p(0,T)} \leq M_1(\left\|\varphi_1^0\right\|_{\mathcal{L}_{p,n}(0,T)} + \left\|\varphi_2^0\right\|_{\mathcal{L}_{p,n}(0,T)}).$$

Therefore

$$\|b_k\|_{\mathcal{L}_{p,n}(0,T)} \leq M_4(\left\|\varphi_1^0\right\|_{\mathcal{L}_{p,n}(0,T)} + \left\|\varphi_2^0\right\|_{\mathcal{L}_{p,n}(0,T)}), k = 1,2, \tag{20}$$

where $M_4 = M_3 M_1$.

Now, let's estimate the norms $\left\|\varphi_k^0\right\|_{\mathcal{L}_{p,n}(0,T)}$, $k = 1,2$. Obviously if the vector $z \in \widehat{W}_{p,n}(G)$ satisfies the conditions (1)-(4), then its independent elements $\hat{b} = (z(0,0), z_t(t,0), z_t(t,\alpha+0), z_x(0,x), z_{tx}(t,x)) = (b_0, b_1(t), b_2(t), b_3(x), b(t,x))$ satisfy the equalities (5)-(8) and therewith

$$\|b_0\| \leq \|\hat{\varphi}\|_{\hat{Q}_{p,n}}$$

$$\|b_3\|_{\mathcal{L}_{p,n}(0,l)} \leq \|\hat{\varphi}\|_{\hat{Q}_{p,n}} \tag{21}$$

where

$$\|\hat{\varphi}\|_{\hat{Q}_{p,n}} = \left\|\hat{L}z\right\|_{\hat{W}_{p,n}(G)} = \left\|\hat{L}(N^{-1}(\hat{b}))\right\|_{\hat{Q}_{p,n}} =$$

$$= \|\varphi_0\| + \|\varphi_1\|_{\mathcal{L}_{p,n}(0,T)} + \|\varphi_2\|_{\mathcal{L}_{p,n}(0,T)} +$$

$$+ \|\varphi_3\|_{\mathcal{L}_{p,n}(0,l)} + \|\varphi\|_{\mathcal{L}_{p,n}(G)}, \ z = N^{-1}\hat{b}.$$

Therefore, from the expressions (6*) of the vectors $\varphi_k^0(t)$ we have

$$\left\|\varphi_k^0(t)\right\| \leq \|\varphi_k(t)\| + \|b_0\| (\|\beta_{0,k}(t)\| + \|\beta_{1,k}(t)\| +$$

$$+ \|\beta_{2,k}(t)\| + \|\beta_{3,k}(t)\|) + \|b_3\|_{\mathcal{L}_{p,n}(0,l)} \alpha^{\frac{1}{q}} \times$$

$$\times (\|\beta_{1,k}(t)\| + \|\beta_{2,k}(t)\|) + l^{\frac{1}{q}} \|b_3\|_{\mathcal{L}_{p,n}(0,l)} \|\beta_{3,k}(t)\| + \|\varphi_{k,b}(t)\|, k = 1,2.$$

Hence by means of the Minkowsky inequality allowing for (20) we have

$$\left\|\varphi_k^0\right\|_{\mathcal{L}_{p,n}(0,T)} \leq \|\varphi_k\|_{\mathcal{L}_{p,n}(0,T)} + \|b_0\| \sum_{i=0}^{3} \|\beta_{i,k}\|_{\mathcal{L}_{p,n\times n}(0,T)} +$$

$$+ \|b_3\|_{\mathcal{L}_{p,n}(0,l)} \alpha^{\frac{1}{q}} \sum_{i=1}^{2} \|\beta_{i,k}\|_{\mathcal{L}_{p,n\times n}(0,T)} +$$

$$+ \|b_3\|_{\mathcal{L}_{p,n}(0,l)} l^{\frac{1}{q}} \|\beta_{3,k}\|_{\mathcal{L}_{p,n\times n}(0,T)} +$$

$$+ \|\varphi_{k,b}\|_{\mathcal{L}_{p,n}(0,T)} \leq \|\hat{\varphi}\|_{\hat{Q}_{p,n}} (1 + \sum_{i=0}^{3} \|\beta_{i,k}\|_{\mathcal{L}_{p,n\times n}(0,T)} +$$

