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An upper bound for the amplitude of limit cycles of Liénard-type differential systems

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Abstract. In this paper, we investigate the position problem of limit cycles for a class of Liénard-type differential systems. By considering the upper bound of the amplitude of limit cycles on $\{(x,y) \in \mathbb{R}^2 : x < 0\}$ and $\{(x,y) \in \mathbb{R}^2 : x > 0\}$ respectively, we provide a criterion concerning an explicit upper bound for the amplitude of the unique limit cycle of the Liénard-type system on the plane. Here the amplitude of a limit cycle on $\{(x,y) \in \mathbb{R}^2 : x < 0\}$ (*resp.* $\{(x,y) \in \mathbb{R}^2 : x > 0\}$) is defined as the minimum (*resp.* maximum) value of the x-coordinate on such a limit cycle. Finally, we give two examples including an application to predator–prey system model to illustrate the obtained theoretical result, and Matlab simulations are presented to show the agreement between our theoretical result with the simulation analysis.

Keywords: Liénard-type system, limit cycle, amplitude, upper bound.

2010 Mathematics Subject Classification: 34A26, 34A34, 34C07.

1 Introduction

Liénard differential equation

$$\frac{d^2x}{dt^2} + f(x)\frac{dx}{dt} + g(x) = 0 \tag{1.1}$$

is one of the more studied differential systems, which is originated from physics and proposed by French physicist Liénard in 1928. By the Liénard transformation, the second order differential equation (1.1) can be transformed into the following equivalent two dimensional Liénard system

$$\frac{dx}{dt} = y - F(x), \qquad \frac{dy}{dt} = -g(x), \tag{1.2}$$

where $F(x) = \int_0^x f(s)ds$. The more general differential system is of the form

$$\frac{dx}{dt} = h(y) - F(x), \qquad \frac{dy}{dt} = -g(x), \tag{1.3}$$

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which is called as Liénard-type system or generalized Liénard system. The qualitative theory of (1.2) and (1.3) is an important and challenging problem which has attracted great interests of many researchers. So far there have been rich achievements concerning the existence and uniqueness of limit cycles, the number of limit cycles, and the bifurcation of limit cycles, see for example [2, 4, 6, 9, 12, 13, 15–17, 24–29], and references therein. The Liénard system (1.2) and Liénard-type system (1.3) are two important classes of nonlinear systems of ordinary differential equations, because these two dynamical systems often appear in several branches of science such as biology, chemistry, mechanics and engineering. Indeed (1.2) and (1.3) are suitable mathematical models for many practical applications in the real world, and the relevant subjects have been the focus of many recent studies. On the application side, many planar models in mechanics, engineering, biology, chemical reaction and ecological models can be transformed into (1.2) or (1.3), and the existence and uniqueness of limit cycles in the original system can be proved via the existence and uniqueness of limit cycles for the transformational system, see for example [5, 7, 10, 14, 21–23] and references therein. Hence it deserves considerable attention to investigate the nonlinear Liénard system (1.2) and Liénard-type system (1.3).

On the other hand, one of the classical problems in the qualitative theory of planar ordinary differential systems is to characterize the number and relative position of limit cycles. This problem restricted to planar polynomial differential systems is the second part of the well known Hilbert's 16th problem, for more details we refer to [11] for example. When the functions $F(x)$ and $g(x)$ in (1.2) are real coefficient polynomials in the variable x and have degree n and m respectively, then (1.2) is called as a generalized Liénard polynomial differential system. In such case, the investigation for the number of limit cycles becomes a discussion to the second part of the Hilbert's 16th problem. Up to now, there are a lot of works concerning the number of limit cycles of the generalized Liénard polynomial differential systems [1, 3, 8, 18–20]. However, to the best of our knowledge there are few papers involving the position problem of limit cycles. Especially, there is hardly any result on the relative position of limit cycles to the nonlinear Liénard-type system (1.3). So we focus on the amplitude of limit cycles of (1.3). Inspired by [24], in which the authors studied the location problem of a limit cycle for a class of Liénard systems (1.2) with symmetry. In this paper we investigate the same problem to the general Liénard-type system (1.3), and we obtain an explicit upper bound for the amplitude of the unique limit cycle of (1.3) under the sufficient assumptions proving the existence and uniqueness of limit cycles of (1.3). Hence, in some sense this paper is a generalization.

As we know, there are many rich achievements concerning the existence and uniqueness of limit cycles of (1.3). In the existing papers, to guarantee the existence and uniqueness of solutions of the system, it is often assumed that $g(x)$ is continuous, $F(x)$ is continuously differentiable on (b, a) with $-\infty \leq b < 0 < a \leq +\infty$, and $h(y)$ is continuously differentiable for $y \in \mathbb{R}$. Beyond that it is always assumed that

(i) $xg(x) > 0$ for $x \neq 0$, and denote by $G(x) = \int_0^x g(s)ds$ satisfying $G(\pm\infty) = +\infty$;

(ii) there exist x_1 and x_0 satisfying $b < x_1 < 0 < x_0 < a$, such that $F(x_1) = F(0) = F(x_0) = 0$ and $xF(x) > 0$ for $x \in (x_0, a) \cup (b, x_1)$, $xF(x) < 0$ for $x \in (x_1, x_0)$;

(iii) $F'(x) > 0$ for $x \in (x_0, a) \cup (b, x_1)$;

(iv) $h'(y) > 0$ for $y \in \mathbb{R}$, $h(0) = 0$ and $h(\pm\infty) = \pm\infty$.

We note that the existence of limit cycles can be proved by various methods based on the well known Poincaré–Bendixson annular region theorem by constructing a trapping zone where a limit cycle is located. While for the uniqueness of the limit cycle, it can be established by various techniques based on the geometric properties of the isocline curve $h(y) = F(x)$. From the above assumptions, it is easy to obtain the existence and uniqueness of a limit cycle of (1.3). Then in this paper, we are concerned with the position problem of the unique limit cycle of (1.3). By considering the amplitudes of the unique limit cycle on half planes $\{(x,y) \in \mathbb{R}^2 : x > 0\}$ and $\{(x,y) \in \mathbb{R}^2 : x < 0\}$ respectively, we obtain an explicit upper bound for the amplitude of the unique limit cycle of (1.3) on $\mathbb{R}^2$.

