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A regularity criterion for the three-dimensional micropolar fluid system in homogeneous Besov spaces

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Abstract. By establishing a new trilinear estimate, we show a regularity criterion for the three dimensional micropolar fluid system via the velocity in homogeneous Besov spaces. This improves [B. Q. Dong, Z. L. Zhang, On the regularity criterion for three-dimensional micropolar fluid flows in Besov spaces, *Nonlinear Anal.* **73**(2010), 2334–2341] in some sense.

Keywords: micropolar fluid flows, regularity criteria, Besov spaces.

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1 Introduction

In this paper, we study the following three-dimensional micropolar fluid system with unit viscosities:

$$\begin{cases} \partial_t \boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} - \triangle \boldsymbol{u} + \nabla P - \nabla\times\boldsymbol{w} = \boldsymbol{0}, \\ \partial_t \boldsymbol{w} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{w} - \triangle \boldsymbol{w} + 2\boldsymbol{w} - \nabla \operatorname{div}\boldsymbol{w} - \nabla\times\boldsymbol{u} = \boldsymbol{0}, \\ \nabla\cdot\boldsymbol{u} = 0, \\ (\boldsymbol{u},\boldsymbol{w})|_{t=0} = (\boldsymbol{u}_0,\boldsymbol{w}_0), \end{cases} \tag{1.1}$$

where $\boldsymbol{u} = (u_1, u_2, u_3)$ is the fluid velocity field, P is the pressure arising from the usual assumption of incompressibility $\operatorname{div}\boldsymbol{u} = 0$, $\boldsymbol{w} = (w_1, w_2, w_3)$ is the micro-rotation vector field, $\boldsymbol{u}_0$ and $\boldsymbol{w}_0$ are the prescribed initial data satisfying $\nabla\cdot\boldsymbol{u}_0 = 0$.

The modern theory of micropolar fluid system was initialled by Eringe [9], which models some physical phenomena that cannot be treated by the classical Navier–Stokes equations, for example, the motion of animal blood, liquid crystals and dilute aqueous polymer solutions for example. Mathematically, many authors [3, 4, 7, 10, 20, 22, 24] treated the well-posedness and large-time behaviour of solutions to system (1.1). However, the issue of global regularity of weak solutions to (1.1) remains an open problem. Therefore it is important to study the

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regularity criterion on the some physical quantities, such as velocity, vorticity and pressure. Dong and Zhang [6] showed the following regularity condition

$$\boldsymbol{u} \in L^{\frac{2}{1+r}}(0,T;B^r_{\infty,\infty}(\mathbb{R}^3)), \qquad -1 < r < 1. \tag{1.2}$$

Here and in what follows, $\dot{B}^s_{p,q}(\mathbb{R}^3)$ (resp. $B^s_{p,q}(\mathbb{R}^3)$) with $s \in \mathbb{R}$, $p,q \in [1,\infty]$ is the inhomogeneous (homogeneous) Besov space, whose definition, fine properties and its utilization in fluid dynamical systems can be found in [1]. Later, Wang and Yuan [23] and He and Wang [17] (with [17, Equation 1.10] replaced by [26]) then considered the following regularity criterion in a logarithmically improved version

$$\int_0^T \frac{\|P\|^2_{\dot{B}^r_{\infty,\infty}}}{1+\ln\left(e+\|P\|_{\dot{B}^r_{\infty,\infty}}\right)}\,\mathrm{d}t < \infty, \qquad -1 \le r < 1. \tag{1.3}$$

For interested readers, please refer to [5, 8, 11–16, 18, 25, 29].

The purpose of this paper is to improve (1.2) from inhomogeneous Besov spaces to homogeneous ones. Before stating the precise result, let us recall the weak formulation of (1.1).

Definition 1.1. Let $\boldsymbol{u}_0 \in L^2(\mathbb{R}^3)$ with $\operatorname{div}\boldsymbol{u}_0 = 0$, $\boldsymbol{w}_0 \in L^2(\mathbb{R}^3)$. A measurable $\mathbb{R}^3$-valued pair $(\boldsymbol{u},\boldsymbol{w})$ is called a weak solution to system (1.1) on $(0,T)$, provided the following two conditions hold,

(1) $(\boldsymbol{u},\boldsymbol{w}) \in L^\infty(0,T;L^2(\mathbb{R}^3)) \cap L^2(0,T;H^1(\mathbb{R}^3))$;

(2) $(\boldsymbol{u},\boldsymbol{w})$ verifies (1.1) in the distributional sense.

Now, our result reads as follows.

Theorem 1.2. *Let $T>0$, $0<r<1$ and $(\boldsymbol{u},\boldsymbol{w})$ be a weak solution pair of system (1.1) with initial data $(\boldsymbol{u}_0,\boldsymbol{w}_0) \in H^1(\mathbb{R}^3)$. If*

$$\boldsymbol{u} \in L^{\frac{2}{1+r}}(0,T;\dot{B}^r_{\infty,\infty}(\mathbb{R}^3)), \tag{1.4}$$

then the solution $(\boldsymbol{u},\boldsymbol{w})$ is regular on $(0,T)$.

