# Solvable product-type system of difference equations whose associated polynomial is of the fourth order

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**Abstract.** The solvability problem for the following system of difference equations

$$z_{n+1} = \alpha z_n^a w_n^b, \quad w_{n+1} = \beta w_{n-1}^c z_{n-2}^d, \quad n \in \mathbb{N}_0,$$

where $a, b, c, d \in \mathbb{Z}$, $\alpha, \beta \in \mathbb{C} \setminus \{0\}$, $z_{-2}, z_{-1}, z_0, w_{-1}, w_0 \in \mathbb{C} \setminus \{0\}$, is solved. In the main case when $bd \neq 0$, a polynomial of the fourth order is associated to the system, and its solutions are represented in terms of the parameters, through the roots of the polynomial in all possible cases (the roots are given in terms of parameters $a, b, c, d$). This is also the first paper which successfully deals with the associated polynomial (to a product-type system) of the fourth order in detail, which is the main achievement of the paper.

**Keywords:** system of difference equations, product-type system, solvable in closed form, polynomial of fourth order.

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## 1  Introduction

Concrete nonlinear difference equations and systems is a research field of some recent interest (see, e.g., [2,4,9–15,17–45]). Among the systems, symmetric and related ones have attracted attention of some experts, especially after the publication of several papers by Papaschinopoulos and Schinas almost twenty years ago (see, e.g., [2,4,9–14,17,18,21,23–28,30,31,34–40,42–45]). On the other hand, solvability of the equations and systems has re-attracted some recent attention (see, e.g., [2,15,21–36,38–45]). Some of them are solved by the method of transformation (see, e.g., [15,21,22,24,38–41] and the references therein). For somewhat more complex methods see [33] and [34]. An interesting related method has been recently applied to partial difference equations in [29] and [32]. Books [1,5–8] contain many classical methods for solving difference equations and systems.

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If initial values and coefficients of product-type systems are positive they can be solved by transforming them to the linear ones with constant coefficients, by using the logarithm. If the initial values and coefficients are not positive the method is not of a special use. Therefore, the solvability of product-type systems with non-positive initial values and coefficients is a problem of interest. We started studying the problem in [36], where we showed the solvability of the system

$$z_{n+1} = \frac{w_n^a}{z_{n-1}^b}, \quad w_{n+1} = \frac{z_n^c}{w_{n-1}^d}, \quad n \in \mathbb{N}_0, \tag{1.1}$$

for $a, b, c, d \in \mathbb{Z}$ and $z_{-1}, z_0, w_{-1}, w_0 \in \mathbb{C} \setminus \{0\}$, and gave many results on the long-term behavior of solutions to (1.1) by using the obtained closed form formulas. Product-type equations appeared also in the study of the difference equation in [33], as its special cases. The max-type system in [23] is solved by reducing it to a product-type one. They also appeared indirectly in the study of some max-type and related difference equations and systems, as their boundary cases (see, e.g., [19, 20, 37]). The study was continued in [42], in [30] where a three-dimensional system was considered, in [28] where it was noticed for the first time that some coefficients can be added to a product-type system so that the solvability is preserved, and later in [31, 35, 43–45] where various new details and methods are presented.

This paper continues investigating the solvability problem, by studying the following product-type system

$$z_{n+1} = \alpha z_n^a w_n^b, \quad w_{n+1} = \beta w_{n-1}^c z_{n-2}^d, \quad n \in \mathbb{N}_0, \tag{1.2}$$

where $a, b, c, d \in \mathbb{Z}$, $\alpha, \beta \in \mathbb{C}$ and $z_{-2}, z_{-1}, z_0, w_{-1}, w_0 \in \mathbb{C}$.

Clearly, the domain of undefinable solutions [24] to system (1.2) is a subset of

$$\mathcal{U} = \{(z_{-2}, z_{-1}, z_0, w_{-1}, w_0) \in \mathbb{C}^5 : z_{-2} = 0 \text{ or } z_{-1} = 0 \text{ or } z_0 = 0 \text{ or } w_{-1} = 0 \text{ or } w_0 = 0\}.$$

Thus, from now on we will assume that $z_{-2}, z_{-1}, z_0, w_{-1}, w_0 \in \mathbb{C} \setminus \{0\}$. Since the cases $\alpha = 0$ and $\beta = 0$ are trivial or produce solutions which are not well-defined we will also assume that $\alpha\beta \neq 0$.

In the main case when $bd \neq 0$, a polynomial of the fourth order is associated to the system, and its solutions are represented in terms of the parameters, through the roots of the polynomial in all the cases (the roots are given in terms of parameters $a, b, c, d$), which is the main achievement of the paper. This is the first paper which deals with the associated polynomial (to a product-type system) of the fourth order in detail. An associated polynomial of the fourth order appears yet in [42], but almost without any analysis of its roots and their influence on the solutions to the system therein.

In this paper, we will use the following standard convention $\sum_{i=k}^{m} a_i = 0$, when $m < k$.

## 2   Auxiliary results

In this section we quote two auxiliary results which are used in the proofs of the main results. The first one is the following lemma which is well-known (see, e.g., [6, 8]). For a proof of a more general result see [35].

**Lemma 2.1.** *Let $i \in \mathbb{N}_0$ and*

$$s_n^{(i)}(z) = 1 + 2^i z + 3^i z^2 + \cdots + n^i z^{n-1}, \quad n \in \mathbb{N} \tag{2.1}$$

*where $z \in \mathbb{C}$.*

*Then*

$$s_n^{(0)}(z) = \frac{1 - z^n}{1 - z}, \tag{2.2}$$

$$s_n^{(1)}(z) = \frac{1 - (n+1)z^n + nz^{n+1}}{(1-z)^2}, \tag{2.3}$$

*for every $z \in \mathbb{C} \setminus \{1\}$ and $n \in \mathbb{N}$.*

The following lemma is also known, and can be proved, for example, by using the Lagrange interpolation polynomial or the calculus of residue (see, for example, [6] and [42]).

**Lemma 2.2.** *Assume that $\lambda_j$, $j = \overline{1, k}$, are pairwise different zeros of the polynomial*

$$P(z) = \alpha_k z^k + \alpha_{k-1} z^{k-1} + \cdots + \alpha_1 z + \alpha_0,$$

*with $\alpha_k \alpha_0 \neq 0$.*

*Then*

$$\sum_{j=1}^{k} \frac{\lambda_j^l}{P'(\lambda_j)} = 0$$

*for $l = \overline{0, k-2}$, and*

$$\sum_{j=1}^{k} \frac{\lambda_j^{k-1}}{P'(\lambda_j)} = \frac{1}{\alpha_k}.$$

## 3 Main results

The main results in this paper are formulated and proved in this section.

### 3.1 Solvability of system (1.2)

The first result concerns the solvability problem of system (1.2).

**Theorem 3.1.** *Assume that $a, b, c, d \in \mathbb{Z}$, $\alpha, \beta \in \mathbb{C} \setminus \{0\}$ and $z_{-2}, z_{-1}, z_0, w_{-1}, w_0 \in \mathbb{C} \setminus \{0\}$. Then system (1.2) is solvable in closed form.*

*Proof.* *Case $b = 0$.* In this case system (1.2) becomes

$$z_{n+1} = \alpha z_n^a, \quad w_{n+1} = \beta w_{n-1}^c z_{n-2}^d, \quad n \in \mathbb{N}_0. \tag{3.1}$$

From the first equation in (3.1) we get

$$z_n = \alpha^{\sum_{j=0}^{n-1} a^j} z_0^{a^n}, \quad n \in \mathbb{N}, \tag{3.2}$$

from which it follows that

$$z_n = \alpha^{\frac{1-a^n}{1-a}} z_0^{a^n}, \quad n \in \mathbb{N}, \tag{3.3}$$

when $a \neq 1$, and

$$z_n = \alpha^n z_0, \quad n \in \mathbb{N}, \tag{3.4}$$

when $a = 1$.

Employing (3.2) in the second equation in (3.1), we get

$$w_n = \beta \alpha^{d \sum_{j=0}^{n-4} a^j} z_0^{da^{n-3}} w_{n-2}^c, \tag{3.5}$$

for $n \geq 4$, from which it follows that

$$w_{2n+i} = \beta \alpha^{d \sum_{j=0}^{2n+i-4} a^j} z_0^{da^{2n+i-3}} w_{2n+i-2}^c, \tag{3.6}$$

for $n \geq 2$ and $i = 0, 1$.

Assume that for some $k \in \mathbb{N}$ it has been proved that

$$w_{2n+i} = \beta^{\sum_{j=0}^{k-1} c^j} \alpha^{d \sum_{j=0}^{k-1} c^j \sum_{l=0}^{2n-2j+i-4} a^l} z_0^{d \sum_{j=0}^{k-1} c^j a^{2n-2j+i-3}} w_{2(n-k)+i}^{c^k}, \tag{3.7}$$

for $n \geq k+1$ and $i = 0, 1$.