The paper is organized as follows. In the next section, we present some relevant preliminaries on the existence and uniqueness of limit cycles of the Liénard-type system. In Section 3, we give several relevant lemmas which can be used to derive the upper bound of amplitudes of the unique limit cycle on $\{(x,y) \in \mathbb{R}^2 : x > 0\}$ and $\{(x,y) \in \mathbb{R}^2 : x < 0\}$ respectively. Then we provide a criterion concerning the position of the unique limit cycle on $\mathbb{R}^2$. In Section 4, we give two examples including an application to predator–prey system model and the corresponding Matlab simulations to illustrate the obtained result. Conclusion is stated in Section 5.

2 Preliminaries

Consider the following Liénard-type system

$$\begin{cases} \dfrac{dx}{dt} = h(y) - F(x), \\ \dfrac{dy}{dt} = -g(x). \end{cases} \tag{2.1}$$

Suppose that the following assumptions hold:

$(H1)$ there exist $x_0 > 0$ and $x_1 < 0$ such that $F(x_1) = F(0) = F(x_0) = 0$, and $F(x)(x - x_0) > 0$ for $x \in (0, +\infty) \setminus \{x_0\}$, $F(x)(x - x_1) > 0$ for $x \in (-\infty, 0) \setminus \{x_1\}$ and $F(\pm\infty) = \pm\infty$;

$(H2)$ $F \in C^1(\mathbb{R}, \mathbb{R})$ satisfying $F'(x) > 0$ for $x \in (x_0, +\infty) \cup (-\infty, x_1)$;

$(H3)$ $g \in C(\mathbb{R}, \mathbb{R})$ satisfying $xg(x) > 0$ for $x \neq 0$, and denote by $G(x) \triangleq \int_0^x g(s)ds$ satisfying $G(\pm\infty) = +\infty$;

$(H4)$ $h \in C^1(\mathbb{R}, \mathbb{R})$ satisfying $h'(y) > 0$, $yh(y) > 0$ for $y \neq 0$, and $h(\pm\infty) = \pm\infty$.

By $(H1)$–$(H4)$, it is easy to obtain that the system (2.1) has a unique limit cycle surrounding the unique equilibrium point $O(0,0)$. Since $x' = h(y)$ for $x = 0$, it follows that all orbits of (2.1) other than the origin run around the origin O in a clockwise fashion on $\mathbb{R}^2$. Moreover, according to the theorem of the existence and uniqueness of solutions then for any initial point $P(x_0, y_0) \in \mathbb{R}^2 \backslash \{O\}$ the system (2.1) has a unique solution $\varphi(P,t)$ satisfying $\varphi(P,0) = P$ for $t \in (-T_1, T_2)$, where $(-T_1, T_2)$ is the maximum existence interval of the solution. The corresponding orbit is denoted by $L_P = \{\varphi(P,t) : -T_1 < t < T_2\}$. Similarly, we let $L_P^+ = \{\varphi(P,t) : 0 \le t < T_2\}$ denote the positive half orbit starting from the point P, and let $L_P^- = \{\varphi(P,t) : -T_1 < t \le 0\}$ denote the negative half orbit. For convenience, we also define the following regions:

$$\begin{aligned}
&\Sigma_0 = \{(x,y) : x = 0, -\infty < y < +\infty\}, && \Sigma_F = \{(x,y) \in \mathbb{R}^2 : h(y) = F(x)\},\\
&\Sigma_0^+ = \{(x,y) : x = 0,\ y > 0\}, && \Sigma_0^- = \{(x,y) : x = 0,\ y < 0\},\\
&\Sigma_+ = \{(x,y) \in \mathbb{R}^2 : x > 0\}, && \Sigma_- = \{(x,y) \in \mathbb{R}^2 : x < 0\},\\
&\Sigma_F^+ = \{(x,y) \in \mathbb{R}^2 : x > 0, F(x) > h(y)\}, && \Sigma_F^- = \{(x,y) \in \mathbb{R}^2 : x > 0, F(x) < h(y)\},\\
&\widetilde{\Sigma}_F^+ = \{(x,y) \in \mathbb{R}^2 : x < 0, F(x) > h(y)\}, && \widetilde{\Sigma}_F^- = \{(x,y) \in \mathbb{R}^2 : x < 0, F(x) < h(y)\}.
\end{aligned}$$

By the geometric properties of solutions of (2.1), it follows that for any orbit starting from $P(0,y) \in \Sigma_0^+$ (*resp.* Σ_0^-) then the positive orbit L_P^+ enters Σ_F^- (*resp.* $\widetilde{\Sigma}_F^+$) when $t > 0$ small, runs around the origin O on Σ_+ (*resp.* Σ_-) in a clockwise fashion, and crosses $\Sigma_F \setminus \{O\}$ transversally for the first time and only once in a finite time. After L_P^+ makes a half turn around the origin on Σ_+ (*resp.* Σ_-), it again intersects with Σ_0^- (*resp.* Σ_0^+) for the first time in a finite time. By the direction of vector field of the system, it follows that L_P^+ continues to enter $\widetilde{\Sigma}_F^+$ (*resp.* Σ_F^-) and runs clockwise on Σ_- (*resp.* Σ_+), intersecting successively with $\Sigma_F \setminus \{O\}$ and Σ_0^+ (*resp.* Σ_0^-) in a finite time. After making a circle on $\mathbb{R}^2$, if L_P^+ returns to the point $P(0,y)$ for the first time, then the orbit corresponding to the initial point P is the unique limit cycle of (2.1). In the next section, we investigate the position of the unique limit cycle of (2.1), *i.e.* we obtain the amplitude of the unique limit cycle, where the amplitude of a limit cycle on $\mathbb{R}^2$ is defined as the maximum absolute value of the x-coordinate on such a limit cycle.

