Electronic Journal of Qualitative Theory of Differential Equations
2010, No. **46**, 1-19; **http://www.math.u-szeged.hu/ejqtde/**

Periodic solutions of differential equations with a general piecewise constant argument and applications

Kuo-Shou Chiu[a] * and Manuel Pinto[b] †

[a]Departamento de Matemática
Facultad de Ciencias Básicas
Universidad Metropolitana de Ciencias de la Educación
José Pedro Alessandri 774 - Santiago - Chile
e-mail: kschiu@umce.cl

[b]Departamento de Matemáticas
Facultad de Ciencias - Universidad de Chile
Casilla 653, Santiago - Chile
e-mail: pintoj@uchile.cl

Abstract

In this paper we investigate the existence of the periodic solutions of a quasilinear differential equation with piecewise constant argument of generalized type. By using some fixed point theorems and some new analysis technique, sufficient conditions are obtained for the existence and uniqueness of periodic solutions of these systems. A new Gronwall type lemma is proved. Some examples concerning biological models as Lasota-Wazewska, Nicholson's blowflies and logistic models are treated.

Keywords: Periodic solutions, Piecewise constant argument of generalized type, Fixed point theorems, Gronwall's inequality.

A.M.S. (2000) Subjects Classification: 34K13, 34A36, 37C25, 26D10.

*Corresponding author.
†This research has been supported by: FONDECYT 1080034 and DI 0052-10 UNAP.

1 Introduction

Differential equations with piecewise constant arguments (briefly DEPCA) arise in an attempt to extend the theory of functional differential equations with continuous arguments to differential equations with discontinuous arguments. This task is of considerable applied interest since DEPCA include, as particular cases, impulsive and loaded equations of control theory and are similar to those found in some biomedical models. The study of such equations

$$\frac{dx(t)}{dt} = f(t, x(t), x(\gamma(t))), \quad \gamma(t) = [t] \quad \text{or} \quad \gamma(t) = 2\left[\frac{t+1}{2}\right], \tag{1.1}$$

where $[\cdot]$ signifies the greatest integer function, has been initiated by Wiener [29], Cooke and Wiener [12], and Shah and Wiener [27] in the 80's; and has been developed by many authors [1]-[6],[11]-[15],[20],[22],[27]-[29],[32],[35]. Applications of DEPCA are discussed in [28]. DEPCA usually describes hybrid dynamical systems (a combination of continuous and discrete) and so combine properties of both differential and difference equations. Over the years, great attention has been paid to the study of the existence of periodic solutions of this type of equations. For specific references (see [1],[5],[6],[11]).

Let $\mathbb{Z}$, $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ be the set of all integers, natural, real and complex numbers, respectively. Fix two real sequences $t_i, \gamma_i, i \in \mathbb{Z}$, such that $t_i < t_{i+1}, t_i \leq \gamma_i \leq t_{i+1}$ for all $i \in \mathbb{Z}$, $t_i \to \pm\infty$ as $i \to \pm\infty$. Let $\gamma : \mathbb{R} \to \mathbb{R}$ be a step function given by $\gamma(t) = \gamma_i$ for $t \in I_i = [t_i, t_{i+1})$ and consider the DEPCA (1.1) with this general γ. In this case we speak of DEPCA of general type, in short DEPCAG. With the general delay case: $\gamma_i = t_i$ this concept was introduced by M. U. Akhmet [2] in 2007 and in several other papers [3]-[6].

Differential equations alternately of retarded and advanced type can occur in many problems of economy, biology and physics, because differential equations of this type are much more suitable than delay differential equations for an adequate treatment of dynamic phenomena. The concept of delay is related to a memory of system, the past events are important for the present current behavior (see for example [8],[17],[19],[23],[33]), and the concept of advance is related to potential future events which can be known at the current present time which could be useful for decision making. The study of various problems for differential equations alternately of retarded and advanced type with piecewise constant arguments can be found in many works, we cite for example [1],[13],[16],[25],[27].

The existence of periodic solutions of ordinary differential equations has been discussed extensively in theory and in practice (for example, see [9],[10],[21],[23],[24],[31],[33],[34] and the references cited therein), but there are few papers considering discontinuous deviations in differential perturbed equations.

In 2008, Akhmet et al. [5] obtained some sufficient conditions for the existence and uniqueness of periodic solutions for the following system,

$$x'(t) = A(t)x(t) + h(t) + \mu g(t, x(t), x(\gamma(t)), \mu), \tag{1.2}$$

where $A : \mathbb{R} \to \mathbb{R}^{n\times n},\ h : \mathbb{R} \to \mathbb{R},\ g : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \times I \to \mathbb{R}^n$ are continuous functions, $\gamma(t) = t_i$ if $t_i \le t < t_{i+1}$ and μ is a small parameter belonging to an interval $I \subset \mathbb{R}$ with $0 \in I$. See also Akhmet et al. [6], where other important results on periodic solutions are obtained.

In an interesting paper, Xia et al. [32], using exponential dichotomy and contraction mapping principle, obtained some sufficient conditions for the existence and uniqueness of almost periodic solutions of general inhomogeneous DEPCA of the form

$$y'(t) = A(t)y(t) + B(t)y([t]) + h(t) + \mu \mathrm{g}(t,y(t),y([t]), \mu),\ t \in \mathbb{R}, \tag{1.3}$$

and the very general nonlinear DEPCA

$$y'(t) = f(t, y(t), y([t])) + \mu \mathrm{g}(t,y(t),y([t]), \mu),\ t \in \mathbb{R}, \tag{1.4}$$

where $A, B : \mathbb{R} \to \mathbb{R}^{q\times q},\ h : \mathbb{R} \to \mathbb{R}^q,\ f : \mathbb{R} \times \mathbb{R}^q \times \mathbb{R}^q \to \mathbb{R}^q$, $g : \mathbb{R} \times \mathbb{R}^q \times \mathbb{R}^q \times I \to \mathbb{R}^q$ are continuous functions and μ is a small parameter belonging to an interval $I \subset \mathbb{R}$ with $0 \in I$.

The main purpose of this paper is to establish some simple criteria for the existence of periodic solutions of quasilinear differential systems with piecewise constant argument of generalized type:

$$y'(t) = A(t)y(t) + f(t, y(t), y(\gamma(t))), \tag{1.5}$$

where $t \in \mathbb{R}$, $y \in \mathbb{C}^p$, $A(t)$ is a $p \times p$ matrix for $p \in \mathbb{N}$, $f(t, x, y)$ is a p dimensional vector and f is continuous in the first argument.

In this paper, the estimates of solutions have been obtained by a new Gronwall's inequality. Under certain conditions on the nonlinearity f, several sufficient conditions for the existence and uniqueness of periodic (or harmonic) and subharmonic solutions of (1.5) are obtained by using Poincaré operator and fixed point theory. The conditions can be checked easily.