By using (3.6) with $n$ replaced by $n - k$ and inserting into (3.7) we get

$$w_{2n+i} = \beta^{\sum_{j=0}^{k-1} c^j} \alpha^{d \sum_{j=0}^{k-1} c^j \sum_{l=0}^{2n-2j+i-4} a^l} z_0^{d \sum_{j=0}^{k-1} c^j a^{2n-2j+i-3}} \left( \beta \alpha^{d \sum_{j=0}^{2(n-k)+i-4} a^j} z_0^{da^{2(n-k)+i-3}} w_{2(n-k-1)+i}^c \right)^{c^k}$$

$$= \beta^{\sum_{j=0}^{k} c^j} \alpha^{d \sum_{j=0}^{k} c^j \sum_{l=0}^{2n-2j+i-4} a^l} z_0^{d \sum_{j=0}^{k} c^j a^{2n-2j+i-3}} w_{2(n-k-1)+i}^{c^{k+1}}, \tag{3.8}$$

for $n \geq k+2$ and $i = 0, 1$, from which along with (3.6) and the method of induction it follows that (3.7) holds for every $n \geq k+1$ and $i = 0, 1$.

Choosing $k = n - 1$ in (3.7) we obtain

$$w_{2n} = \beta^{\sum_{j=0}^{n-2} c^j} \alpha^{d \sum_{j=0}^{n-2} c^j \sum_{i=0}^{2n-2j-4} a^i} z_0^{d \sum_{j=0}^{n-2} c^j a^{2n-2j-3}} w_2^{c^{n-1}}, \tag{3.9}$$

and

$$w_{2n+1} = \beta^{\sum_{j=0}^{n-2} c^j} \alpha^{d \sum_{j=0}^{n-2} c^j \sum_{i=0}^{2n-2j-3} a^i} z_0^{d \sum_{j=0}^{n-2} c^j a^{2n-2j-2}} w_3^{c^{n-1}}, \tag{3.10}$$

for $n \geq 2$.

From the second equation in (3.1), with $n = 0, 1, 2$, we have

$$w_1 = \beta w_{-1}^c z_{-2}^d, \quad w_2 = \beta w_0^c z_{-1}^d, \quad w_3 = \beta w_1^c z_0^d = \beta^{1+c} w_{-1}^{c^2} z_{-2}^{cd} z_0^d. \tag{3.11}$$

Then by using (3.11) into (3.9) and (3.10) we have

$$w_{2n} = \beta^{\sum_{j=0}^{n-2} c^j} \alpha^{d \sum_{j=0}^{n-2} c^j \sum_{i=0}^{2n-2j-4} a^i} z_0^{d \sum_{j=0}^{n-2} c^j a^{2n-2j-3}} (\beta w_0^c z_{-1}^d)^{c^{n-1}}$$

$$= \beta^{\sum_{j=0}^{n-1} c^j} \alpha^{d \sum_{j=0}^{n-2} c^j \sum_{i=0}^{2n-2j-4} a^i} w_0^{c^n} z_{-1}^{dc^{n-1}} z_0^{d \sum_{j=0}^{n-2} c^j a^{2n-2j-3}} \tag{3.12}$$

and

$$w_{2n+1} = \beta^{\sum_{j=0}^{n-2} c^j} \alpha^{d \sum_{j=0}^{n-2} c^j \sum_{i=0}^{2n-2j-3} a^i} z_0^{d \sum_{j=0}^{n-2} c^j a^{2n-2j-2}} (\beta^{1+c} w_{-1}^{c^2} z_{-2}^{cd} z_0^d)^{c^{n-1}}$$

$$= \beta^{\sum_{j=0}^{n} c^j} \alpha^{d \sum_{j=0}^{n-2} c^j \sum_{i=0}^{2n-2j-3} a^i} w_{-1}^{c^{n+1}} z_{-2}^{dc^n} z_0^{d \sum_{j=0}^{n-1} c^j a^{2n-2j-2}}, \tag{3.13}$$

for $n \geq 2$.

Subcase $a \neq 1 \neq c$, $c \neq a^2$. In this case we have

$$
\begin{aligned}
w_{2n} &= \beta^{\sum_{j=0}^{n-1} c^j} \alpha^{d \sum_{j=0}^{n-2} c^j \sum_{i=0}^{2n-2j-4} a^i} w_0^{c^n} z_0^{d \sum_{j=0}^{n-2} c^j a^{2n-2j-3}} z_{-1}^{dc^{n-1}} \\
&= \beta^{\frac{1-c^n}{1-c}} \alpha^{d \sum_{j=0}^{n-2} c^j \frac{1-a^{2n-2j-3}}{1-a}} w_0^{c^n} z_0^{da \frac{a^{2n-2}-c^{n-1}}{a^2-c}} z_{-1}^{dc^{n-1}} \\
&= \beta^{\frac{1-c^n}{1-c}} \alpha^{\frac{d}{1-a} \left( \frac{1-c^{n-1}}{1-c} - \frac{a(a^{2n-2}-c^{n-1})}{a^2-c} \right)} w_0^{c^n} z_0^{da \frac{a^{2n-2}-c^{n-1}}{a^2-c}} z_{-1}^{dc^{n-1}} \\
&= \beta^{\frac{1-c^n}{1-c}} \alpha^{\frac{d(a^2-c+(a+c)(1-a)c^{n-1}-(1-c)a^{2n-1})}{(1-a)(1-c)(a^2-c)}} w_0^{c^n} z_0^{da \frac{a^{2n-2}-c^{n-1}}{a^2-c}} z_{-1}^{dc^{n-1}},
\end{aligned}
\tag{3.14}
$$

for $n \geq 2$, and

$$
\begin{aligned}
w_{2n+1} &= \beta^{\sum_{j=0}^{n} c^j} \alpha^{d \sum_{j=0}^{n-2} c^j \sum_{i=0}^{2n-2j-3} a^i} w_{-1}^{c^{n+1}} z_0^{d \sum_{j=0}^{n-1} c^j a^{2n-2j-2}} z_{-2}^{dc^n} \\
&= \beta^{\frac{1-c^{n+1}}{1-c}} \alpha^{d \sum_{j=0}^{n-2} c^j \frac{1-a^{2n-2j-2}}{1-a}} w_{-1}^{c^{n+1}} z_0^{d \frac{a^{2n}-c^n}{a^2-c}} z_{-2}^{dc^n} \\
&= \beta^{\frac{1-c^{n+1}}{1-c}} \alpha^{\frac{d}{1-a} \left( \frac{1-c^{n-1}}{1-c} - \frac{a^2(a^{2n-2}-c^{n-1})}{a^2-c} \right)} w_{-1}^{c^{n+1}} z_0^{d \frac{a^{2n}-c^n}{a^2-c}} z_{-2}^{dc^n} \\
&= \beta^{\frac{1-c^{n+1}}{1-c}} \alpha^{\frac{d(a^2-c+(1-a^2)c^n-(1-c)a^{2n})}{(1-a)(1-c)(a^2-c)}} w_{-1}^{c^{n+1}} z_0^{d \frac{a^{2n}-c^n}{a^2-c}} z_{-2}^{dc^n},
\end{aligned}
\tag{3.15}
$$

for $n \in \mathbb{N}$.