3 Main results

In this section, we investigate the upper bound for the amplitude of the unique limit cycle of (2.1). Let

$$\lambda(x(t),y(t)) = H(y(t)) + G(x(t)), \tag{3.1}$$

where $H(y) = \int_0^y h(s)ds$ and $G(x) = \int_0^x g(s)ds$ satisfying $H(0) = 0$ and $G(0) = 0$. Along the solutions of (2.1) then

$$\frac{d\lambda(x(t),y(t))}{dt} = -h(y(t))g(x(t)) + g(x(t))[h(y(t)) - F(x(t))] = -g(x(t))F(x(t)). \tag{3.2}$$

Since $F(x)(x - x_0) > 0$ for $x \in (0,+\infty) \setminus \{x_0\}$ and $F(x)(x - x_1) > 0$ for $x \in (-\infty,0) \setminus \{x_1\}$, it follows from $(H3)$ that $\frac{d\lambda(x(t),y(t))}{dt} > 0$ for $x \in (x_1,0) \cup (0,x_0)$. This implies that the unique limit cycle of (2.1) is not completely contained in the strip region $\{(x,y) \in \mathbb{R}^2 : x_1 < x < x_0\}$, *i.e.* the limit cycle must encircle $(x_1,0)$ and $(x_0,0)$ as interior points. Hence for the purpose of the amplitude of the unique limit cycle of (2.1), we only consider the orbits crossing Σ_F transversally for $x > x_0$ and $x < x_1$. Consider a positive orbit L_A^+ starting from $A(0,y_A) \in \Sigma_0^+$ (see Figure 3.1). It follows that L_A^+ enters Σ_F^- when $t > 0$ small, intersects with Σ_F for $x > x_0$ at $B(x_B,y_B)$, and after L_A^+ makes a half turn around the origin on Σ_+ then it eventually intersects with Σ_0^- at $C(0,y_C)$. Later L_A^+ starting from the point C continues to enter $\widetilde{\Sigma}_F^+$, and intersect successively with Σ_F and Σ_0^+ at $D(x_D,y_D)$ with $x_D < x_1$ and $E(0,y_E)$.

By the properties of planar autonomous systems, we let $\Delta_+(x_B) = H(y_A) - H(y_C)$ denote the positive half trajectory arc of (2.1) on Σ_+ crossing Σ_F for $x_B > x_0$ (see $\widehat{ABC}$ in Figure 3.1), and let $\Delta_-(x_D) = H(y_C) - H(y_E)$ denote the positive half trajectory arc on Σ_- crossing Σ_F for $x_D < x_1$ (see $\widehat{CDE}$ in Figure 3.1). The main result concerning an upper bound for the amplitude of the unique limit cycle of (2.1) as follows.

Theorem 3.1. *Let* $(H1)$*–*$(H4)$ *hold, then there exists an explicit upper bound* $x^* = \max\{x_B, |x_D|\}$ *for the amplitude of the unique limit cycle of* (2.1)*, where* x_B *and* x_D *are determined by* $\int_0^x F(x)g(x)dx = 0$*. In other words, the unique limit cycle locates in the strip region* $\{(x,y) \in \mathbb{R}^2 : |x| < x^*\}$*.*

We first give the following several relevant lemmas, which can be used to derive the amplitude of the unique limit cycle of (2.1).

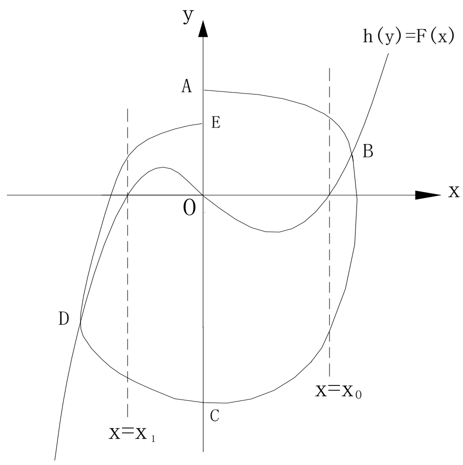

Figure 3.1: An orbit of (2.1) crossing Σ_F transversally when $x > x_0$ and $x < x_1$.

Lemma 3.2. *For any* $x_B \in (x_0, +\infty)$*, then the function* $\Delta_+(x_B)$ *is strictly monotone increasing satisfying* $\Delta_+(x_B) \to +\infty$ *as* $x_B \to +\infty$*; while for any* $x_D \in (-\infty, x_1)$ *it follows that the function* $\Delta_-(x_D)$ *is strictly monotone decreasing satisfying* $\Delta_-(x_D) \to +\infty$ *as* $x_D \to -\infty$*.*

Proof. Consider any two orbits of (2.1) $\widehat{A_1B_1C_1D_1E_1}$ and $\widehat{A_2B_2C_2D_2E_2}$, which cross Σ_F transversally when $x > x_0$ and $x < x_1$ (see Figure 3.2). Let these two orbits start respectively from $A_1(0, y_{A_1}), A_2(0, y_{A_2}) \in \Sigma_0^+$ with $y_{A_2} > y_{A_1} > 0$. When $x > x_0$ the orbits intersect with Σ_F at $B_1(x_{B_1}, y_{B_1})$ and $B_2(x_{B_2}, y_{B_2})$ satisfying $x_{B_2} > x_{B_1} > x_0$, make a half turn in a clockwise fashion on Σ_+ then they again intersect with Σ_0^- at $C_1(0, y_{C_1})$ and $C_2(0, y_{C_2})$. By the properties of planar autonomous systems, it follows that $y_{C_2} < y_{C_1} < 0$. Later the orbits continue to enter Σ_-, and intersect with Σ_F at $D_1(x_{D_1}, y_{D_1})$ and $D_2(x_{D_2}, y_{D_2})$ satisfying $x_{D_2} < x_{D_1} < x_1$. After making a circle then the orbits eventually intersect with Σ_0^+ at $E_1(0, y_{E_1})$ and $E_2(0, y_{E_2})$ satisfying $y_{E_2} > y_{E_1} > 0$. In Figure 3.2, F_1, G_1 and F_2, G_2 are where $\widehat{A_1B_1C_1D_1E_1}$ and $\widehat{A_2B_2C_2D_2E_2}$ intersect with the straight line $l_{x_0} \triangleq \{(x,y) : x = x_0, -\infty < y < +\infty\}$ respectively, and F_1 and M_2, G_1 and N_2 have the same value of y-coordinate. Similarly, H_1 and I_1 are the intersection points of $\widehat{A_1B_1C_1D_1E_1}$ with $l_{x_1} \triangleq \{(x,y) : x = x_1, -\infty < y < +\infty\}$, H_2 and I_2 are the intersection points of $\widehat{A_2B_2C_2D_2E_2}$ with l_{x_1}, and H_1 and J_2, I_1 and K_2 have the same values of y-coordinate.