The following assumptions for equation (1.5) will be necessary throughout the paper:

(P) There exists $\omega > 0$ such that:

1) $A(t)$ and $f(t, y_1, y_2)$ are periodic functions in t with a period ω, for all $t \ge \tau$.

2) There exists $l \in \mathbb{N}$, for which the sequences $\{t_i\}_{i=1}^{j}$, $\{\gamma_i\}_{i=1}^{j}$, $j \leq \infty$, satisfy the (ω, l) condition, that is

$$t_{i+l} = t_i + \omega, \gamma_{i+l} = \gamma_i + \omega, \text{ for } i \in \{1, ..., j\} \text{ with } j < \infty. \tag{1.6}$$

(N) The homogeneous equation

$$y'(t) = A(t)y(t) \tag{1.7}$$

does not admit any nontrivial ω-periodic solution.

(L) $f : [\tau, \infty) \times \mathbb{C}^p \times \mathbb{C}^p \to \mathbb{C}^p$ is a continuous function such that

$$|f(t, x_1, y_1) - f(t, x_2, y_2)| \leq \eta_1(t)|x_1 - x_2| + \eta_2(t)|y_1 - y_2| \tag{1.8}$$

for $t \in \mathbb{R}$, $x_i, y_i \in \mathbb{C}^p$, $i = 1, 2$, $\eta_i : \mathbb{R} \to [0, \infty)$, $i = 1, 2$, are locally integrable functions and $f(t, 0, 0)$ is a continuous function.

(UC) The solution $y(t) = y(t, \tau, \xi)$ is the unique solution of (1.5) such that $y(\tau) = \xi$ and it depends continuously on ξ.

(H1) $f(t, y_1, y_2) = o(|y_1| + |y_2|)$ if $|y_1| + |y_2| \to \infty$, uniformly in $t \in [\tau, \tau + \omega]$.

(H2) $f(t, y_1, y_2) = o(|y_1| + |y_2|)$ if $|y_1| + |y_2| \to 0$, uniformly in $t \in [\tau, \tau + \omega]$.

Suppose that f satisfies the Lipschitz condition (L). Even if $f(t, 0, 0) \equiv 0$, then neither (H1) nor (H2) is necessarily satisfied. On the other hand, conditions (H1) or (H2) hold for functions f which are not of Lipschitz type. Moreover, although η_i, $i = 1, 2$, are bounded, they determine a better precision than to replace them by its bounds (see (2.15) below as example).

The rest of this paper is organized as follows. In the next section, some definitions and lemmas which will be used to prove our main results, are introduced. Section 3 is devoted to prove our main results for systems (1.5). We end this paper with applications to population models as Lasota-Wazewska, Nicholson or logistic type to show the feasibility of our results. For example, some new and interesting sufficient conditions are obtained to guarantee the existence of a positive periodic solution in Lasota-Wazewska model with DEPCAG

$$y'(t) = -\delta(t)y(t) + p(t)e^{-y(\gamma(t))}, \ t \geq 0,$$

where $y(t)$ is the number of red blood cells at time t and $\delta(t), p(t)$ are positive ω-periodic functions, $\{t_i\}_{i\in\{1,..,j\}}$ and $\{\gamma_i\}_{i\in\{1,..,j\}}$ satisfy the property (ω, l).

2 Existence and Uniqueness of the Solutions

First, we prove the existence and uniqueness of solutions of (1.5). A natural extension of the original definition of a solution of DEPCA[1],[11]-[15],[20],[27]-[29],[32],[35] allows us to define a solution of DEPCAG for our general case. Let $J \subset \mathbb{R}$ be a real interval such that $\{t_i\}_{i=1}^{j} \subseteq J$, $j \leq \infty$.

Definition 1 *A function y is a solution of DEPCAG (1.5) in an interval $J \subset \mathbb{R}$ if*

i) y is continuous on J.

ii) The derivative $y'(t)$ exists at each point $t \in J$ with the possible exception of the points $t_i \in J$, $i \in \{1, .., j\}$, where the one-side derivatives exist.

iii) Equation (1.5) is satisfied for y on each interval (t_i, t_{i+1}), $i \in \{1, .., j\}$, and it holds for the right derivative at the points t_i, $i \in \{1, .., j\}$.

Uniqueness, continuity and estimates of the solutions of DEPCAG (1.5) will follow from a DEPCAG integral inequality of Gronwall type. See [22].

For every $t \in \mathbb{R}$, let $i = i(t) \in \mathbb{Z}$ be the unique integer such that $t \in I_i = [t_i, t_{i+1})$.

Lemma 2.1 *Let $u, \eta_i : J \to [0, \infty)$ $i = 1, 2$ be three continuous functions and α be a nonnegative real constant. Suppose that for all $t \geq \tau$ the inequality*

$$u(t) \leq \alpha + \int_{\tau}^{t} [\eta_1(s)u(s) + \eta_2(s)u(\gamma(s))]ds \tag{2.1}$$

holds. Assume

$$v_i = \int_{t_i}^{\gamma_i} \left[\eta_2(s)e^{\int_s^{\gamma_i} \eta_1(\kappa)d\kappa}\right] ds \leq v := \sup_{i \in \mathbb{N}} v_i < 1. \tag{2.2}$$

Then for $t \geq \tau$,

$$u(t) \leq \alpha \exp\left(\int_{\tau}^{t} \eta_1(s)ds + \frac{1}{1-v}\int_{\tau}^{t}\left[\eta_2(s)e^{\int_{t_{i(s)}}^{\gamma(s)} \eta_1(\kappa)d\kappa}\right] ds\right), \tag{2.3}$$

$$u(\gamma(t)) \leq \frac{\alpha}{1-v} \exp\left(\int_{\tau}^{\gamma(t)} \eta_1(s)ds + \frac{1}{1-v}\int_{\tau}^{t_{i(t)}}\left[\eta_2(s)e^{\int_{t_{i(s)}}^{\gamma(s)} \eta_1(\kappa)d\kappa}\right] ds\right), \tag{2.4}$$

and

$$u(\gamma_i) \leq \frac{1}{1-v} u(t_i) \exp\left(\int_{t_i}^{\gamma_i} \eta_1(s)ds\right). \tag{2.5}$$

Proof. Call $\mathrm{v}(t)$ the right member of (2.1). So $u(\tau) \leq \mathrm{v}(\tau) = \alpha$, $u \leq \mathrm{v}$, v is a piecewise differentiable and nondecreasing function. By (2.1), it satisfies

$$\mathrm{v}'(t) \leq \eta_1(t)\mathrm{v}(t) + \eta_1(t)\mathrm{v}(\gamma(t)).$$

Multiply both sides of the previous relation by $\exp\left(-\int_r^t \eta_1(s)ds\right)$, to obtain

$$(\mathrm{v}'(t) - \eta_1(t)\mathrm{v}(t))\left(\exp\left(-\int_r^t \eta_1(s)ds\right)\right) \leq (\eta_2(t)\mathrm{v}(\gamma(t)))\left(\exp\left(-\int_r^t \eta_1(s)ds\right)\right). \tag{2.6}$$