$$+\alpha^{\frac{1}{q}} \sum_{i=1}^{2} \|\beta_{i,k}\|_{\mathcal{L}_{p,n\times n}(0,T)} + l^{\frac{1}{q}} \|\beta_{3,k}\|_{\mathcal{L}_{p,n\times n}(0,T)}) + \|\varphi_{k,b}\|_{\mathcal{L}_{p,n}(0,T)}$$

or

$$\left\|\varphi_k^0\right\|_{\mathcal{L}_{p,n}(0,T)} \leq M_5 \|\hat{\varphi}\|_{\hat{Q}_{p,n}} + \|\varphi_{k,b}\|_{\mathcal{L}_{p,n}(0,T)}, \tag{22}$$

where

$$M_5 = 1 + \sum_{i=0}^{3} \|\beta_{i,k}\|_{\mathcal{L}_{p,n\times n}(0,T)} + \alpha^{\frac{1}{q}} \sum_{i=1}^{2} \|\beta_{i,k}\|_{\mathcal{L}_{p,n\times n}(0,T)} +$$

$$+l^{\frac{1}{q}} \|\beta_{3,k}\|_{\mathcal{L}_{p,n\times n}(0,T)}.$$

Now let's estimate the norm of the vector $\varphi_{k,b}(t)$. Obviously

$$\|\varphi_{k,b}(t)\| \leq (T\alpha)^{\frac{1}{q}} \|\beta_{1,k}\|_{\mathcal{L}_{p,n\times n}(0,T)} \|b\|_{\mathcal{L}_{p,n}(G)} +$$

$$+(T(l-\alpha))^{\frac{1}{q}} \|\beta_{3,k}\|_{\mathcal{L}_{p,n\times n}(0,T)} \|b\|_{\mathcal{L}_{p,n}(G)} +$$

$$+\alpha^{\frac{1}{q}} \|b(t,\cdot)\|_{\mathcal{L}_{p,n}(0,l)} \|g_{1,k}\|_{\mathcal{L}_{\infty,n\times n}(0,T)} +$$

$$+(l-\alpha)^{\frac{1}{q}} \|b(t,\cdot)\|_{\mathcal{L}_{p,n}(0,l)} \|g_{3,k}\|_{\mathcal{L}_{\infty,n\times n}(0,T)}.$$

Therefore

$$\|\varphi_{k,b}\|_{\mathcal{L}_{p,n}(0,T)} \leq c^k \|b\|_{\mathcal{L}_{p,n}(G)}, k = 1, 2, \tag{23}$$

$$c^k = (T\alpha)^{\frac{1}{q}} \|\beta_{1,k}\|_{\mathcal{L}_{p,n\times n}(0,T)} + (T(l-\alpha))^{\frac{1}{q}} \|\beta_{3,k}\|_{\mathcal{L}_{p,n\times n}(0,T)} +$$

$$+\alpha^{\frac{1}{q}} \|g_{1,k}\|_{\mathcal{L}_{\infty,n\times n}(0,T)} + (l-\alpha)^{\frac{1}{q}} \|g_{3,k}\|_{\mathcal{L}_{\infty,n\times n}(0,T)}. \tag{24}$$

Then using (23) we get from (22)

$$\left\|\varphi_k^0\right\|_{\mathcal{L}_{p,n}(0,T)} \leq M_5 \left\|\hat{\varphi}\right\|_{\hat{Q}_{p,n}} + c^k \left\|b\right\|_{\mathcal{L}_{p,n}(G)}, k = 1, 2. \tag{25}$$

Now, equation (5) is written in the form

$$(\Omega b)(t,x) = \varphi^0(t,x), (t,x) \in G, \tag{26}$$

where

$$(\Omega b)(t,x) = b(t,x) + \int_0^t \int_0^x b(\tau,\zeta) q_1(\zeta,x) A_{0,0}(t,x) d\tau d\zeta +$$

$$+ \int_0^x b(t,\zeta) q_1(\zeta,x) A_{1,0}(t,x) d\zeta + \int_0^t b(\tau,x) A_{0,1}(t,x) d\tau, \tag{27}$$

$$\varphi^0(t,x) = \varphi^{0,0}(t,x) + \varphi^{0,1}(t,x);$$

$$\varphi^{0,0}(t,x) = \varphi(t,x) - b_3(x) A_{0,1}(t,x) -$$

$$-b_0 A_{0,0}(t,x) - \int_0^x b_3(\zeta) A_{0,0}(t,x) d\zeta;$$

$$\varphi^{0,1}(t,x) = -\left(\int_0^t b_1(\tau) \theta(\alpha - x) d\tau + \int_0^t b_2(\tau) \theta(x - \alpha) d\tau \right) A_{0,0}(t,x) -$$