We first analyze the right half trajectory arcs $\widehat{A_1B_1C_1}$ and $\widehat{A_2B_2C_2}$ on Σ_+. For $\widehat{A_1F_1}$ and $\widehat{A_2F_2}$, it follows from $h'(y) > 0$ that for the same vale of x-coordinate then the positive value of $h(y) - F(x)$ along $\widehat{A_2F_2}$ is greater than the one along $\widehat{A_1F_1}$. So one has that

$$\lambda(A_2) - \lambda(F_2) = \int_{\widehat{A_2F_2}} \frac{F(x)g(x)}{h(y(x)) - F(x)}dx > \int_{\widehat{A_1F_1}} \frac{F(x)g(x)}{h(y(x)) - F(x)}dx = \lambda(A_1) - \lambda(F_1). \quad (3.3)$$

It is similar to $\widehat{G_1C_1}$ and $\widehat{G_2C_2}$ then

$$\lambda(G_2) - \lambda(C_2) > \lambda(G_1) - \lambda(C_1). \tag{3.4}$$

Along $\widehat{F_1G_1}$ and $\widehat{M_2N_2}$, since $F'(x) > 0$ for $x > x_0$, it follows that for the same value of y-coordinate then the value of $F(x)$ along $\widehat{M_2N_2}$ is greater than the one along $\widehat{F_1G_1}$. Furthermore,

$$\lambda(M_2) - \lambda(N_2) = \int_{\widehat{M_2N_2}} -F(x)dy > \int_{\widehat{F_1G_1}} -F(x)dy = \lambda(F_1) - \lambda(G_1) > 0. \tag{3.5}$$

Moreover, since $F(x)(x - x_0) > 0$ for $x \in (0, +\infty) \setminus \{x_0\}$, it follows that

$$\begin{aligned} \lambda(F_2) - \lambda(M_2) &= \int_{\widehat{F_2M_2}} -F(x)dy = \int_{y_{M_2}}^{y_{F_2}} F(x(y))dy > 0, \\ \lambda(N_2) - \lambda(G_2) &= \int_{\widehat{N_2G_2}} -F(x)dy = \int_{y_{G_2}}^{y_{N_2}} F(x(y))dy > 0. \end{aligned} \tag{3.6}$$

Adding from (3.3) to (3.6) then

$$\lambda(A_2) - \lambda(C_2) > \lambda(A_1) - \lambda(C_1). \tag{3.7}$$

By (3.1) it follows that $H(y_{A_2}) - H(y_{C_2}) > H(y_{A_1}) - H(y_{C_1})$, i.e. $\Delta_+(x_{B_2}) > \Delta_+(x_{B_1})$. Which together with $x_{B_2} > x_{B_1} > x_0$ then the function $\Delta_+(x_B)$ is strictly monotone increasing for any $x_B > x_0$. On the other hand, from $h(+\infty) = +\infty$ it follows that

$$\lambda(A_2) - \lambda(F_2) = \int_{\widehat{A_2F_2}} \frac{F(x)g(x)}{h(y(x)) - F(x)} dx \to 0,$$

as the value of y-coordinate along $\widehat{A_2F_2}$ tends to $+\infty$ uniformly, and $\lambda(G_2) - \lambda(C_2) \to 0$ due to $h(-\infty) = -\infty$. While for $\widehat{F_2G_2}$ it follows from $F(+\infty) = +\infty$ that

$$0 < \lambda(F_2) - \lambda(G_2) = \int_{\widehat{F_2G_2}} -F(x)dy = \int_{y_{G_2}}^{y_{F_2}} F(x(y))dy \to +\infty,$$

as the value of y-coordinate of F_2, $y_{F_2} \to +\infty$ and the value of y-coordinate of G_2, $y_{G_2} \to -\infty$. Hence the first part of Lemma 3.2 holds.

For the left half trajectory arcs $\widehat{C_1D_1E_1}$ and $\widehat{C_2D_2E_2}$ on Σ_-. With the similar analysis then

$$\lambda(C_2) - \lambda(H_2) > \lambda(C_1) - \lambda(H_1), \qquad \lambda(I_2) - \lambda(E_2) > \lambda(I_1) - \lambda(E_1),$$
$$\lambda(J_2) - \lambda(K_2) > \lambda(H_1) - \lambda(I_1), \qquad \lambda(H_2) - \lambda(J_2) > 0, \qquad \lambda(K_2) - \lambda(I_2) > 0,$$

and so

$$\lambda(C_2) - \lambda(E_2) > \lambda(C_1) - \lambda(E_1). \tag{3.8}$$

That is $\Delta_-(x_{D_2}) > \Delta_-(x_{D_1})$ and then the conclusion holds. The proof is complete. □

Remark 3.3. If we let $\Delta_-(|x_D|) = H(y_C) - H(y_E)$ with $|x_D| \in (|x_1|, +\infty)$ denote the positive half trajectory arc $\widehat{CDE}$ on Σ_-, it follows from (3.8) that the function $\Delta_-(|x_D|)$ is strictly monotone increasing tending $+\infty$ as $|x_D| \to +\infty$.