Integrating (2.6) we have for $r, t \in I_i,\ i \in \mathbb{N}$:

$$\mathrm{v}(t)\left(\exp\left(-\int_r^t \eta_1(s)ds\right)\right) - \mathrm{v}(r) \leq \int_r^t (\eta_2(s)\mathrm{v}(\gamma(s)))\left(\exp\left(-\int_r^s \eta_1(\kappa)d\kappa\right)\right)ds. \tag{2.7}$$

With $t = \gamma_i$ and $r = t_i$ in (2.7) for $t \in [t_i, \gamma_i]$, we get

$$\mathrm{v}(\gamma_i) \leq \mathrm{v}(t_i)\left(\exp\left(\int_{t_i}^{\gamma_i} \eta_1(s)ds\right)\right) + \mathrm{v}(\gamma_i)\int_{t_i}^{\gamma_i}(\eta_2(s))\left(\exp\left(\int_s^{\gamma_i}\eta_1(\kappa)d\kappa\right)\right)ds$$

and, by (2.2), estimate (2.5) follows. Then, for $t \in I_i$, we obtain

$$\begin{aligned}
&\mathrm{v}(t)\left(\exp\left(-\int_{t_i}^t \eta_1(s)ds\right)\right)\\
\leq&\mathrm{v}(t_i) + \int_{t_i}^t \eta_2(s)\mathrm{v}(\gamma(s))\left(\exp\left(-\int_{t_i}^s \eta_1(\kappa)d\kappa\right)\right)ds\\
\leq&\mathrm{v}(t_i) + \left(\frac{1}{1-v}\right)\int_{t_i}^t\left[\eta_2(s)\mathrm{v}(t_i)\left(\exp\left(\int_{t_i}^{\gamma_i}\eta_1(\kappa)d\kappa - \int_{t_i}^s \eta_1(\kappa)d\kappa\right)\right)\right]ds\\
\leq&\mathrm{v}(t_i) + \left(\frac{1}{1-v}\right)\int_{t_i}^t\left[\left(\eta_2(s)\exp\left(\int_{t_i}^{\gamma_i}\eta_1(\kappa)d\kappa\right)\right)\left(\mathrm{v}(s)\exp\left(-\int_{t_i}^s\eta_1(\kappa)d\kappa\right)\right)\right]ds,
\end{aligned}$$

because v is a nondecreasing function.
Now, we can apply the classical Gronwall's Lemma and to get:

$$\mathrm{v}(t)\exp\left(-\int_{t_i}^t \eta_1(s)ds\right) \leq \mathrm{v}(t_i)\exp\left\{\left(\frac{1}{1-v}\right)\int_{t_i}^t\left[\eta_2(s)\exp\left(\int_{t_i}^{\gamma_i}\eta_1(\kappa)d\kappa\right)\right]ds\right\}$$

for $t \in I_i$. By the continuity of v, we have:

$$\begin{aligned}
\mathrm{v}(t_{i+1}) \leq \mathrm{v}(t_i)\exp&\left\{\left(\int_{t_i}^{t_{i+1}}\eta_1(s)ds\right)\right.\\
&\left.+\left(\frac{1}{1-v}\right)\int_{t_i}^{t_{i+1}}\left[\eta_2(s)\exp\left(\int_{t_i}^{\gamma_i}\eta_1(\kappa)d\kappa\right)\right]ds\right\}.
\end{aligned} \tag{2.8}$$

From (2.5) and (2.8), recursively we obtain (2.3) and (2.4). The proof is complete. ■

Remark 1 *This DEPCAG inequality of Gronwall type seems to be new. Lemma 2.1 extends Lemma 1 [22], since (2.2) is weaker than* $\int_{t_i}^{\gamma_i}(\eta_1(s)+\eta_2(s))ds \le v < 1$.

Corollary 2.1 *Let* α *and* λ *be nonnegative real constants and* $u : J \to [0,\infty)$ *be a continuous function. Suppose that for all* $t \ge \tau$ *the inequality*

$$u(t) \le \alpha + \int_{\tau}^{t} \lambda\left[u(s) + u(\gamma(s))\right] ds \tag{2.9}$$

holds. Assume

$$\lambda(\gamma_i - t_i) \le \widehat{v} < \ln 2. \tag{2.10}$$

Then for $t \ge \tau$,

$$u(t) \le \alpha e^{\lambda\tilde{v}(t-\tau)},\ \ \tilde{v} = \frac{2}{2-\exp(\widehat{v})}. \tag{2.11}$$

Akhmet et al. [6] have an analogue of Gronwall-Bellman Lemma. Let $\|v\|_t = \max_{s\in[t_j,t]} |v(s)|$ and $\chi(t) = \max\{t, \gamma(t)\}$.

Lemma 2.2 *[6] Let* $u(t)$ *be continuous,* $\eta_1(t)$ *and* $\eta_2(t)$ *nonnegative piecewise continuous scalar functions defined for* $t \ge t_j$. *Suppose that* α *is a nonnegative real constant and that* $u(t)$ *satisfies the inequality*

$$|u(t)| \le \alpha + \int_{t_j}^{t} \left[\eta_1(s)|u(s)| + \eta_2(s)|u(\gamma(s))|\right] ds$$

for $t \ge t_j$. *Then the inequality*

$$\|u\|_{\chi(t)} \le \alpha \exp\left(\int_{t_j}^{\chi(t)} [\eta_1(s) + \eta_2(s)]ds\right) \tag{2.12}$$

is satisfied for $t \ge t_j$.

Remark 2 *Consider the linear DEPCA with constant coefficients* $a = 0.2$, $b = 0.5$

$$u'(t) = au(t) + bu\left(\gamma(t)\right),\ \gamma(t) = 2[\tfrac{t+1}{2}],$$

which is equivalent to

$$u(t) = \alpha + \int_{\tau}^{t} [au(s) + bu\left(\gamma(s)\right)]ds, \quad t \in [\tau, \infty). \tag{2.13}$$

Then, we have

$$u(t) = \alpha \frac{\hat{\lambda}\left(t-\gamma(t)\right)}{\hat{\lambda}\left(\tau-\gamma(\tau)\right)} \left(\frac{\hat{\lambda}\left(1\right)}{\hat{\lambda}\left(-1\right)}\right)^{i(t)-i(\tau)}, \tag{2.14}$$

where $\hat{\lambda}(t) = e^{0.2t} + \frac{5}{2}(e^{0.2t} - 1)$.
Let $\tau = 1$ *and* $t \in [1,2]$. *In this particular case, we have* $\gamma(t) = \gamma(\tau) = 2$ *and (2.14) implies*

$$u(t) = \frac{\alpha}{\hat{\lambda}\left(-1\right)} \hat{\lambda}\left(t-2\right).$$

As $\hat{\lambda}(t)$ *is a nondecreasing function, we have*

$$\max_{t\in[1,2]} u(t) = \alpha \frac{\hat{\lambda}\left(0\right)}{\hat{\lambda}\left(-1\right)} \approx 2.7355\alpha.$$