Remark 1.3. Due to the fact that $B^s_{p,q}(\mathbb{R}^3) = L^p(\mathbb{R}^3) \cap \dot{B}^s_{p,q}(\mathbb{R}^3)$ for any $s>0$ and $1 \le p,q \le \infty$ (see [2, Theorem 6.3.2]), we point that, in the case $0<r<1$, (1.4) is an improvement of (1.2).

Remark 1.4. The refinement $\boldsymbol{u} \in L^1(0,T;\dot{B}^1_{\infty,\infty}(\mathbb{R}^3))$ of (1.2) in case $r=1$ was already established in [5, Theorem 1.3]; while the gain

$$\boldsymbol{u} \in L^2(0,T;\dot{B}^0_{\infty,\infty}(\mathbb{R}^3)) \tag{1.5}$$

of (1.2) in case $r=0$ can be similarly verified just as [27, Theorem 1.1].

Remark 1.5. By observing the following fact from [21, Section 1.3]:

$$s<0,\ p,q \ge 1 \Rightarrow \dot{B}^s_{p,q}(\mathbb{R}^3) \subset B^s_{p,q}(\mathbb{R}^3),$$

we see that the condition $\boldsymbol{u} \in L^{\frac{2}{1+r}}(0,T;\dot{B}^r_{\infty,\infty}(\mathbb{R}^3))$ with $-1<r<0$ could also ensure the smoothness of the solution. One is referred to [19, 28] for similar results of the Navier–Stokes equations.

Before showing Theorem 1.2 in Section 2, let us end this introduction by proving the following trilinear estimate, which could have its own interest.

Lemma 1.6. *For $f \in \dot{B}^r_{\infty,\infty}(\mathbb{R}^3)$, $g, h \in H^1(\mathbb{R}^3)$ and any $\varepsilon > 0$, $0 < r < 1$, $k \in \{1,2,3\}$, we have*

$$\int_{\mathbb{R}^3} \partial_k f \cdot gh \,\mathrm{d}x \leq C \|f\|^{\frac{2}{1+r}}_{\dot{B}^r_{\infty,\infty}} \|(g,h)\|^2_{L^2} + \varepsilon \|\nabla(g,h)\|^2_{L^2}. \tag{1.6}$$

Proof.

$$\begin{aligned}
\int_{\mathbb{R}^3} \partial_k f \cdot gh \,\mathrm{d}x &= -\int_{\mathbb{R}^3} f \cdot \partial_k(gh)\,\mathrm{d}x = -\int_{\mathbb{R}^3} \Lambda^r f \cdot \Lambda^{-r}\partial_k(gh)\,\mathrm{d}x \qquad \left(\Lambda = (-\triangle)^{\frac{1}{2}}\right)\\
&\leq C \|\Lambda^r f\|_{\dot{B}^0_{\infty,\infty}} \left\|\Lambda^{-r}\partial_k(gh)\right\|_{\dot{B}^0_{1,1}} \quad \text{(by [1, Proposition 2.29])}\\
&\leq C \|f\|_{\dot{B}^r_{\infty,\infty}} \|gh\|_{\dot{B}^{1-r}_{1,1}} \quad \text{(by [1, Lemma 2.1])}\\
&\leq C \|f\|_{\dot{B}^r_{\infty,\infty}} \left(\|g\|_{L^2} \|h\|_{\dot{B}^{1-r}_{2,1}} + \|g\|_{\dot{B}^{1-r}_{2,1}} \|h\|_{L^2}\right)\\
&\quad \text{(by analogues of [1, Corollary 2.54])}\\
&\leq C \|f\|_{\dot{B}^r_{\infty,\infty}} \left(\|g\|_{L^2} \|h\|^r_{\dot{B}^0_{2,\infty}} \|h\|^{1-r}_{\dot{B}^1_{2,\infty}} + \|g\|^r_{\dot{B}^0_{2,\infty}} \|g\|^{1-r}_{\dot{B}^1_{2,\infty}} \|h\|_{L^2}\right)\\
&\quad \text{(by [1, Proposition 2.22])}\\
&\leq C \|f\|_{\dot{B}^r_{\infty,\infty}} \left(\|g\|_{L^2} \|h\|^r_{L^2} \|\nabla h\|^{1-r}_{L^2} + \|g\|^r_{L^2} \|\nabla g\|^{1-r}_{L^2} \|h\|_{L^2}\right)\\
&\quad \text{(by [1, Proposition 2.39])}\\
&\leq C \|f\|_{\dot{B}^r_{\infty,\infty}} \|(g,h)\|^{1+r}_{L^2} \|\nabla(g,h)\|^{1-r}_{L^2}\\
&\leq C \|f\|^{\frac{2}{1+r}}_{\dot{B}^r_{\infty,\infty}} \|(g,h)\|^2_{L^2} + \varepsilon \|\nabla(g,h)\|^2_{L^2}.
\end{aligned}$$