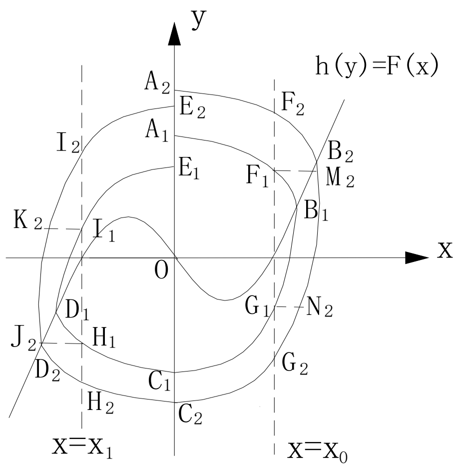

Figure 3.2: Two orbits of (2.1) which cross Σ_F transversally when $x > x_0$ and $x < x_1$.

Similarly, we denote by $\Delta(x) = \Delta_+(x_B) + \Delta_-(x_D) = H(y_A) - H(y_E)$ for the orbit $\widehat{ABCDE}$ (see Figure 3.1) of (2.1). Note that if $y_A = y_E$ i.e. $\Delta(x) = 0$, the corresponding orbit is a periodic orbit. Furthermore, the amplitude of the periodic orbit is exactly $\max\{x_B, |x_D|\}$ with $x_D < x_1 < x_0 < x_B$. If $0 < y_E < y_A$ i.e. $\Delta(x) > 0$, we claim that there exists at least one periodic orbit contained in the compact region Ω, and $\max\{x_B, |x_D|\}$ is a upper bound of the amplitude of the periodic orbit, where Ω denotes the closed region encircled by $\widehat{ABCDE} \cup \overline{EA}$ with $\widehat{ABCDE}$ being a directed arc from A to E successively and $\overline{EA}$ being a directed line segment from E to A. It follows from (3.2) that the origin $O(0,0)$ is a source point, and then Ω is a positive invariant region. Hence according to the planar Poincaré–Bendixson annular region theorem then (2.1) has at least one periodic orbit in Ω. If $0 < y_A < y_E$ i.e. $\Delta(x) < 0$, it follows that Ω is encircled by a periodic orbit. So $\max\{x_B, |x_D|\}$ is a low bound for the amplitude of such a periodic orbit. Our next task is to find $\max\{x_B, |x_D|\}$ as small as possible such that $\Delta(x) \geq 0$.

We only analyze in detail the relevant properties to the right half trajectory arcs of (2.1) on Σ_+, while for the left half trajectory arcs on Σ_- is symmetric. For convenience, we let $L_B = \{\varphi(B,t) : \alpha < t < \beta\}$ denote the right half trajectory arc $\widehat{ABC}$ starting from $B(x_B, y_B) \in \Sigma_F$ (see Figure 3.1), and denote by $L_B^+ = \{\varphi(B,t) : 0 \leq t < \beta\}$ the positive trajectory arc $\widehat{BC}$ contained in Σ_F^+ and $L_B^- = \{\varphi(B,t) : \alpha < t \leq 0\}$ is denoted by the negative trajectory arc $\widehat{AB}$ contained in Σ_F^-. Then $L_B = L_B^+ \cup L_B^-$. Here $\varphi(B,t)$ is the unique solution of (2.1) satisfying $\varphi(B,0) = B$, and (α, β) denotes the existence interval of the solution on Σ_+. Moreover, we also present these two trajectory arcs L_B^- and L_B^+ by the graphs of the functions $y = \overline{y}(x)$ and $y = \underline{y}(x)$ for $x \in [0, x_B]$ respectively. Hence the following properties are true:

(1) $\overline{y}(x)$ is strictly monotone decreasing for $x \in (0, x_B)$, satisfying $h(\overline{y}(x_B)) = F(x_B)$.

(2) $\underline{y}(x) < h^{-1}(F(x)) < \overline{y}(x)$ for $x \in (0, x_B)$, satisfying $\underline{y}(0) = y_C < 0 < \overline{y}(0) = y_A$.

(3) $\underline{y}(x)$ is strictly monotone increasing for $x \in (0, x_B)$, satisfying $h(\underline{y}(x_B)) = F(x_B)$.

The following several lemmas are used to determine the upper bound of the amplitude for the unique limit cycle of (2.1) on Σ_+. Let $\overline{v}(x) = h(\overline{y}(x)) - F(x)$ and $\underline{v}(x) = F(x) - h(\underline{y}(x))$, then $\overline{v}(x_B) = \underline{v}(x_B) = 0$ and $\overline{v}(x) > 0$, $\underline{v}(x) > 0$ for $x \in [0, x_B)$.

Lemma 3.4. *Consider $y = \overline{y}(x)$ for $x \in [0, x_B]$, when $x_B > x_0$ it follows that*

$$H(y_A) > \frac{1}{\overline{v}(x_0)} \int_0^{x_B} F(x)g(x)dx + G(x_B) + H(y_B). \tag{3.9}$$

Proof. By differentiating $\lambda(x, y) = H(y) + G(x)$ with respect to x and taking $y = \overline{y}(x)$ then

$$\frac{d}{dx}[H(\overline{y}(x)) + G(x)] = -\frac{g(x)F(x)}{\overline{v}(x)}. \tag{3.10}$$

Take integral from $x = 0$ to $x = x_B$ in (3.10), we obtain that

$$H(y_B) = -\int_0^{x_B} \frac{g(x)F(x)}{\overline{v}(x)} dx - G(x_B) + H(y_A). \tag{3.11}$$

Since $F(x)(x - x_0) > 0$ for $x \in (0, x_B) \setminus \{x_0\}$, it follows from the property (1) that

$$\overline{v}(x) = h(\overline{y}(x)) - F(x) > \overline{v}(x_0), \quad x \in [0, x_0),$$
$$\overline{v}(x) = h(\overline{y}(x)) - F(x) < \overline{v}(x_0), \quad x \in (x_0, x_B].$$

Furthermore $-\frac{g(x)F(x)}{\overline{v}(x)} < -\frac{g(x)F(x)}{\overline{v}(x_0)}$ for $x \in (0, x_B] \setminus \{x_0\}$, and then (3.9) holds due to (3.11). □

Lemma 3.5. *The function $\underline{v}(x)$ is monotone decreasing for $x \in [0, x_B]$.*

Proof. We first claim that there exists at least one $x_1 \in (0, x_0)$ such that $F'(x_1) = 0$.