However, according to (2.12) in Lemma 2.2, the solutions u *of (2.13) satisfy*

$$\max_{t\in[1,2]} |u(t)| \le \alpha \exp\left(\int_1^2 [0.2+0.5]ds\right) \approx 2.01375\alpha.$$

So, we can conclude that the DEPCAG inequality of Gronwall type in [6] is not true.
Note that if now we apply Lemma 1 [22] to equation (2.13), we obtain

$$\max_{t\in[1,2]} |u(t)| \le \alpha \exp\left(\int_1^2 \left[0.2 + \frac{0.5}{1-\int_1^2 [0.2+0.5]ds}\right] ds\right) \approx 6.4667\alpha$$

and if we apply Lemma 2.1, we get

$$\max_{t\in[1,2]} |u(t)| \le \alpha \exp\left(\int_1^2 \left[0.2 + \frac{0.5}{1-\int_1^2 [0.5 e^{\int_k^2 0.2dt}]dk} e^{\int_1^2 0.2du}\right] ds\right) \approx 4.796\alpha.$$

Then, these two DEPCAG inequalities of Gronwall type are true and Lemma 2.1 has not only a weaker condition than Lemma 1 [22], but also has a better estimate.

Now let us see an estimate of the solution of DEPCAG (1.5).

The following notation is needed in the paper. Let $\Phi(t)$ be a fundamental solution of (1.7) and $\Phi(t,s) = \Phi(t)\Phi^{-1}(s)$, $t,s \in J$. Denote $c_\Phi = \max_{t,s\in[\tau,\tau+\omega]} |\Phi(t,s)|$

Lemma 2.3 *Suppose that condition (L) holds and let* $y_i(\cdot) = y(\cdot, \tau, \xi_i)$*,* $i = 1, 2$ *be solutions of DEPCAG (1.5). Assume*

$$\varsigma := c_\Phi \int_\tau^{\tau+\omega} [\eta_1(s) + \eta_2(s)]ds < 1. \tag{2.15}$$

Then for $t \in [\tau, \tau + \omega]$*, we have the following estimation:*

$$||y_1 - y_2|| \leq \left(\frac{c_\Phi}{1 - \varsigma} \right) |\xi_1 - \xi_2| \,. \tag{2.16}$$

Proof. For DEPCAG variation of constants formula, the solution $y_i(t, \tau, \xi_i)$, $i = 1, 2$ of Eq.(1.5) satisfies

$$y_i(t, \tau, \xi_i) = \Phi(t, \tau)\xi_i + \int_\tau^t \Phi(t, s) f(s, y_i(s), y_i(\gamma(s)))ds, \tag{2.17}$$

where $\Phi(t, \tau) = \Phi(t)\Phi^{-1}(\tau)$. Then,

$$\begin{aligned}
&|y_1(t) - y_2(t)| \\
&\leq |\Phi(t, \tau)| \, |\xi_1 - \xi_2| + \int_\tau^t |\Phi(t, s)| \, |f(s, y_1(s), y_1(\gamma(s))) - f(s, y_2(s), y_2(\gamma(s)))| \, ds \\
&\leq c_\Phi \, |\xi_1 - \xi_2| + c_\Phi \int_\tau^t [\eta_1(s) \, |y_1(s) - y_2(s)| + \eta_2(s) \, |y_1(\gamma(s)) - y_2(\gamma(s))|] \, ds.
\end{aligned}$$

Then,

$$\begin{aligned}
\max_{t \in [\tau, \tau+\omega]} |y_1(t) - y_2(t)| &\leq c_\Phi \, |\xi_1 - \xi_2| \\
&\quad + \left(c_\Phi \int_\tau^{\tau+\omega} [\eta_1(s) + \eta_2(s)]ds \right) \max_{t \in [\tau, \tau+\omega]} |y_1(t) - y_2(t)| \,.
\end{aligned}$$

By (2.15), we have (2.16). ■

As in [22], we obtain:

Proposition 2.1 *Let* $(\tau, \xi) \in J \times \mathbb{C}^p$*. The function* $y(\cdot) = y(\cdot, \tau, \xi)$ *is a solution on* J *of the DEPCAG (1.5) in the sense of Definition 1 if and only if it is a solution of the integral equation*

$$y(t) = \Phi(t, \tau)\xi + \int_\tau^t \Phi(t, s) f(s, y(s), y(\gamma(s)))ds, \; t \geq \tau. \tag{2.18}$$

Moreover, if Hypothesis (L) and $\int_{t_i}^{\gamma_i} \max_{t \in [t_i, \gamma_i]} |\Phi(t, s)| (\eta_1(s) + \eta_2(s))ds < 1$ *hold, then for every* $(\tau, \xi) \in J \times \mathbb{C}^p$*, there exists a unique solution* $y(\cdot) = y(\cdot, \tau, \xi)$ *with* $y(\tau) = \xi$ *in the sense of Definition 1.*

Remark 3 *Recently, in [3]-[4], Akhmet obtained fundamental results about the variation of constants formula, existence and uniqueness of solutions of the perturbed system (1.5). He did not consider in special form the advanced and delayed intervals* $I_i = [t_i, \gamma_i] \cup [\gamma_i, t_{i+1})$ *(See [22],[25]). The advanced intervals determine the better precision to find the contraction condition and it is not necessary to study estimate of solutions of (1.5) with the maxima norm for all intervals* I_i *as Akhmet's approach. Then, Akhmet's existence and uniqueness result for the perturbed system (1.5) is obtained under stronger conditions than ours. Akhmet had many difficulties since he did not have a global Gronwall-type lemma as Lemma 2.1 or Lemma 1 [22].*

Lemma 2.4 *Suppose that conditions (P) and (L)(or (UC)) hold. Then, a solution* $y(\cdot) = y(\cdot, \tau, \xi)$ *of DEPCAG (1.5) is* ω*-periodic if and only if* $y(\tau + \omega) = y(\tau)$*.*

Proof. If $y(t, \tau, \xi)$ is ω-periodic, then $y(\tau + \omega) = y(\tau)$ is obviously satisfied. Suppose that $y(\tau + \omega) = y(\tau)$ holds. Let $\kappa(t) = y(t + \omega)$ on J. Then, $y(\tau+\omega) = y(\tau)$ can be written as $\kappa(\tau) = y(\tau)$. (P2) implies $\gamma(t+\omega) = \gamma(t)+\omega$ for all $t \in J$. Hence, $\kappa(t)$ is a solution of DEPCAG (1.5). By the uniqueness of solutions, we have $\kappa(t) = y(t)$ on J. The lemma is proved. ∎

To prove some existence criteria for ω-periodic solutions of Equation (1.5) we use the Banach's fixed point theorem and Brouwer's fixed point theorem.