Subcase $a^2 \neq 1 \neq c$, $c = a^2$. In this case we have

$$
\begin{aligned}
w_{2n} &= \beta^{\sum_{j=0}^{n-1} a^{2j}} \alpha^{d \sum_{j=0}^{n-2} a^{2j} \sum_{i=0}^{2n-2j-4} a^i} w_0^{a^{2n}} z_0^{d \sum_{j=0}^{n-2} a^{2j} a^{2n-2j-3}} z_{-1}^{da^{2n-2}} \\
&= \beta^{\frac{1-a^{2n}}{1-a^2}} \alpha^{d \sum_{j=0}^{n-2} a^{2j} \frac{1-a^{2n-2j-3}}{1-a}} w_0^{a^{2n}} z_0^{d(n-1)a^{2n-3}} z_{-1}^{da^{2n-2}} \\
&= \beta^{\frac{1-a^{2n}}{1-a^2}} \alpha^{\frac{d}{1-a} \left( \frac{1-a^{2n-2}}{1-a^2} - (n-1)a^{2n-3} \right)} w_0^{a^{2n}} z_0^{d(n-1)a^{2n-3}} z_{-1}^{da^{2n-2}} \\
&= \beta^{\frac{1-a^{2n}}{1-a^2}} \alpha^{\frac{d(1-(n-1)a^{2n-3}-a^{2n-2}+(n-1)a^{2n-1})}{(a-1)^2(a+1)}} w_0^{a^{2n}} z_0^{d(n-1)a^{2n-3}} z_{-1}^{da^{2n-2}}
\end{aligned}
\tag{3.16}
$$

for $n \geq 2$, and

$$
\begin{aligned}
w_{2n+1} &= \beta^{\sum_{j=0}^{n} a^{2j}} \alpha^{d \sum_{j=0}^{n-2} a^{2j} \sum_{i=0}^{2n-2j-3} a^i} w_{-1}^{a^{2n+2}} z_0^{d \sum_{j=0}^{n-1} a^{2j} a^{2n-2j-2}} z_{-2}^{da^{2n}} \\
&= \beta^{\frac{1-a^{2n+2}}{1-a^2}} \alpha^{d \sum_{j=0}^{n-2} a^{2j} \frac{1-a^{2n-2j-2}}{1-a}} w_{-1}^{a^{2n+2}} z_0^{dna^{2n-2}} z_{-2}^{da^{2n}} \\
&= \beta^{\frac{1-a^{2n+2}}{1-a^2}} \alpha^{\frac{d}{1-a} \left( \frac{1-a^{2n-2}}{1-a^2} - (n-1)a^{2n-2} \right)} w_{-1}^{a^{2n+2}} z_0^{dna^{2n-2}} z_{-2}^{da^{2n}} \\
&= \beta^{\frac{1-a^{2n+2}}{1-a^2}} \alpha^{\frac{d(1-na^{2n-2}+(n-1)a^{2n})}{(a-1)^2(a+1)}} w_{-1}^{a^{2n+2}} z_0^{dna^{2n-2}} z_{-2}^{da^{2n}},
\end{aligned}
\tag{3.17}
$$

for $n \geq 2$.

Subcase $a^2 \neq 1 = c$. In this case we have

$$
\begin{aligned}
w_{2n} &= \beta^{\sum_{j=0}^{n-1} 1} \alpha^{d \sum_{j=0}^{n-2} \sum_{i=0}^{2n-2j-4} a^i} w_0 z_0^{d \sum_{j=0}^{n-2} a^{2n-2j-3}} z_{-1}^{d} \\
&= \beta^n \alpha^{d \sum_{j=0}^{n-2} \frac{1-a^{2n-2j-3}}{1-a}} w_0 z_0^{da \frac{a^{2n-2}-1}{a^2-1}} z_{-1}^{d} \\
&= \beta^n \alpha^{\frac{d}{1-a} \left( n-1 - \frac{a(a^{2n-2}-1)}{a^2-1} \right)} w_0 z_0^{da \frac{a^{2n-2}-1}{a^2-1}} z_{-1}^{d} \\
&= \beta^n \alpha^{\frac{d(a^{2n-1}+(n-1)(1-a^2)-a)}{(a-1)^2(a+1)}} w_0 z_0^{da \frac{a^{2n-2}-1}{a^2-1}} z_{-1}^{d},
\end{aligned}
\tag{3.18}
$$

for $n \geq 2$, and

$$
\begin{aligned}
w_{2n+1} &= \beta^{\sum_{j=0}^{n} 1} \alpha^{d \sum_{j=0}^{n-2} \sum_{i=0}^{2n-2j-3} a^i} w_{-1} z_0^{d \sum_{j=0}^{n-1} a^{2n-2j-2}} z_{-2}^d \\
&= \beta^{n+1} \alpha^{d \sum_{j=0}^{n-2} \frac{1-a^{2n-2j-2}}{1-a}} w_{-1} z_0^{d \frac{a^{2n}-1}{a^2-1}} z_{-2}^d \\
&= \beta^{n+1} \alpha^{\frac{d}{1-a} \left(n-1-\frac{a^2(a^{2n-2}-1)}{a^2-1}\right)} w_{-1} z_0^{d \frac{a^{2n}-1}{a^2-1}} z_{-2}^d \\
&= \beta^{n+1} \alpha^{\frac{d(a^{2n}-na^2+n-1)}{(a-1)^2(a+1)}} w_{-1} z_0^{d \frac{a^{2n}-1}{a^2-1}} z_{-2}^d,
\end{aligned}
\tag{3.19}
$$

for $n \in \mathbb{N}$.

*Subcase $a = -1$, $c = 1$.* In this case we have

$$
\begin{aligned}
w_{2n} &= \beta^n \alpha^{d \sum_{j=0}^{n-2} \sum_{i=0}^{2n-2j-4}(-1)^i} w_0 z_0^{d \sum_{j=0}^{n-2}(-1)^{2n-2j-3}} z_{-1}^d \\
&= \beta^n \alpha^{d \sum_{j=0}^{n-2} \frac{1-(-1)^{2n-2j-3}}{1-(-1)}} w_0 z_0^{d(1-n)} z_{-1}^d \\
&= \beta^n \alpha^{d(n-1)} w_0 z_0^{d(1-n)} z_{-1}^d,
\end{aligned}
\tag{3.20}
$$

for $n \in \mathbb{N}$, and

$$
\begin{aligned}
w_{2n+1} &= \beta^{\sum_{j=0}^{n} 1} \alpha^{d \sum_{j=0}^{n-2} \sum_{i=0}^{2n-2j-3} a^i} w_{-1} z_0^{d \sum_{j=0}^{n-1} a^{2n-2j-2}} z_{-2}^d \\
&= \beta^{n+1} \alpha^{d \sum_{j=0}^{n-2} \sum_{i=0}^{2n-2j-3}(-1)^i} w_{-1} z_0^{d \sum_{j=0}^{n-1}(-1)^{2n-2j-2}} z_{-2}^d \\
&= \beta^{n+1} \alpha^{d \sum_{j=0}^{n-2} \frac{1-(-1)^{2n-2j-2}}{1-(-1)}} w_{-1} z_0^{dn} z_{-2}^d \\
&= \beta^{n+1} w_{-1} z_0^{dn} z_{-2}^d,
\end{aligned}
\tag{3.21}
$$

for $n \in \mathbb{N}_0$.

*Subcase $a = 1$, $c \neq 1$.* In this case, by using formula (2.3), we get

$$
\begin{aligned}
w_{2n} &= \beta^{\sum_{j=0}^{n-1} c^j} \alpha^{d \sum_{j=0}^{n-2} c^j \sum_{i=0}^{2n-2j-4} 1} w_0^{c^n} z_0^{d \sum_{j=0}^{n-2} c^j} z_{-1}^{dc^{n-1}} \\
&= \beta^{\frac{1-c^n}{1-c}} \alpha^{d \sum_{j=0}^{n-2}(2n-2j-3)c^j} w_0^{c^n} z_0^{d \frac{1-c^{n-1}}{1-c}} z_{-1}^{dc^{n-1}} \\
&= \beta^{\frac{1-c^n}{1-c}} \alpha^{d\left((2n-1)\frac{1-c^{n-1}}{1-c} - 2\frac{1-nc^{n-1}+(n-1)c^n}{(1-c)^2}\right)} w_0^{c^n} z_0^{d \frac{1-c^{n-1}}{1-c}} z_{-1}^{dc^{n-1}} \\
&= \beta^{\frac{1-c^n}{1-c}} \alpha^{d \frac{2n-3-(2n-1)c+c^{n-1}+c^n}{(1-c)^2}} w_0^{c^n} z_0^{d \frac{1-c^{n-1}}{1-c}} z_{-1}^{dc^{n-1}},
\end{aligned}
\tag{3.22}
$$

for $n \geq 2$, and

$$
\begin{aligned}
w_{2n+1} &= \beta^{\sum_{j=0}^{n} c^j} \alpha^{d \sum_{j=0}^{n-2} c^j \sum_{i=0}^{2n-2j-3} 1} w_{-1}^{c^{n+1}} z_0^{d \sum_{j=0}^{n-1} c^j} z_{-2}^{dc^n} \\
&= \beta^{\frac{1-c^{n+1}}{1-c}} \alpha^{d \sum_{j=0}^{n-2}(2n-2j-2)c^j} w_{-1}^{c^{n+1}} z_0^{d \frac{1-c^n}{1-c}} z_{-2}^{dc^n} \\
&= \beta^{\frac{1-c^{n+1}}{1-c}} \alpha^{d\left(2n\frac{1-c^{n-1}}{1-c} - 2\frac{1-nc^{n-1}+(n-1)c^n}{(1-c)^2}\right)} w_{-1}^{c^{n+1}} z_0^{d \frac{1-c^n}{1-c}} z_{-2}^{dc^n} \\
&= \beta^{\frac{1-c^{n+1}}{1-c}} \alpha^{\frac{2d(n-1-nc+c^n)}{(1-c)^2}} w_{-1}^{c^{n+1}} z_0^{d \frac{1-c^n}{1-c}} z_{-2}^{dc^n},
\end{aligned}
\tag{3.23}
$$

for $n \in \mathbb{N}$.