Suppose on the contrary that $F'(x) \neq 0$ for any $x \in (0, x_0)$. By $F(0) = 0$ and $F(x)(x - x_0) > 0$ for $x \in (0, +\infty) \setminus \{x_0\}$, it follows that $F'(0) < 0$. Together with the continuous dependence of the derived function $F'(x)$ on $(0, x_0)$ then $F'(x) < 0$ for $x \in U^+(0)$, where $U^+(0)$ denotes some small right-neighborhood of 0. Hence we suppose that $F'(x) < 0$ for every $x \in (0, x_0)$. Let $U^-(x_0)$ denote some small left-neighborhood of x_0, it follows from $F(x_0) = 0$ that $F(x) > 0$ for every $x \in U^-(x_0)$. This contradicts with $F(x) < 0$ for $x \in (0, x_0)$. Hence the claim holds.

Without loss of generality, we suppose that there exist x_i, $i = 1, 2, \ldots, m$ with m being an odd integer satisfying $0 < x_1 < x_2 < \cdots < x_m < x_0$ such that $F'(x_i) = 0$. By $F'(x) < 0$ for $x \in U^+(0)$, $F'(x) > 0$ for $x \in U^-(x_0) \cup [x_0, x_B]$ and $F \in C^1((0, +\infty), \mathbb{R})$, it follows that

$$F'(x) < 0 \qquad \text{for } \ x \in (0, x_1) \cup (x_2, x_3) \cup \cdots \cup (x_{m-1}, x_m),$$
$$F'(x) > 0 \qquad \text{for } \ x \in (x_1, x_2) \cup (x_3, x_4) \cup \cdots \cup (x_m, x_B].$$

For convenience, let us introduce the notations $\underline{I} = [0, x_1] \cup [x_2, x_3] \cup \cdots \cup [x_{m-1}, x_m]$ and $\overline{I} = (x_1, x_2) \cup (x_3, x_4) \cup \cdots \cup (x_m, x_B]$. We next show that

$$\frac{d\underline{v}(x)}{dx} = F'(x) - h'(y)\frac{d\underline{y}(x)}{dx} \leq 0 \qquad \text{for } \ x \in [0, x_B]. \tag{3.12}$$

By the property (3), it follows that $\frac{d\underline{y}(x)}{dx} > 0$ for $x \in (0, x_B]$. Since again $h'(y) > 0$, it follows that $\frac{d\underline{v}(x)}{dx} < 0$ for any $x \in \underline{I}$. While for $x \in \overline{I}$, we from three steps to analyze.

(*i*) If $F'(x) \leq h'(y)\frac{d\underline{y}(x)}{dx}$ for any $x \in (x_1, x_2)$, it follows that $\frac{d\underline{v}(x)}{dx} \leq 0$ for $x \in (x_1, x_2)$. Otherwise there exist a finite number $x^i \in (x_1, x_2)$ with x^1 being the first point such that $F'(x^i) > h'(y)\frac{d\underline{y}(x^i)}{dx}$. By the continuous dependence of the derived function $F'(x)$, $F'(x) > h'(y)\frac{d\underline{y}(x)}{dx}$ for $x \in (x^1, x_2)$. Furthermore $\frac{d\underline{y}(x_2)}{dx} < 0$, this is a contradiction. Hence $\frac{d\underline{v}(x)}{dx} \leq 0$ for any $x \in (x_1, x_2)$.

(ii) For $x \in (x_{2i-1}, x_{2i})$, $i = 2, 3, \ldots, \frac{m-1}{2}$, it is similar to the case (i) and so omitted.

(iii) If $F'(x) \le h'(y)\frac{dy(x)}{dx}$ for any $x \in (x_m, x_B]$, it follows that $\frac{d\underline{v}(x)}{dx} \le 0$ for $x \in (x_m, x_B]$. Otherwise there exist some points $\tilde{x}^i \in (x_m, x_B]$ such that $F'(\tilde{x}^i) > h'(y)\frac{dy(\tilde{x}^i)}{dx}$. By the continuous dependence of the derived function $F'(x)$, then $F'(x) > h'(y)\frac{dy(x)}{dx}$ for $x \in U^-(x_B)$ with $U^-(x_B)$ being a small left-neighborhood of x_B. Furthermore $F'(x) \to +\infty$ as $x \to x_B^-$, it is a contradiction. Hence $\frac{d\underline{v}(x)}{dx} \le 0$ for any $x \in (x_m, x_B]$.

In conclusion (3.12) holds. The proof is complete. □

Lemma 3.6. *Consider $y = \underline{y}(x)$ for $x \in [0, x_B]$, when $x_B > x_0$ it follows that*

$$H(y_C) \le -\frac{1}{\underline{v}(x_0)} \int_0^{x_B} F(x)g(x)dx + G(x_B) + H(y_B). \tag{3.13}$$

Proof. With the similar way to Lemma 3.4, it follows that

$$\frac{d}{dx}[H(\underline{y}(x)) + G(x)] = \frac{g(x)F(x)}{\underline{v}(x)},$$

and take integral from $x = 0$ to $x = x_B$ in the above equality then

$$H(y_C) = G(x_B) + H(y_B) - \int_0^{x_B} \frac{g(x)F(x)}{\underline{v}(x)} dx. \tag{3.14}$$

From Lemma 3.5 and $F(x)(x - x_0) > 0$ for $x \in (0, x_B) \setminus \{x_0\}$, it follows that

$$\frac{-g(x)F(x)}{\underline{v}(x)} \le \frac{-g(x)F(x)}{\underline{v}(x_0)}, \qquad x \in (0, x_B].$$

Hence together with (3.14) then (3.13) holds. The proof is complete. □

Correspondingly, for the left half trajectory arcs of (2.1) on Σ_- then we have also several relevant lemmas. Let $L_D = \{\varphi(D, t) : \mu < t < \nu\}$ denote the left half trajectory arc $\widehat{CDE}$ starting from $D(x_D, y_D) \in \Sigma_F$ (see Figure 3.1), and denote by $L_D^+ = \{\varphi(D, t) : 0 \le t < \nu\}$ the positive trajectory arc $\widehat{DE}$ contained in $\tilde{\Sigma}_F^-$, and $L_D^- = \{\varphi(D, t) : \mu < t \le 0\}$ is denoted by the negative trajectory arc $\widehat{CD}$ contained in $\tilde{\Sigma}_F^+$. Moreover, we also present these two half orbits L_D^- and L_D^+ by the graphs of the functions $y = \underline{y}(x)$ and $y = \overline{y}(x)$ for $x \in [x_D, 0]$ respectively. Hence the following properties are true:

(1) $\overline{y}(x)$ is strictly monotone increasing for $x \in (x_D, 0)$, satisfying $h(\overline{y}(x_D)) = F(x_D)$.