Theorem A (Banach's fixed point theorem [9]): *Let* $\mathfrak{F}$ *be a complete metric space and* $f : \mathfrak{F} \to \mathfrak{F}$ *is a contraction operator. Then there is a unique* $x \in \mathfrak{F}$ *with* $f(x) = x$*.*

Theorem B (Brouwer's fixed point theorem [7,9]): *Let* $\mathbb{B}$ *be a closed ball in* $\mathbb{R}^n$*. Any continuous function* $h : \mathbb{B} \to \mathbb{B}$ *has a fixed point.*

3 Main Results

For $\omega > 0$, define $\mathbb{P}_\omega = \{\phi \in C(\mathbb{R}, \mathbb{R}) : \phi(t + \omega) = \phi(t)\}$, where $C(\mathbb{R}, \mathbb{R})$ is the space of all real valued continuous functions. Then $\mathbb{P}_\omega$ is a Banach space when it is endowed with the supremum norm

$$\|y\| = \sup_{t\in\mathbb{R}} |y(t)| = \sup_{t\in[\tau,\tau+\omega]} |y(t)| \,.$$

Suppose that condition (N) holds. If $y(t, \tau, \xi)$ is the unique solution of DEPCAG (1.5) such that $y(\tau) = \xi$, we define the Poincaré's operator P :

$\mathbb{C}^p \to \mathbb{C}^p$ such that

$$P\xi = D\left(y(\tau+\omega,\tau,\xi) - C\xi\right), \tag{3.1}$$

where $C = \Phi(\tau+\omega,\tau),\ D = (I-C)^{-1}$. The condition (N) implies that D exists.

Lemma 3.1 *Under conditions (P), (L)(or (UC)) and (N) we have that the operator P has a fixed point ξ if and only if $y(\cdot) = y(\cdot,\tau,\xi)$ is ω-periodic.*

Proof. If $P\xi = \xi$, then $y(\tau+\omega,\tau,\xi) - C\xi = D^{-1}\xi$, i.e., $y(\tau+\omega,\tau,\xi) = \xi = y(\tau)$. By Lemma 2.4, the conclusion is proved. ■

Applying the Banach's fixed point theorem we have:

Theorem 3.1 *Suppose that the conditions (N), (P), (L) and (2.15) hold and let*

$$\frac{c_\Phi \varsigma}{1-\varsigma}|D| < 1, \tag{3.2}$$

where ς is defined in (2.15). Then, the DEPCAG (1.5) has only one ω-periodic solution.

Proof. By Proposition 2.1, we obtain

$$y(t) = \Phi(t,\tau)\xi + \int_\tau^t \Phi(t,s) f(s, y(s), y(\gamma(s)))ds,\ t \ge \tau,$$

where $\tau \in J$, $y(\tau) = \xi$. Therefore

$$y(\tau+\omega,\tau,\xi) - C\xi = \int_\tau^{\tau+\omega} \Phi(\tau+\omega,s) f(s,y(s),y(\gamma(s)))ds.$$

Let ξ_1, ξ_2 two initial conditions, then by (3.1), we obtain

$$\begin{aligned}
&|P\xi_1 - P\xi_2| = |D\left(y_1(\tau+\omega,\tau,\xi_1) - y_2(\tau+\omega,\tau,\xi_2)\right) - DC\left(\xi_1-\xi_2\right)| \\
&\le |D| \int_\tau^{\tau+\omega} |\Phi(\tau+\omega,s)|\, |f(s,y_1(s),y_1(\gamma(s))) - f(s,y_2(s),y_2(\gamma(s)))|\, ds \\
&\le c_\Phi\, |D| \int_\tau^{\tau+\omega} \left[\eta_1(s)\, |y_1(s)-y_2(s)| + \eta_2(s)\, |y_1(\gamma(s)) - y_2(\gamma(s))|\right] ds \\
&\le \left(c_\Phi\, |D| \int_\tau^{\tau+\omega} \left[\eta_1(s) + \eta_2(s)\right] ds \right) \max_{t\in[\tau,\tau+\omega]} |y_1(t) - y_2(t)|\,.
\end{aligned}$$

By Lemma 2.3, we get

$$|P\xi_1 - P\xi_2| \le \left(\frac{c_\Phi \varsigma |D|}{1-\varsigma}\right) |\xi_1 - \xi_2|\,, \tag{3.3}$$

where ς is defined in (2.15). This proves that the operator P is a continuous function. By the Banach's fixed point theorem, there is a unique fixed point ξ^*: $P\xi^* = \xi^*$, and the corresponding solution $y(t) = y(t, \tau, \xi^*)$ is ω-periodic, i.e., the equation (1.5) has only one ω-periodic solution. ■

Corollary 3.1 *Suppose that the conditions (N), (P) and (L), with $\eta_1 = \eta_2 = L$ constant, hold. Assume*

$$2c_\Phi L\omega < 1 \text{ and } \frac{2c_\Phi L\omega |D|}{1 - 2c_\Phi L\omega} < 1.$$

Then, the DEPCAG (1.5) has only one ω-periodic solution.

If (L) holds, then, by Lemma 2.3, $y(t, \tau, \xi)$ depends continuously on ξ. However, this continuity follows from other conditions as the local Lipschitz condition or smoothness conditions of the function f, etc. Therefore, from now on, we use the condition (UC).

Theorem 3.2 *Suppose that the conditions (N), (P), (UC) and (H1) hold. Then the DEPCAG (1.5) has an ω-periodic solution.*

Proof. If $y(t, \tau, \xi)$ is the solution of (1.5), then it follows from the variation of constants formula that $y(t, \tau, \xi)$ satisfies for all $\xi \in \mathbb{C}^p$ the integral equation (2.17). Let $\lambda \in (0, 1)$ be arbitrary such that $\lambda c_\Phi \theta < \ln 2$, where $\theta := \max_{i \in \mathbb{N}} (\gamma_i - t_i)$. Because of (H1), there exists some $\beta_\lambda > 0$ so that

$$|f(t, y_1, y_2)| \leq \beta_\lambda + \lambda (|y_1| + |y_2|), \text{ for all } t \in [\tau, \tau + \omega], \ y_1, y_2 \in \mathbb{C}^p. \quad (3.4)$$

Thus from (2.17) and (3.4) it follows that for $t \in [\tau, \tau + \omega]$,

$$\begin{aligned} |y(t, \tau, \xi)| &\leq |\Phi(t, \tau)| |\xi| + \int_\tau^t |\Phi(t, s)| |f(s, y(s), y(\gamma(s)))| ds \\ &\leq |\Phi(t, \tau)| |\xi| + c_\Phi \int_\tau^t [\beta_\lambda + \lambda (|y(s)| + |y(\gamma(s))|)] ds \\ &\leq c_\Phi |\xi| + c_\Phi \beta_\lambda \omega + c_\Phi \int_\tau^t \lambda (|y(s)| + |y(\gamma(s))|) ds. \end{aligned}$$