$$-(b_1(t)\theta(\alpha - x) + b_2(t)\theta(x - \alpha)) A_{1,0}(t,x).$$

In the expression of the vector $\varphi^{0,0}(t,x)$ there are the vectors $\varphi(t,x), b_3(x), b_0$ and the given matrices $A_{0,1}(t,x), A_{0,0}(t,x)$. Above we have estimated the norms of the vectors $\varphi(t,x), b_3(x), b_0$ by $\|\hat{\varphi}\|_{\hat{Q}_{p,n}}$. Therefore, from the expression of the vector $\varphi^{0,0}(t,x)$ by means of the Holder and Minkowsky inequalities we can easily get the estimate

$$\left\|\varphi^{0,0}\right\|_{\mathcal{L}_{p,n}(G)} \leq M_6 \left\|\hat{\varphi}\right\|_{\hat{Q}_{p,n}}, \tag{28}$$

where $M_6 > 0$ is a suitable constant.

Further, from the expression of the vector $\varphi^{0,1}(t,x)$ it is seen that

$$\left\|\varphi^{0,1}(t,x)\right\| \leq (T^{\frac{1}{q}} \left\|b_1\right\|_{\mathcal{L}_{p,n}(0,T)} \theta(\alpha - x) +$$

$$+T^{\frac{1}{q}} \left\|b_2\right\|_{\mathcal{L}_{p,n}(0,T)} \theta(x - \alpha)) \|A_{0,0}(t,x)\| +$$

$$+(\left\|b_1\right\|_{\mathcal{L}_{p,n}(0,T)} \theta(\alpha - x) + \left\|b_2\right\|_{\mathcal{L}_{p,n}(0,T)} \theta(x - \alpha)) A_{1,0}^0(x).$$

Hence we get

$$\left\|\varphi^{0,1}\right\|_{\mathcal{L}_{p,n}(G)} \leq (T^{\frac{1}{q}} \left\|A_{0,0}\right\|_{\mathcal{L}_{p,n\times n}(G)} +$$

$$+\left\|A^0_{1,0}\right\|_{\mathcal{L}_p(0,l)})(\|b_1\|_{\mathcal{L}_{p,n}(0,T)}+\|b_2\|_{\mathcal{L}_{p,n}(0,T)})=$$

$$=M_7(\|b_1\|_{\mathcal{L}_{p,n}(0,T)}+\|b_2\|_{\mathcal{L}_{p,n}(0,T)}), \tag{29}$$

where

$$M_7=T^{\frac{1}{q}}\|A_{0,0}\|_{\mathcal{L}_{p,n\times n}(G)}+\left\|A^0_{1,0}\right\|_{\mathcal{L}_p(0,l)}.$$

The operator Ω, defined by the equality (27) acts in $\mathcal{L}_{p,n}(G)$, is bounded and has a bounded inverse in it [3]. Therefore from (26) we have

$$\|b\|_{\mathcal{L}_{p,n}(G)}\le\left\|\Omega^{-1}\right\|\left\|\varphi^0\right\|_{\mathcal{L}_{p,n}(G)}.$$

Hence by (28) and (29) we get

$$\|b\|_{\mathcal{L}_{p,n}(G)}\le\left\|\Omega^{-1}\right\|(M_6\|\hat{\varphi}\|_{\hat{Q}_{p,n}}+M_7(\|b_1\|_{\mathcal{L}_{p,n}(0,T)}+\|b_2\|_{\mathcal{L}_{p,n}(0,T)})). \tag{30}$$

Take into account, (25) in (20) and get

$$\|b_1\|_{\mathcal{L}_{p,n}(0,T)}+\|b_2\|_{\mathcal{L}_{p,n}(0,T)}\le 2M_4(2M_5\|\hat{\varphi}\|_{\hat{Q}_{p,n}}+(c^1+c^2)\|b\|_{\mathcal{L}_{p,n}(G)}). \tag{31}$$

Hence substituting (31) into (30) we get

$$\|b\|_{\mathcal{L}_{p,n}(G)}\le\left\|\Omega^{-1}\right\|\left(M_8\|\hat{\varphi}\|_{\hat{Q}_{p,n}}+2M_4M_7(c^1+c^2)\|b\|_{\mathcal{L}_{p,n}(G)}\right),$$

where $M_8=M_6+4M_4M_5M_7>0$.