(2) $\underline{y}(x) < h^{-1}(F(x)) < \overline{y}(x)$ for $x \in (x_D, 0)$, satisfying $\underline{y}(0) = y_C < 0 < \overline{y}(0) = y_E$.

(3) $\underline{y}(x)$ is strictly monotone decreasing for $x \in (x_D, 0)$, satisfying $h(\underline{y}(x_D)) = F(x_D)$.

Similarly, let $\overline{v}(x) = h(\overline{y}(x)) - F(x)$ and $\underline{v}(x) = F(x) - h(\underline{y}(x))$. Then $\overline{v}(x_D) = \underline{v}(x_D) = 0$ and $\overline{v}(x) > 0$, $\underline{v}(x) > 0$ for any $x \in (x_D, 0]$.

Lemma 3.7. *Consider $y = \underline{y}(x)$ for $x \in [x_D, 0]$, when $x_D < x_1$ it follows that*

$$H(y_C) > \frac{1}{\underline{v}(x_1)} \int_{x_D}^0 F(x)g(x)dx + G(x_D) + H(y_D).$$

Lemma 3.8. *The function $\overline{v}(x)$ is monotone increasing for $x \in [x_D, 0]$.*

Lemma 3.9. *Consider $y = \overline{y}(x)$ for $x \in [x_D, 0]$, when $x_D < x_1$ it follows that*

$$H(y_E) \leq -\frac{1}{\overline{v}(x_1)} \int_{x_D}^{0} F(x)g(x)dx + G(x_D) + H(y_D).$$

Based on the above several lemmas, now we are ready to prove our main result (*i.e.* Theorem 3.1) concerning the position of the unique limit cycle of (2.1).

Proof of Theorem 3.1. Consider the right half trajectory arcs of (2.1) on Σ_+. We first show that there exists a unique x_+^* satisfying $x_+^* > x_0$ such that $\int_0^{x_+^*} g(x)F(x)dx = 0$.

By $F(x)(x - x_0) > 0$ for $x \in (0, +\infty) \setminus \{x_0\}$, it follows that $\int_0^x g(x)F(x)dx < 0$ for $0 < x \leq x_0$. Since again $\frac{d}{dx}(\int_0^x g(x)F(x)dx) > 0$ for $x > x_0$, it follows from $F(+\infty) = +\infty$ that there exist sufficiently large x with $x > x_0$ such that $\int_0^x g(x)F(x)dx > 0$. Hence by the property of strictly monotone increasing of the continuous function $\int_0^x g(x)F(x)dx$, there exists a unique $x_+^* > x_0$ such that $\int_0^{x_+^*} g(x)F(x)dx = 0$ and $\int_0^x g(x)F(x)dx > 0$ for $x > x_+^*$. On the other hand, it follows from Lemma 3.2 that there exists a unique $x_B^*(> x_0)$ such that $\Delta_+(x_B^*) = 0$ and $\Delta_+(x_B) > 0$ for $x_B > x_B^*$. Moreover, by Lemmas 3.4 and 3.6 one has that

$$\Delta_+(x_B) = H(y_A) - H(y_C) > \left(\frac{1}{\overline{v}(x_0)} + \frac{1}{\underline{v}(x_0)}\right) \int_0^{x_B} F(x)g(x)dx. \tag{3.15}$$

Since the function $\int_0^x g(x)F(x)dx$ is strictly monotone increasing for $x > x_0$, we let $x_B = x_+^*$ such that $\int_0^{x_B} F(x)g(x)dx = 0$. Furthermore $\Delta_+(x_+^*) > 0$, *i.e.* x_+^* is an upper bound for the amplitude of the unique limit cycle of (2.1) on Σ_+.

Similar to the left half trajectory arcs of (2.1) on Σ_-. Then there exists a unique x_-^* satisfying $x_-^* < x_1$ such that $\int_{x_-^*}^0 g(x)F(x)dx = 0$, and from Lemmas 3.7 and 3.9 it follows that

$$\Delta_-(x_D) = H(y_C) - H(y_E) > \left(\frac{1}{\overline{v}(x_1)} + \frac{1}{\underline{v}(x_1)}\right) \int_{x_D}^{0} F(x)g(x)dx. \tag{3.16}$$

So we let $x_D = x_-^*$ such that $\Delta_-(x_-^*) > 0$. By combining with (3.15) and (3.16) then the unique limit cycle of (2.1) locates in the strip region $\{(x, y) \in \mathbb{R}^2 : |x| < x^*\}$ with $x^* = \max\{x_B, |x_D|\}$. The proof is complete. □

Remark 3.10. In [24], the authors studied the location problem of a limit cycle for a class of Liénard systems with symmetry. However, in this paper we investigate the same problem for a more general Liénard-type differential system (2.1), and it is worth noting that we do not require that the orbits of the system is symmetric in the origin. Moreover, we obtain an explicit upper bound for the amplitude of the unique limit cycle of (2.1) merely under the sufficient conditions proving the existence and uniqueness of limit cycles of (2.1). Hence in some sense this paper improves and generalizes the result in [24].