Hence by Corollary 2.1 of DEPCAG Gronwall's inequality we get for $t \in [\tau, \tau + \omega]$,

$$|y(t, \tau, \xi)| \leq c_\Phi (|\xi| + \beta_\lambda \omega) e^\kappa, \ \kappa = \frac{2\lambda c_\Phi \omega}{2 - \exp(\lambda c_\Phi \theta)}. \quad (3.5)$$

Therefore we obtain, again based on (2.17) and (3.5), that

$$\begin{aligned}|y(t,\tau,\xi)-\Phi(t,\tau)\xi| &\leq c_\Phi\int_\tau^t \lambda\left(|y(s)|+|y(\gamma(s))|\right)ds\\ &\leq c_\Phi\int_\tau^t 2\lambda c_\Phi\left(|\xi|+\beta_\lambda\omega\right)e^\kappa ds\\ &\leq 2\lambda c_\Phi^2\omega e^\kappa|\xi|+2\lambda c_\Phi^2\beta_\lambda\omega^2e^\kappa.\end{aligned}$$

Hence

$$\limsup_{|\xi|\to\infty}\frac{|y(t,\tau,\xi)-\Phi(t,\tau)\xi|}{|\xi|}\leq 2\lambda c_\Phi^2\omega e^\kappa$$

uniformly in $t\in[\tau,\tau+\omega]$. It follows from the arbitrary choice of $\lambda\in(0,1)$ that

$$y(t,\tau,\xi)-\Phi(t,\tau)\xi=o(|\xi|)\ \text{as}\quad |\xi|\to\infty \tag{3.6}$$

uniformly in $t\in[\tau,\tau+\omega]$.
By condition (UC), the Poincaré operator

$$P\xi=\left(I-\Phi(\tau+\omega,\tau)\right)^{-1}\left(y(\tau+\omega,\tau,\xi)-\Phi(\tau+\omega,\tau)\xi\right)$$

is continuous. Moreover, from (3.6) we get that

$$P(\xi)=o(|\xi|)\ \text{as}\ |\xi|\to\infty.$$

Consequently, there exists a $\rho>0$ such that

$$|P(\xi)|\leq\rho+\frac{|\xi|}{2},\quad \forall\xi\in\mathbb{C}^p.$$

Therefore, P maps the closed ball $\bar{B}[0,2\rho]$ into itself, by Brouwer's fixed point theorem it has a fixed point ξ^*, $P\xi^*=\xi^*$, and the corresponding solution $y(t)=y(t,\tau,\xi^*)$ is ω-periodic, i.e., the equation (1.5) has an ω-periodic solution. ∎

Theorem 3.3 *Suppose that the conditions (N), (P), (UC) and (H2) hold. Then the DEPCAG (1.5) has an ω-periodic solution.*

Proof. Let $\varepsilon\in(0,1)$ be arbitrary such that $\varepsilon c_\Phi\theta<\ln 2$, where $\theta:=\max_{i\in\mathbb{N}}(\gamma_i-t_i)$. Because of (H2), there exists some $\delta>0$ so that

$$|f(t,y_1,y_2)|\leq\varepsilon(|y_1|+|y_2|), \tag{3.7}$$

for $t\in[\tau,\tau+\omega]$, $|y_1|+|y_2|\leq\delta$. Thus from (2.17) it follows that for $|\xi|$ small enough there is t^* such that for $t\in[\tau,t^*)$

$$\begin{aligned}|y(t,\tau,\xi)| &\leq |\Phi(t,\tau)|\,|\xi|+\int_\tau^t|\Phi(t,s)|\,|f(s,y(s),y(\gamma(s)))|\,ds\\ &\leq c_\Phi|\xi|+\varepsilon c_\Phi\int_\tau^t\left(|y(s)|+|y(\gamma(s))|\right)ds.\end{aligned}$$

Hence by DEPCAG Gronwall's inequality for $t \in [\tau, t^*]$, we get

$$|y(t,\tau,\xi)| \leq c_\Phi \, |\xi| \, e^{\vartheta}, \ \vartheta = \frac{2\varepsilon c_\Phi \omega}{2 - \exp(\varepsilon c_\Phi \theta)}. \tag{3.8}$$

So, choosing $|\xi|$ small enough, we obtain $|y(t^*)| + |y(\gamma(t^*))| \leq \delta$. Then (3.8) is true for $t \in [\tau, \tau + \omega]$.
Now, using the same technique in the proof of Theorem 3.2, we can conclude

$$y(t,\tau,\xi) - \Phi(t,\tau)\xi = o(|\xi|) \ \text{ as } \ |\xi| \to 0$$

uniformly in $t \in [\tau, \tau + \omega]$, and the equation (1.5) has an ω-periodic solution.
■

Remark 4 *Suppose that (P1) is satisfied by $\omega = \omega_1$ and (P2) by $\omega = \omega_2$, if $\frac{\omega_2}{\omega_1}$ is a rational number, then both (P1) and (P2) are simultaneously satisfied by $\omega = l.c.m.\{\omega_1, \omega_2\}$, where $l.c.m.\{\omega_1, \omega_2\}$ denotes the least common multiple between ω_1 and ω_2. In the general case it is possible that there exist three possible periods: ω_1 for $\{t_i\}$, $\{\gamma_i\}$, ω_2 for A and ω_3 for f, if $\frac{\omega_i}{\omega_j}$ is a rational number for all $i, j = 1, 2, 3$. So, in this situation our results insure the existence of an ω-periodic solution with $\omega = l.c.m.\{\omega_1, \omega_2, \omega_3\}$. Therefore the above results insure the existence of ω-periodic solutions of the DEPCAG (1.5). These solutions are called subharmonic solutions. See section 4.*

4 Applications and Examples

In order to illustrate some features of our main results, in this section, we apply the criteria established above to some mathematical models arising in biology, which have been widely explored in the literature.

These mathematical models were investigated by Lasota-Wazewska, Gurney and Richards, which are well-known models in hematopoiesis and population dynamics. We extend these models to anticipatory models.

The mathematical biologist Robert Rosen introduced the concept of anticipatory systems in 1985,defining anticipatory systems as systems that contain a representation of the system itself. The internal representations can be used by the system for the anticipation because the system's parameters can be varied and recombined within the system. A biological system can use this degree of freedom for anticipatory adaptation, that is, by making a selection in the present among its possible representations in a next (phenotypical) exhibition.

The study of anticipatory systems requires a model that is sufficiently complex to accommodate representations of the system within the system under study. These systems no longer model an external world, but they entertain internal representations of their relevant environments in terms of the ranges of possible further developments. In other words, the possibility of anticipation in systems can be considered as a consequence of the complexity of the analytical model. This additional complexity is found by using the time dimension not as a (historical) given, but as another degree of freedom available to the system.

In our modeling of population dynamics, anticipation means a qualitative kind of prediction enriched by the active moment of decisions made at present real time. Moreover, it seems that complex factors which are not necessarily subjective can be considered as a reason of an anticipation.

We propose, for the first time apparently, to consider equations of the alternately advanced and retarded generalized type, which can represent anticipatory models.