*Subcase $a = c = 1$.* In this case we have

$$
\begin{aligned}
w_{2n} &= \beta^{\sum_{j=0}^{n-1} 1} \alpha^{d \sum_{j=0}^{n-2} \sum_{i=0}^{2n-2j-4} 1} w_0 z_0^{d \sum_{j=0}^{n-2} 1} z_{-1}^d \\
&= \beta^n \alpha^{d \sum_{j=0}^{n-2}(2n-2j-3)} w_0 z_0^{d(n-1)} z_{-1}^d \\
&= \beta^n \alpha^{d(n-1)^2} w_0 z_0^{d(n-1)} z_{-1}^d,
\end{aligned}
\tag{3.24}
$$

for $n \in \mathbb{N}$, and

$$
\begin{aligned}
w_{2n+1} &= \beta^{\sum_{j=0}^{n} 1} \alpha^{d \sum_{j=0}^{n-2} \sum_{i=0}^{2n-2j-3} 1} w_{-1} z_0^{d \sum_{j=0}^{n-1} 1} z_{-2}^d \\
&= \beta^{n+1} \alpha^{d \sum_{j=0}^{n-2}(2n-2j-2)} w_{-1} z_0^{dn} z_{-2}^d \\
&= \beta^{n+1} \alpha^{dn(n-1)} w_{-1} z_0^{dn} z_{-2}^d,
\end{aligned}
\tag{3.25}
$$

for $n \in \mathbb{N}_0$.

*Case $d = 0$.* In this case system (1.2) becomes

$$
z_{n+1} = \alpha z_n^a w_n^b, \quad w_{n+1} = \beta w_{n-1}^c, \quad n \in \mathbb{N}_0.
\tag{3.26}
$$

The solvability of system (3.26) was proved in [35], hence we will only sketch the proof here.

The second equation in (3.26) yields

$$
w_{2n} = \beta^{\sum_{j=0}^{n-1} c^j} w_0^{c^n} \quad n \in \mathbb{N}, \quad \text{and} \quad w_{2n+1} = \beta^{\sum_{j=0}^{n} c^j} w_{-1}^{c^{n+1}},
\tag{3.27}
$$

for $n \in \mathbb{N}_0$.

Hence, if $c \neq 1$ we have

$$
w_{2n} = \beta^{\frac{1-c^n}{1-c}} w_0^{c^n} \quad n \in \mathbb{N}, \quad \text{and} \quad w_{2n+1} = \beta^{\frac{1-c^{n+1}}{1-c}} w_{-1}^{c^{n+1}}, \quad n \in \mathbb{N}_0,
\tag{3.28}
$$

while, if $c = 1$, we have

$$
w_{2n} = \beta^n w_0 \quad \text{and} \quad w_{2n+1} = \beta^{n+1} w_{-1}, \quad n \in \mathbb{N}_0.
\tag{3.29}
$$

Using (3.27) in the first equation in (3.26) it follows that

$$
z_{2n} = \alpha \beta^{b \sum_{j=0}^{n-1} c^j} w_{-1}^{bc^n} z_{2n-1}^a
\tag{3.30}
$$

$$
z_{2n+1} = \alpha \beta^{b \sum_{j=0}^{n-1} c^j} w_0^{bc^n} z_{2n}^a,
\tag{3.31}
$$

for $n \in \mathbb{N}$.

From (3.30) and (3.31) we get

$$
\begin{aligned}
z_{2n} &= \alpha \beta^{b \sum_{j=0}^{n-1} c^j} w_{-1}^{bc^n} \left( \alpha \beta^{b \sum_{j=0}^{n-2} c^j} w_0^{bc^{n-1}} z_{2n-2}^a \right)^a \\
&= \alpha^{1+a} \beta^{b \sum_{j=0}^{n-1} c^j + ab \sum_{j=0}^{n-2} c^j} (w_{-1}^{bc} w_0^{ab})^{c^{n-1}} z_{2n-2}^{a^2},
\end{aligned}
\tag{3.32}
$$

for $n \geq 2$, and

$$
\begin{aligned}
z_{2n+1} &= \alpha \beta^{b \sum_{j=0}^{n-1} c^j} w_0^{bc^n} \left( \alpha \beta^{b \sum_{j=0}^{n-1} c^j} w_{-1}^{bc^n} z_{2n-1}^a \right)^a \\
&= \alpha^{1+a} \beta^{b(1+a) \sum_{j=0}^{n-1} c^j} (w_{-1}^{ab} w_0^b)^{c^n} z_{2n-1}^{a^2},
\end{aligned}
\tag{3.33}
$$

for every $n \in \mathbb{N}$.

By induction it is proved that

$$z_{2n} = \alpha^{(1+a)\sum_{j=0}^{k-1} a^{2j}} \beta^{b\sum_{i=0}^{k-1} a^{2i}\left(\sum_{j=0}^{n-i-1} c^j + a\sum_{j=0}^{n-i-2} c^j\right)} (w_{-1}^{bc} w_0^{ab})^{\sum_{j=0}^{k-1} a^{2j} c^{n-j-1}} z_{2n-2k}^{a^{2k}}, \qquad (3.34)$$

and

$$z_{2n+1} = \alpha^{(1+a)\sum_{j=0}^{k-1} a^{2j}} \beta^{b(1+a)\sum_{i=0}^{k-1} a^{2i} \sum_{j=0}^{n-i-1} c^j} (w_{-1}^{ab} w_0^b)^{\sum_{j=0}^{k-1} a^{2j} c^{n-j}} z_{2n-2k+1}^{a^{2k}}, \qquad (3.35)$$

for every $k, n \in \mathbb{N}$ such that $n \geq k$.

Choosing $k = n$ in (3.34) and (3.35) it follows that

$$z_{2n} = \alpha^{(1+a)\sum_{j=0}^{n-1} a^{2j}} \beta^{b\sum_{i=0}^{n-1} a^{2i}\left(\sum_{j=0}^{n-i-1} c^j + a\sum_{j=0}^{n-i-2} c^j\right)} (w_{-1}^{bc} w_0^{ab})^{\sum_{j=0}^{n-1} a^{2j} c^{n-j-1}} z_0^{a^{2n}}, \qquad (3.36)$$

$$\begin{aligned} z_{2n+1} &= \alpha^{(1+a)\sum_{j=0}^{n-1} a^{2j}} \beta^{b(1+a)\sum_{i=0}^{n-1} a^{2i} \sum_{j=0}^{n-i-1} c^j} (w_{-1}^{ab} w_0^b)^{\sum_{j=0}^{n-1} a^{2j} c^{n-j}} z_1^{a^{2n}} \\ &= \alpha^{(1+a)\sum_{j=0}^{n-1} a^{2j}} \beta^{b(1+a)\sum_{i=0}^{n-1} a^{2i} \sum_{j=0}^{n-i-1} c^j} (w_{-1}^{ab} w_0^b)^{\sum_{j=0}^{n-1} a^{2j} c^{n-j}} (\alpha z_0^a w_0^b)^{a^{2n}} \\ &= \alpha^{\sum_{j=0}^{2n} a^j} \beta^{b(1+a)\sum_{i=0}^{n-1} a^{2i} \sum_{j=0}^{n-i-1} c^j} w_{-1}^{ab\sum_{j=0}^{n-1} a^{2j} c^{n-j}} w_0^{b\sum_{j=0}^{n} a^{2j} c^{n-j}} z_0^{a^{2n+1}}, \end{aligned} \qquad (3.37)$$

for every $n \in \mathbb{N}$.

From formulas (3.36), (3.37), by using Lemma 2.1 and some calculations the following formulas are obtained.

*Subcase $c \neq a^2 \neq 1 \neq c$.* We have

$$z_{2n} = \alpha^{\frac{1-a^{2n}}{1-a}} \beta^{\frac{b(c-a^2+(a-1)(a+c)c^n+(1-c)a^{2n+1})}{(1-a)(1-c)(c-a^2)}} (w_{-1}^{bc} w_0^{ab})^{\frac{c^n-a^{2n}}{c-a^2}} z_0^{a^{2n}}, \qquad (3.38)$$

$$z_{2n+1} = \alpha^{\frac{1-a^{2n+1}}{1-a}} \beta^{\frac{b(1+a)(c-a^2+(a^2-1)c^{n+1}+(1-c)a^{2n+2})}{(1-c)(1-a^2)(c-a^2)}} w_{-1}^{abc\frac{c^n-a^{2n}}{c-a^2}} w_0^{b\frac{c^{n+1}-a^{2n+2}}{c-a^2}} z_0^{a^{2n+1}}, \qquad (3.39)$$

for every $n \in \mathbb{N}$.