□

2 Proof of Theorem 1.2

In this section, we give the proof of Theorem 1.2. Taking the inner product of $(1.1)_1$ with $-\triangle \boldsymbol{u}$ in $L^2(\mathbb{R}^3)$, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \|\nabla \boldsymbol{u}\|^2_{L^2} + \|\triangle \boldsymbol{u}\|^2_{L^2} = \int_{\mathbb{R}^3} [(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}]\cdot\triangle\boldsymbol{u}\,\mathrm{d}x - \int_{\mathbb{R}^3} (\nabla\times\boldsymbol{w})\cdot\triangle\boldsymbol{u}\,\mathrm{d}x. \tag{2.1}$$

On the other hand, testing $(1.1)_2$ by $-\triangle \boldsymbol{w}$, we get

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \|\nabla \boldsymbol{w}\|^2_{L^2} + \|\triangle \boldsymbol{w}\|^2_{L^2} + 2\|\nabla \boldsymbol{w}\|^2_{L^2} + \|\nabla \mathrm{div}\, \boldsymbol{w}\|^2_{L^2}\\
&\quad = \int_{\mathbb{R}^3} [(\boldsymbol{u}\cdot\nabla)\boldsymbol{w}]\cdot\triangle\boldsymbol{w}\,\mathrm{d}x - \int_{\mathbb{R}^3} (\nabla\times\boldsymbol{u})\cdot\triangle\boldsymbol{w}\,\mathrm{d}x.
\end{aligned} \tag{2.2}$$

Plugging (2.1) and (2.2) together, integrating by parts then yields

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \|\nabla(\boldsymbol{u},\boldsymbol{w})\|^2_{L^2} + \|\triangle(\boldsymbol{u},\boldsymbol{w})\|^2_{L^2} + 2\|\nabla \boldsymbol{w}\|^2_{L^2} + \|\nabla \mathrm{div}\, \boldsymbol{w}\|^2_{L^2}\\
&\quad = \left[-\sum_{i=1}^3 \int_{\mathbb{R}^3} [(\partial_i\boldsymbol{u}\cdot\nabla)\boldsymbol{u}]\cdot\partial_i\boldsymbol{u}\,\mathrm{d}x - \sum_{i=1}^3 \int_{\mathbb{R}^3} [(\partial_i\boldsymbol{u}\cdot\nabla)\boldsymbol{w}]\cdot\partial_i\boldsymbol{w}\,\mathrm{d}x\right]\\
&\qquad -2\int_{\mathbb{R}^3} (\nabla\times\boldsymbol{w})\cdot\triangle\boldsymbol{u}\,\mathrm{d}x\\
&\quad \equiv I + J.
\end{aligned} \tag{2.3}$$

By Lemma 1.6,

$$I \leq C \|\boldsymbol{u}\|_{\dot{B}^r_{\infty,\infty}}^{\frac{2}{1+r}} \|\nabla(\boldsymbol{u},\boldsymbol{w})\|_{L^2}^2 + \frac{1}{4}\|\triangle(\boldsymbol{u},\boldsymbol{w})\|_{L^2}^2. \tag{2.4}$$

Meanwhile,

$$J \leq 2\|\nabla\times\boldsymbol{w}\|_{L^2}\|\triangle\boldsymbol{u}\|_{L^2} \leq 2\|\nabla\boldsymbol{w}\|_{L^2}^2 + \frac{1}{2}\|\triangle\boldsymbol{u}\|_{L^2}^2. \tag{2.5}$$

Putting (2.4) and (2.5) into (2.3), we deduce

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla(\boldsymbol{u},\boldsymbol{w})\|_{L^2}^2 + \frac{1}{4}\|\triangle(\boldsymbol{u},\boldsymbol{w})\|_{L^2}^2 + \|\nabla\mathrm{div}\,\boldsymbol{w}\|_{L^2}^2 \leq C\|\boldsymbol{u}\|_{\dot{B}^r_{\infty,\infty}}^{\frac{2}{1+r}}\|\nabla(\boldsymbol{u},\boldsymbol{w})\|_{L^2}^2.$$

Applying Gronwall's inequality with the fact (1.4), we find

$$\|\nabla(\boldsymbol{u},\boldsymbol{w})(t)\|_{L^2}^2 \leq \|\nabla(\boldsymbol{u}_0,\boldsymbol{w}_0)\|_{L^2}^2\cdot\exp\left[\int_0^T\|\boldsymbol{u}(\tau)\|_{\dot{B}^r_{\infty,\infty}}^{\frac{2}{1+r}}\,\mathrm{d}\tau\right] < \infty, \qquad \forall\, 0\leq t<T.$$

By Sobolev's inequality,

$$\boldsymbol{u}\in L^\infty(0,T;L^6(\mathbb{R}^3)) \subset L^4(0,T;L^6(\mathbb{R}^3)).$$

By [5, Theorem 1.1], we complete the proof of Theorem 1.2.

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