4 Application

Example 4.1. Consider the following celebrated predator–prey system model with non-monotonic functional response

$$\begin{cases} \dfrac{dx}{dt} = rx\left(1 - \dfrac{x}{K}\right) - \dfrac{xy}{c + x^2}, \\ \dfrac{dy}{dt} = y\left(\dfrac{\mu x}{c + x^2} - D\right), \end{cases} \tag{4.1}$$

where r, c, μ, K and D are positive parameters. In [21], Ruan and Xiao have studied that (4.1) has a unique limit cycle surrounding the unique positive equilibrium point (x_1, y_1) with $x_1 = \frac{\mu - \sqrt{\mu^2 - 4cD^2}}{2D}$, $y_1 = r(1 - \frac{x_1}{K})(c + x_1^2)$ if

$$\mu^2 > \frac{16}{3}cD^2, \qquad \frac{\mu + \sqrt{\mu^2 - 4cD^2}}{2D} > K > \frac{2\mu - \sqrt{\mu^2 - 4cD^2}}{2D}.$$

Now we verify the position of the unique limit cycle of (4.1). By making a transformation

$$x - x_1 = -X, \qquad y - y_1 = y_1(e^Y - 1), \qquad xdt = (c + x^2)dT,$$

then (4.1) is transformed into the Liénard-type system form

$$\begin{cases} \dfrac{dX}{dT} = h(Y) - F(X), \\ \dfrac{dY}{dT} = -g(X), \end{cases} \tag{4.2}$$

with $h(Y) = y_1(e^Y - 1)$, $F(X) = \frac{rX}{K}(X^2 + (K - 3x_1)X + c + 3x_1^2 - 2Kx_1)$ and $g(X) = \frac{DX(X - x_1 + x_2)}{x_1 - X}$. Let $c = 1/2$, $D = 1$, $\mu = 2$ and choose $K = 3/2$, $r = 3/2$, it follows that $x_1 = 1 - \sqrt{2}/2$, $x_2 = 1 + \sqrt{2}/2$, $x_3 = 2 - \sqrt{2}/2$ and $y_1 = \sqrt{2}/2$. We still let x, y, t be the variables of (4.2) and then $h(y) = \frac{\sqrt{2}}{2}(e^y - 1)$, $F(x) = x[x^2 - \frac{3}{2}(1 - \sqrt{2})x + 2 - \frac{3\sqrt{2}}{2}]$ and $g(x) = \frac{x(x + \sqrt{2})}{1 - \sqrt{2}/2 - x}$. By some simple but tedious computations it follows that

$$\begin{aligned} \int_0^x F(x)g(x)dx = \int_A^{A-x} \Big[& t^4 - (5A + B)t^3 + (10A^2 + 4AB + C)t^2 - (10A^3 + 6A^2B + 3AC + E)t \\ & + (5A^4 + 4A^3B + 3A^2C + 2AE) - \frac{A^5 + A^4B + A^3C + A^2E}{t} \Big] dt, \end{aligned}$$

with $A = 1 - \frac{\sqrt{2}}{2}$, $B = \frac{5}{2}\sqrt{2} - \frac{3}{2}$, $C = 5 - 3\sqrt{2}$ and $E = 2\sqrt{2} - 3$. By Matlab and Theorem 3.1, it follows that the unique limit cycle of (4.2) locates in the strip region $\{(x, y) \in \mathbb{R}^2 : |x| < 0.89\}$. Indeed Matlab simulation shows the result shown in Figure 4.1.

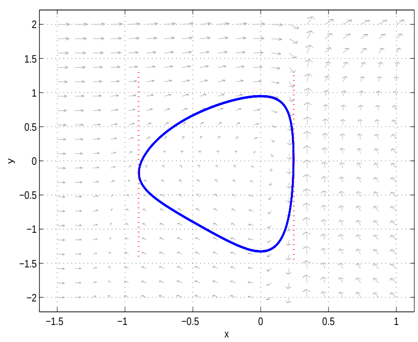

Figure 4.1: The unique limit cycle of (4.2) surrounding the origin.

Example 4.2. Consider the following Liénard-type system

$$\begin{cases} \dfrac{dx}{dt} = y^3 - (x^3 - x), \\ \dfrac{dy}{dt} = -x. \end{cases} \tag{4.3}$$

Since $F(x) = x^3 - x$ and $g(x) = x$ are odd functions, it easily verify that the system (4.3) has a unique symmetric limit cycle surrounding the origin (see Figure 4.2). From

$$\int_0^x F(x)g(x)dx = x^3\Big(\frac{1}{5}x^2 - \frac{1}{3}\Big) = 0,$$

it follows that $x = \frac{\sqrt{15}}{3}$ is the unique positive root. Hence by Theorem 3.1, the unique limit cycle of (4.3) locates in the strip region $\{(x, y) \in \mathbb{R}^2 : |x| < \frac{\sqrt{15}}{3}\}$ (see Figure 4.2).

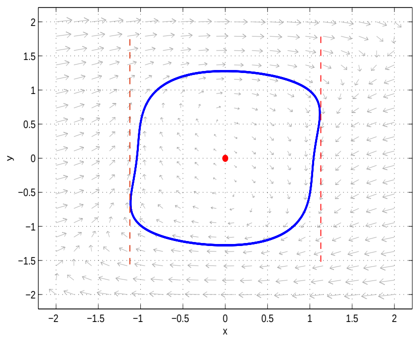

Figure 4.2: The unique symmetric limit cycle of (4.3).

Remark 4.3. From the above two Figures we note that the result obtained from Theorem 3.1 is not the best upper bound. The interested readers can further seek some new methods to study a good upper bound for the amplitude of a limit cycle in Liénard-type system (2.1).

5 Conclusion

In this paper, we have investigated the position problem of the unique limit cycle of Liénard-type system (2.1). By considering the amplitudes of the unique limit cycle on Σ_- and Σ_+ respectively, we have obtained an explicit upper bound for the amplitude of the unique limit cycle on the plane. Compared with the assumptions in [24], we have obtained the result merely under the sufficient conditions of the existence and uniqueness of limit cycles of (2.1). Hence in some sense, we have generalized and improved the result in [24]. Finally, we have given two examples including an application to predator–prey system model and the corresponding Matlab simulations to illustrate the effectiveness of our theoretical result. However, it should be noted that from these two figures then the result (*i.e.* Theorem 3.1) is not the best. Hence the interested readers can seek some new methods to further study the upper bound of the amplitude of limit cycles of (2.1).

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