*Subcase $a^2 \neq 1 \neq c$, $c = a^2$.* We have

$$z_{2n} = \alpha^{\frac{1-a^{2n}}{1-a}} \beta^{\frac{b(1-na^{2n-1}-a^{2n}+na^{2n+1})}{(1-a)^2(1+a)}} (w_{-1}^{ba^2} w_0^{ab})^{na^{2n-2}} z_0^{a^{2n}}, \qquad (3.40)$$

$$z_{2n+1} = \alpha^{\frac{1-a^{2n+1}}{1-a}} \beta^{\frac{b(1+a)(1-(n+1)a^{2n}+na^{2n+2})}{(1-a^2)^2}} w_{-1}^{bna^{2n+1}} w_0^{b(n+1)a^{2n}} z_0^{a^{2n+1}}, \qquad (3.41)$$

for every $n \in \mathbb{N}$.

*Subcase $a^2 \neq 1 = c$.* In this case, by using (2.3), we have

$$z_{2n} = \alpha^{\frac{1-a^{2n}}{1-a}} \beta^{\frac{b(n-a-na^2+a^{2n+1})}{(1-a^2)(1-a)}} (w_{-1}^b w_0^{ab})^{\frac{1-a^{2n}}{1-a}} z_0^{a^{2n}}, \quad n \in \mathbb{N}, \qquad (3.42)$$

$$z_{2n+1} = \alpha^{\frac{1-a^{2n+1}}{1-a}} \beta^{\frac{b(1+a)(n-(n+1)a^2+a^{2n+2})}{(1-a^2)^2}} w_{-1}^{ab\frac{1-a^{2n}}{1-a^2}} w_0^{b\frac{1-a^{2n+2}}{1-a^2}} z_0^{a^{2n+1}}, \qquad (3.43)$$

for every $n \in \mathbb{N}_0$.

*Subcase $a = -1$, $c = 1$.* In this case we have

$$z_{2n} = \beta^{bn} (w_{-1}^b w_0^{-b})^n z_0, \qquad (3.44)$$

$$z_{2n+1} = \alpha w_{-1}^{-bn} w_0^{b(n+1)} z_{-1}^{-1}, \qquad (3.45)$$

for every $n \in \mathbb{N}$.

*Subcase $a = 1 \neq c$.* In this case we have

$$z_{2n} = \alpha^{2n} \beta^{\frac{b(2n-1-(2n+1)c+c^n+c^{n+1})}{(1-c)^2}} \left(w_{-1}^{bc} w_0^b\right)^{\frac{1-c^n}{1-c}} z_0, \tag{3.46}$$

$$z_{2n+1} = \alpha^{2n+1} \beta^{\frac{2b(n-(n+1)c+c^{n+1})}{(1-c)^2}} w_{-1}^{bc\frac{1-c^n}{1-c}} w_0^{b\frac{1-c^{n+1}}{1-c}} z_0, \tag{3.47}$$

for every $n \in \mathbb{N}$.

*Subcase $a = c = 1$.* We have

$$z_{2n} = \alpha^{2n} \beta^{bn^2} (w_{-1} w_0)^{bn} z_0, \tag{3.48}$$

$$z_{2n+1} = \alpha^{2n+1} \beta^{bn(n+1)} w_{-1}^{bn} w_0^{b(n+1)} z_0, \tag{3.49}$$

for every $n \in \mathbb{N}$.

*Case $a = 0$, $bd \neq 0$.* In this case system (1.2) becomes

$$z_{n+1} = \alpha w_n^b, \quad w_{n+1} = \beta w_{n-1}^c z_{n-2}^d, \quad n \in \mathbb{N}_0. \tag{3.50}$$

By using the first equation in (3.50) into the second one is obtained

$$w_{n+1} = \beta w_{n-1}^c z_{n-2}^d = \beta w_{n-1}^c (\alpha w_{n-3}^b)^d = \alpha^d \beta w_{n-1}^c w_{n-3}^{bd},$$

for $n \geq 3$, from which it follows that

$$w_{2(m+1)+i} = \alpha^d \beta w_{2m+i}^c w_{2(m-1)+i}^{bd}, \tag{3.51}$$

for every $m \in \mathbb{N}$ and each $i = 0, 1$.

Let $\gamma := \alpha^d \beta$,

$$a_1^{(i)} = c, \quad b_1^{(i)} = bd, \quad x_1^{(i)} = 1, \quad \text{for} \quad i = 0, 1. \tag{3.52}$$

Then (3.51) can be written as

$$w_{2(m+1)+i} = \gamma^{x_1^{(i)}} w_{2m+i}^{a_1^{(i)}} w_{2(m-1)+i}^{b_1^{(i)}}, \tag{3.53}$$

for every $m \in \mathbb{N}$ and each $i = 0, 1$.

Using (3.53) with $m \to m - 1$ into itself we get

$$
\begin{aligned}
w_{2(m+1)+i} &= \gamma^{x_1^{(i)}} w_{2m+i}^{a_1^{(i)}} w_{2(m-1)+i}^{b_1^{(i)}} \\
&= \gamma^{x_1^{(i)}} \left(\gamma w_{2(m-1)+i}^{a_1^{(i)}} w_{2(m-2)+i}^{b_1^{(i)}}\right)^{a_1^{(i)}} w_{2(m-1)+i}^{b_1^{(i)}} \\
&= \gamma^{x_1^{(i)}+a_1^{(i)}} w_{2(m-1)+i}^{a_1^{(i)} a_1^{(i)} + b_1^{(i)}} w_{2(m-2)+i}^{b_1^{(i)} a_1^{(i)}} \\
&= \gamma^{x_2^{(i)}} w_{2(m-1)+i}^{a_2^{(i)}} w_{2(m-2)+i}^{b_2^{(i)}}, 
\end{aligned} \tag{3.54}
$$

for every $m \geq 2$ and each $i = 0, 1$, where

$$a_2^{(i)} := a_1^{(i)} a_1^{(i)} + b_1^{(i)}, \quad b_2^{(i)} := b_1^{(i)} a_1^{(i)}, \quad x_2^{(i)} := x_1^{(i)} + a_1^{(i)}. \tag{3.55}$$

Assume that for a $k \in \mathbb{N}$ we have proved that

$$w_{2(m+1)+i} = \gamma^{x_k^{(i)}} w_{2(m-k+1)+i}^{a_k^{(i)}} w_{2(m-k)+i}^{b_k^{(i)}}, \tag{3.56}$$

for $m \geq k$ and each $i = 0, 1$, and that

$$a_k^{(i)} := a_1^{(i)} a_{k-1}^{(i)} + b_{k-1}^{(i)}, \quad b_k^{(i)} := b_1^{(i)} a_{k-1}^{(i)}, \quad x_k^{(i)} := x_{k-1}^{(i)} + a_{k-1}^{(i)}. \tag{3.57}$$

Then, by using (3.53) with $m \to m - k$ into (3.56) we get

$$\begin{aligned}
w_{2(m+1)+i} &= \gamma^{x_k^{(i)}} w_{2(m-k+1)+i}^{a_k^{(i)}} w_{2(m-k)+i}^{b_k^{(i)}} \\
&= \gamma^{x_k^{(i)}} (\gamma w_{2(m-k)+i}^{a_1^{(i)}} w_{2(m-k-1)+i}^{b_1^{(i)}})^{a_k^{(i)}} w_{2(m-k)+i}^{b_k^{(i)}} \\
&= \gamma^{x_k^{(i)} + a_k^{(i)}} w_{2(m-k)+i}^{a_1^{(i)} a_k^{(i)} + b_k^{(i)}} w_{2(m-k-1)+i}^{b_1^{(i)} a_k^{(i)}} \\
&= \gamma^{x_{k+1}^{(i)}} w_{2(m-k)+i}^{a_{k+1}^{(i)}} w_{2(m-k-1)+i}^{b_{k+1}^{(i)}}, 
\end{aligned} \tag{3.58}$$

for $m \geq k + 1$ and each $i = 0, 1$, where

$$a_{k+1}^{(i)} := a_1^{(i)} a_k^{(i)} + b_k^{(i)}, \quad b_{k+1}^{(i)} := b_1^{(i)} a_k^{(i)}, \quad x_{k+1}^{(i)} := x_k^{(i)} + a_k^{(i)}. \tag{3.59}$$

From (3.54), (3.55), (3.58), (3.59) and the induction, we see that (3.56) and (3.57) hold for every $k$ and $m$ such that $2 \leq k \leq m$ for each $i = 0, 1$. In fact, (3.56) holds for $1 \leq k \leq m$, because of (3.53).