If we assume

$$\gamma=2M_4M_7\left\|\Omega^{-1}\right\|(c^1+c^2)<1, \tag{$*$}$$

then we can obtain

$$\|b\|_{\mathcal{L}_{p,n}(G)}\le M_9\|\hat{\varphi}\|_{\hat{Q}_{p,n}}, \tag{32}$$

with constant

$$M_9=(1-\gamma)^{-1}\left\|\Omega^{-1}\right\|M_8.$$

Taking into account (32) in (31) we have

$$\|b_1\|_{\mathcal{L}_{p,n}(0,T)}+\|b_2\|_{\mathcal{L}_{p,n}(0,T)}\le 2M_4(2M_5+M_9(c^1+c^2))\|\hat{\varphi}\|_{\hat{Q}_{p,n}}. \tag{33}$$

Now, summing up the inequalities (21), (32), (33) for the totality

$\hat{b} = (b_0, b_1(t), b_2(t), b_3(x), b(t,x))$ we have the estimate

$$\left\|\hat{b}\right\|_{\hat{Q}_{p,n}} = \|b_0\| + \|b_1\|_{\mathcal{L}_{p,n}(0,T)} + \|b_2\|_{\mathcal{L}_{p,n}(0,T)} + \|b_3\|_{\mathcal{L}_{p,n}(0,l)} +$$

$$+ \|b\|_{\mathcal{L}_{p,n}(G)} \leq M_{10} \|\hat{\varphi}\|_{\hat{Q}_{p,n}} = M_{10} \left\|\hat{L}z\right\|_{\hat{Q}_{p,n}},$$

where

$$M_{10} = 2M_4(2M_5 + M_9(c^1 + c^2)) > 0$$

and

$$\widehat{L}z = \widehat{\varphi},\ \widehat{b} = Nz.$$

Using the last inequality we get

$$\|z\|_{\widehat{W}_{p,n}(G)} \leq M_{11} \|Nz\|_{\widehat{Q}_{p,n}} \leq M_{11}M_{10} \left\|\widehat{L}z\right\|_{\widehat{Q}_{p,n}},$$

with suitable constant $M_{11} > 0$ independent on z. Hence the following theorem is true.

Theorem 1. *Let the matrix $\Delta(t)$ be invertible for almost all $t \in (0,T)$ and conditions (11) and (*) be fulfilled, where M_4 and M_7 are the constants defined above by using the number M_1, the constants c^k $(k = 1,2)$ are given by the formula (24), and the operator Ω is given by the relation (27). Then, for every solution z of problem* (1)-(4)*, the a priori estimate $\|z\|_{\widehat{W}_{p,n}(G)} \leq M \left\|\widehat{L}z\right\|_{\widehat{Q}_{p,n}}$ holds, where $M > 0$ is a positive constant independent on z.*

The operator $\widehat{L}$ is a linear and bounded operator from $\widehat{W}_{p,n}(G)$ to $\widehat{Q}_{p,n}$. Therefore, there exists a bounded, conjugated operator $\widehat{L}^* : \left(\widehat{Q}_{p,n}\right)^* \to \left(\widehat{W}_{p,n}(G)\right)^*$. Using general forms of linear bounded functional determined on $\widehat{Q}_{p,n}$ and $\widehat{W}_{p,n}(G)$ we can prove that $\widehat{L}^*$ is a bounded vector operator of the form $\widehat{L}^* = (\omega_0, \omega_1, \omega_2, \omega_3, \omega)$ acting in the space $\widehat{Q}_{q,n}$, where $1/p + 1/q = 1$. Therefore, we can consider the equation $\widehat{L}^*\widehat{f} = \widehat{\psi}$ as a conjugated equation for problem(1)-(4), where $\widehat{f}$ is a desired solution, $\widehat{\psi}$ is an element from $\left(\widehat{W}_{p,n}(G)\right)^*$. It follows from Theorem 1 that the following theorem is true.

Theorem 2. *Let the conditions of Theorem 1 be satisfied. Then problem* (1)-(4) *may have at most one solution $z \in \widehat{W}_{p,n}(G)$, and the conjugated equation $\widehat{L}^*\widehat{f} = \widehat{\psi}$ for any right hand side $\widehat{\psi} \in \left(\widehat{W}_{p,n}(G)\right)^*$ has at least one solution $\widehat{f} \in \widehat{Q}_{q,n}$.*

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