1. In 1976, Wazewska and Lasota [30] proposed a mathematical model

$$y'(t) = -\delta y(t) + pe^{-\gamma y(t-\tau)},\ t \geq 0$$

to describe the survival of red blood cells in an animal; here, $y(t)$ denotes the number of red blood cells at time t, $\delta > 0$ is the probability of death of a red blood cell, p and γ are positive constants related to the production of red blood cells per unit time, and τ is the time required to produce a red blood cell.

We consider the anticipatory Lasota-Wazewska model with DEPCAG as follows:

$$y'(t) = -\delta(t)y(t) + p(t)e^{-y(\gamma(t))},\ t \geq 0, \tag{4.1}$$

where $y(t)$ is the number of red blood cells at time t and $\delta(t), p(t)$ are positive ω-periodic functions, $\{t_i\}_{i\in\{1,..,j\}}$ and $\{\gamma_i\}_{i\in\{1,..,j\}}$ satisfy the property (ω, l). We note that if the initial number of red blood cells $y(0) > 0$ Eq.(4.1) has a positive solution. Indeed, it is easy to verify that for every $t \in J$,

$$\begin{aligned} y(t) &= e^{-\int_0^t \delta(s)ds} y(0) + e^{-y(\gamma_{i(0)}) - \int_{t_{i(0)+1}}^t \delta(s)ds} \left(\int_0^{t_{i(0)+1}} e^{-\int_s^{t_{i(0)+1}} \delta(s)d\kappa} p(s)ds \right) \\ &+ \sum_{j=i(0)+1}^{j=i(t)-1} \left[e^{-y(\gamma_j) - \int_{t_{j+1}}^t \delta(s)ds} \left(\int_{t_j}^{t_{j+1}} e^{-\int_s^{t_{j+1}} \delta(\kappa)d\kappa} p(s)ds \right) \right] \\ &+ e^{-y(\gamma(t))} \left(\int_{t_{i(t)}}^t e^{-\int_s^t \delta(s)d\kappa} p(s)ds \right). \end{aligned}$$

So, the solution y of DEPCAG (4.1) has to be positive. Note that if $\delta(t)$ is a positive ω-periodic function, then the condition (N) is satisfied and thus, using the same technique in the proof of Theorem 3.1 we have:

Theorem 4.1 *Suppose that $\delta(t), p(t)$ are positive ω-periodic functions, $\{t_i\}_{i\in\{1,..,j\}}$ and $\{\gamma_i\}_{i\in\{1,..,j\}}$ satisfy the property (ω, l) and assume that*

$$c_\Phi \int_\tau^{\tau+\omega} p(s)ds < 1 \ \text{and} \ \frac{c_\Phi \int_\tau^{\tau+\omega} p(s)ds}{1 - c_\Phi \int_\tau^{\tau+\omega} p(s)ds}|D| < 1$$

hold, where $c_\Phi = \frac{\exp(\int_\tau^{\tau+\omega}\delta(s)ds)}{\exp(\int_\tau^{\tau+\omega}\delta(s)ds)-1}$ and $|D| = \frac{1}{\exp(\int_\tau^{\tau+\omega}\delta(s)ds)-1}$. Then the DEPCAG (4.1) has a unique positive ω-periodic solution.

As the function $f(t, y_2) = p(t)e^{-y_2}$ satisfies (H1), using the same technique of Theorem 3.2, we have:

Theorem 4.2 *Suppose that $\delta(t), p(t)$ are positive ω-periodic functions, $\{t_i\}_{i\in\{1,..,j\}}$ and $\{\gamma_i\}_{i\in\{1,..,j\}}$ satisfy the property (ω, l), then the DEPCAG (4.1) has a positive ω-periodic solution.*

Remark 5 *Let $\{t_i\}, \{\gamma_i\}$ fulfill property (ω_1, l), $\delta(t)$ is ω_2-periodic and $p(t)$ is ω_3-periodic. Therefore, as there is not the same frequency between $\delta(t)$, $p(t)$ and (ω_1, l), consider $\omega_0 = l.c.m.\{\omega_1, \omega_2, \omega_3\}$ if $\frac{\omega_i}{\omega_j}$ is a rational number for all $i, j = 1, 2, 3$. Replace ω by ω_0, and applying Theorem 4.1, we obtain the existence of a unique positive ω_0-periodic, i.e., a subharmonic solution of Eq.(4.1). This holds in each of our Theorems.*

2. In 1980, Gurney et al.[18] proposed a mathematical model

$$y'(t) = -\delta y(t) + Py(t-\tau)e^{-ay(t-\tau)},$$

to describe the dynamics of Nicholson's blowflies. Here, $y(t)$ is the size of the population at time t, P is the maximum per capita daily egg production, $\frac{1}{a}$ is the size at which the population reproduces at its maximum rate, δ is the per capita daily adult death rate, and τ is the generation time.

Nicholson's blowflies model belongs to a class of biological systems and it has attracted more attention because of its extensively realistic significance, its greater details and discrete analogues.

We consider a class of the anticipatory Nicholson's blowflies model with DEPCAG as follows:

$$y'(t) = -\delta(t)y(t) + p(t)y(\gamma(t))e^{-\beta(t)y(\gamma(t))}, t \geq 0, \tag{4.2}$$

where $\delta(t), p(t), \beta(t)$ are positive ω-periodic functions, $\{t_i\}_{i\in\{1,..,j\}}$ and $\{\gamma_i\}_{i\in\{1,..,j\}}$ satisfy the property (ω, l). Again a solution y of DEPCAG (4.2) is a positive solution if $y(0) > 0$.

As the function $f(t, y_2) = p(t)y_2e^{-\beta(t)y_2}$ satisfies (H1), using the same technique in the proof of Theorem 3.2, we have:

Theorem 4.3 *Suppose that $\delta(t)$, $p(t)$, $\beta(t)$ are positive ω-periodic functions, $\{t_i\}_{i\in\{1,..,j\}}$ and $\{\gamma_i\}_{i\in\{1,..,j\}}$ satisfy the property (ω, l), then the DEPCAG (4.2) has a positive ω-periodic solution.*

3. In 1959, Richards [26] proposed a mathematical model

$$y'(t) = y(t)\left(a - \left(\frac{y(t)}{K}\right)^k\right), \ t, k \geq 0$$

to describe the growth of a single individual or the growth of identical individuals, where, a, K, k are positive constants. This equation is the original model proposed by Richards [26], also known as the Gilpin–Ayala model.

We consider a class of the anticipatory Gilpin-Ayala model with DEPCAG as follows:

$$y'(t) = y(t)\left(a(t) - \left(\frac{y(\gamma(t))}{b(t)}\right)^k\right), \ t \geq 0, \tag{4.3}$$

where $a(t), b(t)$, are positive ω-periodic functions, $k > 0$ $\{t_i\}_{i\in\{1,..,j\}}$ and $\{\gamma_i\}_{i\in\{1,..,j\}}$ satisfy the property (ω, l). This solution of DEPCAG (4.3) is a positive solution if $y(0) > 0$.