The first two equations in (3.57) yield

$$a_k^{(i)} = a_1^{(i)} a_{k-1}^{(i)} + b_1^{(i)} a_{k-2}^{(i)}, \quad k \geq 3. \tag{3.60}$$

The equalities in (3.57) with $k = 1$ yield

$$a_1^{(i)} = a_1^{(i)} a_0^{(i)} + b_0^{(i)}, \quad b_1^{(i)} = b_1^{(i)} a_0^{(i)}, \quad x_1^{(i)} = x_0^{(i)} + a_0^{(i)}. \tag{3.61}$$

Since $b_1^{(i)} = bd \neq 0$, from the second equation in (3.61) we get $a_0^{(i)} = 1$. This, along with $x_1^{(i)} = 1$ and the other two relations in (3.61) implies $b_0^{(i)} = x_0^{(i)} = 0$.

From this and (3.57) with $k = 0$ is obtained

$$1 = a_0^{(i)} = a_1^{(i)} a_{-1}^{(i)} + b_{-1}^{(i)}, \quad 0 = b_0^{(i)} = b_1^{(i)} a_{-1}^{(i)}, \quad 0 = x_0^{(i)} = x_{-1}^{(i)} + a_{-1}^{(i)}. \tag{3.62}$$

Since $b_1^{(i)} \neq 0$, from the second equation in (3.62) we get $a_{-1}^{(i)} = 0$. This along with the other two relations in (3.62) implies $b_{-1}^{(i)} = 1$ and $x_{-1}^{(i)} = 0$.

Using these facts along with the second equation in (3.57) we have that $(a_k^{(i)})_{k \geq -1}$ and $(b_k^{(i)})_{k \geq -1}$, $i = 0, 1$, are solutions to linear equation (3.60) satisfying the initial conditions

$$a_{-1}^{(i)} = 0, \quad a_0^{(i)} = 1; \qquad b_{-1}^{(i)} = 1, \quad b_0^{(i)} = 0, \tag{3.63}$$

respectively, and that $(x_k^{(i)})_{k \geq -1}$, $i = 0, 1$, satisfies the third recurrent relation in (3.57) and

$$x_{-1}^{(i)} = x_0^{(i)} = 0, \quad x_1^{(i)} = 1. \tag{3.64}$$

Since the initial values in (3.52) are the same for $i = 0$ and $i = 1$, and the sequences $a_k^{(0)}$, $b_k^{(0)}$, $x_k^{(0)}$, and $a_k^{(1)}$, $b_k^{(1)}$, $x_k^{(1)}$, satisfy the same system, that is, system (3.57), we have that $a_k^{(0)} = a_k^{(1)}$, $b_k^{(0)} = b_k^{(1)}$ and $x_k^{(0)} = x_k^{(1)}$ for every $k \geq -1$. Thus, from now on we will simply denote these three pairs of sequences, by $a_k$, $b_k$ and $x_k$ respectively.

From (3.56) with $m \to m - 1$ and $k = m - 1$, we have that

$$w_{2m+i} = \gamma^{x_{m-1}} w_{2+i}^{a_{m-1}} w_i^{b_{m-1}}, \tag{3.65}$$

for $m \in \mathbb{N}$ and $i = 0, 1$.

Using the relations in (3.57) in (3.65) it follows that

$$
\begin{aligned}
w_{2m} &= \gamma^{x_{m-1}} w_2^{a_{m-1}} w_0^{b_{m-1}} \\
&= (\alpha^d \beta)^{x_{m-1}} (\beta w_0^c z_{-1}^d)^{a_{m-1}} w_0^{b_{m-1}} \\
&= \alpha^{d x_{m-1}} \beta^{x_{m-1} + a_{m-1}} w_0^{c a_{m-1} + b_{m-1}} z_{-1}^{d a_{m-1}} \\
&= \alpha^{d x_{m-1}} \beta^{x_m} w_0^{a_m} z_{-1}^{d a_{m-1}},
\end{aligned} \tag{3.66}
$$

for $m \in \mathbb{N}$, and

$$
\begin{aligned}
w_{2m+1} &= \gamma^{x_{m-1}} w_3^{a_{m-1}} w_1^{b_{m-1}} \\
&= (\alpha^d \beta)^{x_{m-1}} (\beta^{1+c} w_{-1}^{c^2} z_{-2}^{cd} z_0^d)^{a_{m-1}} (\beta w_{-1}^c z_{-2}^d)^{b_{m-1}} \\
&= \alpha^{d x_{m-1}} \beta^{x_{m-1} + (1+c) a_{m-1} + b_{m-1}} w_{-1}^{c^2 a_{m-1} + c b_{m-1}} z_{-2}^{c d a_{m-1} + d b_{m-1}} z_0^{d a_{m-1}} \\
&= \alpha^{d x_{m-1}} \beta^{x_{m+1}} w_{-1}^{c a_m} z_{-2}^{d a_m} z_0^{d a_{m-1}},
\end{aligned} \tag{3.67}
$$

for $m \in \mathbb{N}$, from which along with the first equation in (3.50) we have that

$$z_{2m+1} = \alpha^{1 + b d x_{m-1}} \beta^{b x_m} w_0^{b a_m} z_{-1}^{b d a_{m-1}}, \tag{3.68}$$

$$z_{2m+2} = \alpha^{1 + b d x_{m-1}} \beta^{b x_{m+1}} w_{-1}^{b c a_m} z_{-2}^{b d a_m} z_0^{b d a_{m-1}}, \tag{3.69}$$

for $m \in \mathbb{N}$.

From the third equation in (3.57) and since $x_1 = 1$ and $a_0 = 1$, we get

$$x_m = \sum_{j=0}^{m-1} a_j, \quad m \in \mathbb{N}. \tag{3.70}$$

Now note that the characteristic equation associated to difference equation (3.60) is $\lambda^2 - c\lambda - bd = 0$, from which it follows that

$$\lambda_{1,2} = \frac{c \pm \sqrt{c^2 + 4bd}}{2}.$$

Hence if $c^2 + 4bd \neq 0$, then

$$a_n = c_1 \lambda_1^n + c_2 \lambda_2^n.$$

From this and since $a_{-1} = 0$ and $a_0 = 1$, we have that

$$a_n = \frac{\lambda_1^{n+1} - \lambda_2^{n+1}}{\lambda_1 - \lambda_2}, \tag{3.71}$$

which along with the second equation in (3.57) implies

$$b_n = bd \frac{\lambda_1^n - \lambda_2^n}{\lambda_1 - \lambda_2}. \tag{3.72}$$

Using (3.71) in (3.70) with $m = n$, for the case when $\lambda_1 \neq 1 \neq \lambda_2$, which is equivalent to $c + bd \neq 1$, we get

$$
\begin{aligned}
x_n &= \sum_{j=0}^{n-1} \frac{\lambda_1^{j+1} - \lambda_2^{j+1}}{\lambda_1 - \lambda_2} = \frac{1}{(\lambda_1 - \lambda_2)} \left( \lambda_1 \frac{\lambda_1^n - 1}{\lambda_1 - 1} - \lambda_2 \frac{\lambda_2^n - 1}{\lambda_2 - 1} \right) \\
&= \frac{(\lambda_2 - 1)\lambda_1^{n+1} - (\lambda_1 - 1)\lambda_2^{n+1} + \lambda_1 - \lambda_2}{(\lambda_1 - 1)(\lambda_2 - 1)(\lambda_1 - \lambda_2)},
\end{aligned} \tag{3.73}
$$

while if $c + bd = 1$, that is, if one of the characteristic roots is equal to one, say $\lambda_2$, which implies that $\lambda_1 = -bd$, we get

$$
\begin{aligned}
x_n &= \sum_{j=0}^{n-1} \frac{\lambda_1^{j+1} - 1}{\lambda_1 - 1} = \frac{1}{(\lambda_1 - 1)} \left( \lambda_1 \frac{\lambda_1^n - 1}{\lambda_1 - 1} - n \right) \\
&= \frac{\lambda_1^{n+1} - (n+1)\lambda_1 + n}{(\lambda_1 - 1)^2} = \frac{(-bd)^{n+1} + (n+1)bd + n}{(1 + bd)^2}.
\end{aligned} \tag{3.74}
$$

If $c^2 + 4bd = 0$, then

$$a_n = (\widehat{c}_1 + \widehat{c}_2 n) \left( \frac{c}{2} \right)^n. \tag{3.75}$$

Using the facts $a_{-1} = 0$ and $a_0 = 1$ in (3.75), we get

$$a_n = (n+1) \left( \frac{c}{2} \right)^n, \tag{3.76}$$

which along with the second equation in (3.57) and $bd = -c^2/4$, implies

$$b_n = bdn \left( \frac{c}{2} \right)^{n-1} = -n \left( \frac{c}{2} \right)^{n+1}. \tag{3.77}$$

Using (3.76) in (3.70) and employing (2.3), for the case $c \neq 2$, we get

$$x_n = \sum_{j=0}^{n-1} (j+1) \left( \frac{c}{2} \right)^j = \frac{1 - (n+1)(\frac{c}{2})^n + n(\frac{c}{2})^{n+1}}{\left( 1 - \frac{c}{2} \right)^2}, \tag{3.78}$$

while if $c = 2$, we get

$$x_n = \sum_{j=0}^{n-1} (j+1) = \frac{n(n+1)}{2}. \tag{3.79}$$

*Case $bd \neq 0$.* First note that $\alpha, \beta \in \mathbb{C} \setminus \{0\}$ and $z_{-2}, z_{-1}, z_0, w_{-1}, w_0 \in \mathbb{C} \setminus \{0\}$ along with (1.2) and a simple inductive argument shows that $z_n w_n \neq 0$ for $n \geq -1$. For such a solution from the first equation in (1.2) we have

$$w_n^b = \frac{z_{n+1}}{\alpha z_n^a}, \quad n \in \mathbb{N}_0, \tag{3.80}$$

while from the second one it follows that

$$w_{n+1}^b = \beta^b w_{n-1}^{bc} z_{n-2}^{bd}, \quad n \in \mathbb{N}_0. \tag{3.81}$$

Using (3.80) into (3.81), we obtain

$$z_{n+2} = \alpha^{1-c}\beta^b z_{n+1}^a z_n^c z_{n-1}^{-ac} z_{n-2}^{bd}, \quad n \in \mathbb{N}, \tag{3.82}$$

which is a fourth order product-type difference equation.

Let $\delta = \alpha^{1-c}\beta^b$,

$$a_1 := a, \quad b_1 := c, \quad c_1 := -ac, \quad d_1 := bd, \quad y_1 := 1. \tag{3.83}$$

Then equation (3.82) can be written as

$$z_{n+2} = \delta^{y_1} z_{n+1}^{a_1} z_n^{b_1} z_{n-1}^{c_1} z_{n-2}^{d_1}, \quad n \in \mathbb{N}. \tag{3.84}$$

Using (3.84) with $n \to n-1$ into (3.84), we get

$$\begin{aligned}
z_{n+2} &= \delta^{y_1}\left(\delta z_n^{a_1} z_{n-1}^{b_1} z_{n-2}^{c_1} z_{n-3}^{d_1}\right)^{a_1} z_n^{b_1} z_{n-1}^{c_1} z_{n-2}^{d_1}, \\
&= \delta^{y_1+a_1} z_n^{a_1 a_1 + b_1} z_{n-1}^{b_1 a_1 + c_1} z_{n-2}^{c_1 a_1 + d_1} z_{n-3}^{d_1 a_1} \\
&= \delta^{y_2} z_n^{a_2} z_{n-1}^{b_2} z_{n-2}^{c_2} z_{n-3}^{d_2}, \tag{3.85}
\end{aligned}$$

for $n \geq 2$, where

$$a_2 := a_1 a_1 + b_1, \quad b_2 := b_1 a_1 + c_1, \quad c_2 := c_1 a_1 + d_1, \quad d_2 := d_1 a_1, \quad y_2 := y_1 + a_1. \tag{3.86}$$

Assume that for a $k \in \mathbb{N}$ such that $2 \leq k \leq n$, we have proved that

$$z_{n+2} = \delta^{y_k} z_{n+2-k}^{a_k} z_{n+1-k}^{b_k} z_{n-k}^{c_k} z_{n-k-1}^{d_k}, \tag{3.87}$$

for $n \geq k$, and that

$$a_k = a_1 a_{k-1} + b_{k-1}, \quad b_k = b_1 a_{k-1} + c_{k-1}, \quad c_k = c_1 a_{k-1} + d_{k-1}, \quad d_k = d_1 a_{k-1}, \tag{3.88}$$

$$y_k = y_{k-1} + a_{k-1}. \tag{3.89}$$

Using (3.84) with $n \to n-k$ into (3.87), we obtain

$$\begin{aligned}
z_{n+2} &= \delta^{y_k}\left(\delta z_{n+1-k}^{a_1} z_{n-k}^{b_1} z_{n-k-1}^{c_1} z_{n-k-2}^{d_1}\right)^{a_k} z_{n+1-k}^{b_k} z_{n-k}^{c_k} z_{n-k-1}^{d_k} \\
&= \delta^{y_k+a_k} z_{n+1-k}^{a_1 a_k + b_k} z_{n-k}^{b_1 a_k + c_k} z_{n-k-1}^{c_1 a_k + d_k} z_{n-k-2}^{d_1 a_k} \\
&= \delta^{y_{k+1}} z_{n+1-k}^{a_{k+1}} z_{n-k}^{b_{k+1}} z_{n-k-1}^{c_{k+1}} z_{n-k-2}^{d_{k+1}}, \tag{3.90}
\end{aligned}$$

for $n \geq k+1$, where

$$a_{k+1} := a_1 a_k + b_k, \quad b_{k+1} := b_1 a_k + c_k, \quad c_{k+1} := c_1 a_k + d_k, \quad d_{k+1} := d_1 a_k,$$

$$y_{k+1} := y_k + a_k. \tag{3.91}$$

From (3.85), (3.86), (3.90), (3.91) and the method of induction we get that (3.87), (3.88) and (3.89) hold for every $k$ and $n$ such that $2 \leq k \leq n$. Moreover, (3.87) holds also for $1 \leq k \leq n$, because of (3.84).

By setting $k = n$ in (3.87) and using $z_1 = \alpha z_0^a w_0^b$, $z_2 = \alpha^{1+a}\beta^b z_{-2}^{bd} z_0^{a^2} w_{-1}^{bc} w_0^{ab}$, (3.88) and (3.89), we get

$$
\begin{aligned}
z_{n+2} &= \delta^{y_n} z_2^{a_n} z_1^{b_n} z_0^{c_n} z_{-1}^{d_n} \\
&= (\alpha^{1-c}\beta^b)^{y_n} (\alpha^{1+a}\beta^b z_{-2}^{bd} z_0^{a^2} w_{-1}^{bc} w_0^{ab})^{a_n} (\alpha z_0^a w_0^b)^{b_n} z_0^{c_n} z_{-1}^{d_n} \\
&= \alpha^{(1-c)y_n+(1+a)a_n+b_n} \beta^{by_n+ba_n} z_{-2}^{bda_n} z_{-1}^{d_n} z_0^{a^2 a_n+ab_n+c_n} w_{-1}^{bca_n} w_0^{aba_n+bb_n} \\
&= \alpha^{y_{n+2}-cy_n} \beta^{by_{n+1}} z_{-2}^{bda_n} z_{-1}^{bda_{n-1}} z_0^{a_{n+2}-ca_n} w_{-1}^{bca_n} w_0^{ba_{n+1}}, \quad n \in \mathbb{N}. \quad (3.92)
\end{aligned}
$$

From (3.88) we easily obtain that $(a_k)_{k\in\mathbb{N}}$ satisfies the difference equation

$$
a_k = a_1 a_{k-1} + b_1 a_{k-2} + c_1 a_{k-3} + d_1 a_{k-4}, \quad k \geq 5. \quad (3.93)
$$

From (3.91) with $k = 0$ we get

$$
a_1 = a_1 a_0 + b_0, \; b_1 = b_1 a_0 + c_0, \; c_1 = c_1 a_0 + d_0, \; d_1 = d_1 a_0, \; y_1 = y_0 + a_0. \quad (3.94)
$$

Since $d_1 = bd \neq 0$, from the fourth equation in (3.94) we get $a_0 = 1$. Using this fact and $y_1 = 1$ in the other equalities in (3.94) we get $b_0 = c_0 = d_0 = y_0 = 0$.

From this and by (3.91) with $k = -1$ we get

$$
1 = a_0 = a_1 a_{-1} + b_{-1}, \quad 0 = b_0 = b_1 a_{-1} + c_{-1}, \quad 0 = c_0 = c_1 a_{-1} + d_{-1}
$$
$$
0 = d_0 = d_1 a_{-1}, \quad 0 = y_0 = y_{-1} + a_{-1}. \quad (3.95)
$$

Since $d_1 \neq 0$, from the fourth equation in (3.95) we get $a_{-1} = 0$. Using this fact in the other equalities in (3.95) we get $b_{-1} = 1$, $c_{-1} = d_{-1} = y_{-1} = 0$.