As the function $f(t, y_1, y_2) = y_1\left(\frac{y_2}{b(t)}\right)^k$ satisfies (H2), by Theorem 3.3, we have:

Theorem 4.4 *Suppose that $a(t), b(t)$ are positive ω-periodic functions, $k > 0$ and $\{t_i\}_{i\in\{1,..,j\}}$ and $\{\gamma_i\}_{i\in\{1,..,j\}}$ satisfy the property (ω, l), then the DEPCAG (4.3) has a positive ω-periodic solution.*

Remark 6 *If $k = 1$, Eq.(4.3) is a generalized logistic model with DEPCAG.*

Acknowledgements

The authors wish to express their sincere gratitude to the referee for the valuable suggestions, which helped to improve the paper.

References

[1] A. R. Aftabizadeh and J. Wiener, Oscillatory and periodic solutions of an equation alternately of retarded and advanced types, Appl. Anal., 23 (1986), 219-231.

[2] M. U. Akhmet, Integral manifolds of differential equations with piecewise constant argument of generalized type, Nonlinear Anal. TMA 66 (2007) 367-383.

[3] M. U. Akhmet, On the reduction principle for differential equations with piecewise constant argument of generalized type, J. Math. Anal. Appl., 336 (2007) 646-663.

[4] M. U. Akhmet, Almost periodic solutions of differential equation with piecewise constant argument, Nonlinear Analysis: Hybrid Systems, 2 (2008) 456-467.

[5] M. U. Akhmet, C. Buyukadali and T. Ergenc, Periodic solutions of the hybrid system with small parameter, Nonlinear Anal. Hybrid Syst. 2(2008) 532-543.

[6] M. U. Akhmet and C. Buyukadali, On periodic solutions of differential equations with piecewise constant argument, Comput. Math. Appl., 56 (2008) 2034-2042.

[7] L. E. J. Brouwer, Über Abbildung von Mannigfaltigkeiten, Mathematische Annalen, 71 (1910), 97-115.

[8] S. Busenberg and K. Cooke, Vertically transmitted diseases: models and dynamics, in: Biomathematics, vol. 23, Springer-Verlag, Berlin, 1993.

[9] T. A. Burton, Stability and Periodic Solutions of Ordinary and Functional Differential Equations, Academic Press Inc., New York, 1985.

[10] T. A. Burton and T. Furumochi, Periodic and asymptotically periodic solutions of neutral integral equations, Electronic J. Qual. Th. Diff. Equs. 10 (2000), 1-19.

[11] A. Cabada, J. B. Ferreiro and J. J. Nieto, Green's function and comparison principles for first order periodic differential equations with piecewise constant arguments, J. Math. Anal. Appl. 291 (2004), 690-697.

[12] K. L. Cooke and J. Wiener, Retarded differential equations with piecewise constant delays, J. Math. Anal. Appl. 99(1984), 265-297.

[13] K. L. Cooke and J. Wiener, An equation alternately of retarded and advanced type, Proceedings of the American Mathematical Society. 99 (1987), 726-732.

[14] K. L. Cooke and J. Wiener, Oscillations in systems of differential equations with piecewise constant argument, J. Math. Anal. Appl. 137 (1989), 221-239.

[15] K.L. Cooke and J.Wiener, A survey of differential equations with piecewise continuous argument, in: Lecture Notes in Math., vol. 1475, Springer, Berlin, 1991, pp. 1-15.

[16] Kuo-Shou Chiu and M. Pinto, Oscillatory and periodic solutions of differential equations with piecewise constant argument of generalized type. Submitted.

[17] K. Gopalsamy, Stability and Oscillation in Delay Differential Equations of Population Dynamics, Math. Appl., vol. 74, Kluwer Academic, Dordrecht, 1992.

[18] W. S. Gurney, S. P. Blythe and R. M. Nisbet, Nicholson's blowflies (revisited), Nature 287, 1980, 17-21.

[19] Y. Kuang, Delay Differential Equations with Applications in Population Dynamics, New York, Academic Press, 1993.

[20] J. J. Nieto and R. Rodríguez-López, Green's function for second-order periodic boundary value problems with piecewise constant arguments, J. Math. Anal. Appl. 304 (2005), 33-57.

[21] A. Omon and M. Pinto, Asymptotics for solutions of periodic difference systems, J. Difference Eqs. Appl., 15(5) (2009), 461-472.

[22] M. Pinto, Asymptotic equivalence of nonlinear and quasilinear differential equations with piecewise constant arguments, Mathematical and Computer Modelling, 49 (2009), 1750-1758.

[23] M. Pinto, Bounded and periodic solutions of nonlinear integro-differential equations with infinite delay, Electronic J. Qual. Th. Diff. Equs. 46 (2009), 1-20.

[24] M. Pinto, Dichotomy and existence of periodic solutions of quasilinear functional differential equations, Nonlinear Anal., TMA, 72 (2010), 1227-1234.

[25] M. Pinto, Cauchy and Green matrices type and stability in alternately advanced and delayed differential systems. J. Difference Eqs. Appl. (in press).

[26] F. J. Richards, A flexible growth function for empirical use, J. Exp. Botany 10 (29) (1959), 290-300.

[27] S. M. Shah and J. Wiener, Advanced differential equations with piecewise constant argument deviations, Internat. J. Math. and Math. Sci. 6 (1983), 671-703.

[28] J. Wiener, Generalized Solutions of Functional Differential Equations, World Scientific, Singapore, 1993.

[29] J. Wiener, Differential equations with piecewise constant delays, in Trends in the Theory and Practice of Nonlinear Differential Equations, ed. V. Lakshmikantham (Marcel Dekker, New York, 1983) pp. 547-580.

[30] M. Wazewska-Czyzewska, A. Lasota, Mathematical problems of the dynamics of red blood cells system, Ann. Polish Math. Soc. Ser. III Appl. Math. 6, 1976, 23-40.

[31] Y. H. Xia, M. Lina and J. Cao, The existence of almost periodic solutions of certain perturbation systems, J. Math. Anal. Appl. 310 (2005), 81-96.

[32] Y. H. Xia, Z. Huang and M. Han, Existence of almost periodic solutions for forced perturbed systems with piecewise constant argument, J. Math. Anal. Appl. 333 (2007), 798-816.

[33] Y. H. Xia, Positive periodic solutions for a neutral impulsive delayed Lotka-Volterra competition system with the effect of toxic substance, Nonlinear Analysis: RWA, 8, No 2 (2007), 204-221.

[34] Y.H. Xia and J. Cao, Global attractivity of a periodic ecological model with m-predators and n-preys by "pure-delay type" system, Comput. Math. Appl. 52 (6-7) (2006), 829-852.

[35] P. Yang, Y. Liu and W. Ge, Green's function for second order differential equations with piecewise constant argument, Nonlinear Anal. 64 (2006), 1812-1830.

(Received July 7, 2010)