From this and by (3.91) with $k = -2$ we get

$$
0 = a_{-1} = a_1 a_{-2} + b_{-2}, \quad 1 = b_{-1} = b_1 a_{-2} + c_{-2}, \quad 0 = c_{-1} = c_1 a_{-2} + d_{-2}
$$
$$
0 = d_{-1} = d_1 a_{-2}, \quad 0 = y_{-1} = y_{-2} + a_{-2}. \quad (3.96)
$$

Since $d_1 \neq 0$, from the fourth equation in (3.96) we get $a_{-2} = 0$. Using this fact in the other equalities in (3.96) we get $b_{-2} = d_{-2} = y_{-2} = 0$ and $c_{-2} = 1$.

From this and by (3.91) with $k = -3$ we get

$$
0 = a_{-2} = a_1 a_{-3} + b_{-3}, \quad 0 = b_{-2} = b_1 a_{-3} + c_{-3}, \quad 1 = c_{-2} = c_1 a_{-3} + d_{-3}
$$
$$
0 = d_{-2} = d_1 a_{-3}, \quad 0 = y_{-2} = y_{-3} + a_{-3}. \quad (3.97)
$$

Since $d_1 \neq 0$, from the fourth equation in (3.97) we get $a_{-3} = 0$. Using this fact in the other equalities in (3.96) we get $b_{-3} = c_{-3} = y_{-3} = 0$ and $d_{-3} = 1$.

Hence, $(a_k)_{k\geq-3}$ is a solution to (3.93) satisfying the next initial conditions

$$
a_{-3} = 0, \quad a_{-2} = 0, \quad a_{-1} = 0, \quad a_0 = 1. \quad (3.98)
$$

Note that by using (3.98) and

$$
y_{-i} = 0, \quad i = \overline{0,3}, \quad (3.99)
$$

we see that (3.92) holds also for $n = -2, -1$.

Since difference equation (3.93) is solvable, closed form formula for $(a_k)_{k \geq -3}$ can be found. From this, since

$$y_k = 1 + \sum_{j=1}^{k-1} a_j = \sum_{j=0}^{k-1} a_j, \quad k \in \mathbb{N}, \tag{3.100}$$

and since the sums can be calculated it follows that closed form formulas for $(y_k)_{k \in \mathbb{N}}$ can be found too. Using these facts and (3.92) we see that equation (3.82) is solvable too.

From the second equation in (1.2), we have that for every well-defined solution

$$z_{n-2}^d = \frac{w_{n+1}}{\beta w_{n-1}^c}, \quad n \in \mathbb{N}_0, \tag{3.101}$$

while from the first one it follows that

$$z_{n+1}^d = \alpha^d z_n^{ad} w_n^{bd}, \quad n \in \mathbb{N}_0. \tag{3.102}$$

Using (3.101) into (3.102) we obtain

$$w_{n+4} = \beta^{1-a} \alpha^d w_{n+3}^a w_{n+2}^c w_{n+1}^{-ac} w_n^{bd}, \quad n \in \mathbb{N}_0, \tag{3.103}$$

which is a related equation to (3.82) (with shifted indices forward for two and with a different coefficient).

Hence, the above presented procedure for getting $z_n$ can be repeated and obtained that for a $k$ such that $1 \leq k \leq n$

$$w_{n+4} = \eta^{y_k} w_{n+4-k}^{a_k} w_{n+3-k}^{b_k} w_{n+2-k}^{c_k} w_{n+1-k}^{d_k}, \quad n \geq k-1, \tag{3.104}$$

where $\eta = \beta^{1-a} \alpha^d$, sequences $(a_k)_{k \in \mathbb{N}}$, $(b_k)_{k \in \mathbb{N}}$, $(c_k)_{k \in \mathbb{N}}$, $(d_k)_{k \in \mathbb{N}}$ satisfy system (3.88) with initial conditions (3.83), and $(y_k)_{k \in \mathbb{N}}$ is given by (3.100). These sequences can be prolonged for $k \geq -3$, so that (3.98) and (3.99) hold.

From (3.104) with $k = n+1$ and by using (3.11) we get

$$\begin{aligned}
w_{n+4} &= \eta^{y_{n+1}} w_3^{a_{n+1}} w_2^{b_{n+1}} w_1^{c_{n+1}} w_0^{d_{n+1}} \\
&= (\beta^{1-a} \alpha^d)^{y_{n+1}} (\beta^{1+c} w_{-1}^{c^2} z_{-2}^{cd} z_0^d)^{a_{n+1}} (\beta w_0^c z_{-1}^d)^{b_{n+1}} (\beta w_{-1}^c z_{-2}^d)^{c_{n+1}} w_0^{d_{n+1}} \\
&= \alpha^{d y_{n+1}} \beta^{(1-a)y_{n+1}+(1+c)a_{n+1}+b_{n+1}+c_{n+1}} w_{-1}^{c^2 a_{n+1}+cc_{n+1}} w_0^{cb_{n+1}+d_{n+1}} \\
&\quad \times z_{-2}^{cda_{n+1}+dc_{n+1}} z_{-1}^{db_{n+1}} z_0^{da_{n+1}} \\
&= \alpha^{d y_{n+1}} \beta^{y_{n+4}-ay_{n+3}} w_{-1}^{c(a_{n+3}-aa_{n+2})} w_0^{a_{n+4}-aa_{n+3}} z_{-2}^{d(a_{n+3}-aa_{n+2})} \\
&\quad \times z_{-1}^{d(a_{n+2}-aa_{n+1})} z_0^{da_{n+1}}, \tag{3.105}
\end{aligned}$$

for every $n \in \mathbb{N}_0$.

From (3.98) and (3.99) it is seen that (3.105) holds also for $n = -4, -3, -2, -1$.

As above the solvability of equation (3.93) shows that closed form formula for $(a_k)_{k \geq -3}$ can be found. Using the formula in (3.100) is obtained closed form formula for $(y_k)_{k \in \mathbb{N}}$. These facts along with (3.105) imply that equation (3.103) is solvable too. A direct calculation shows that the sequences $(z_n)_{n \geq -2}$ in (3.92) and $(w_n)_{n \geq -1}$ in (3.105) are solutions to system (1.2) with initial values $w_{-1}, w_0, z_{-2}, z_{-1}, z_0$. Hence, system (1.2) is also solvable in this case, finishing the proof of the theorem. □

From the proof of Theorem 3.1 we obtain the following corollary.

**Corollary 3.2.** *Consider system* (1.2) *with* $a, b, c, d \in \mathbb{Z}$. *Assume that* $\alpha, \beta \in \mathbb{C} \setminus \{0\}$ *and* $z_{-2}, z_{-1},$ $z_0, w_{-1}, w_0 \in \mathbb{C} \setminus \{0\}$. *Then the following statements are true.*

(a) *If* $b = 0$, $a \neq 1 \neq c$ *and* $c \neq a^2$, *then the general solution to system* (1.2) *is given by* (3.3), (3.14) *and* (3.15).

(b) *If* $b = 0$, $a^2 \neq 1 \neq c$ *and* $c = a^2$, *then the general solution to system* (1.2) *is given by* (3.3), (3.16) *and* (3.17).

(c) *If* $b = 0$ *and* $a^2 \neq 1 = c$, *then the general solution to system* (1.2) *is given by* (3.3), (3.18) *and* (3.19).

(d) *If* $b = 0$, $a = -1$ *and* $c = 1$, *then the general solution to system* (1.2) *is given by* (3.3), (3.20) *and* (3.21).

(e) *If* $b = 0$, $a = 1$ *and* $c \neq 1$, *then the general solution to system* (1.2) *is given by* (3.4), (3.22) *and* (3.23).

(f) *If* $b = 0$, $a = c = 1$, *then the general solution to system* (1.2) *is given by* (3.4), (3.24) *and* (3.25).

(g) *If* $d = 0$, $c \neq a^2 \neq 1 \neq c$, *then the general solution to system* (1.2) *is given by* (3.28), (3.38) *and* (3.39).

(h) *If* $d = 0$, $c = a^2 \neq 1 \neq c$, *then the general solution to system* (1.2) *is given by* (3.28), (3.40) *and* (3.41).

(i) *If* $d = 0$, $a^2 \neq 1 = c$, *then the general solution to system* (1.2) *is given by* (3.29), (3.42) *and* (3.43).

(j) *If* $d = 0$, $a = -1$ *and* $c = 1$, *then the general solution to system* (1.2) *is given by* (3.29), (3.44) *and* (3.45).

(k) *If* $d = 0$, $a = 1$ *and* $c \neq 1$, *then the general solution to system* (1.2) *is given by* (3.28), (3.46) *and* (3.47).

(l) *If* $d = 0$, $a = c = 1$, *then the general solution to system* (1.2) *is given by* (3.29), (3.48) *and* (3.49).

(m) *If* $a = 0$, $bd \neq 0$, $c^2 + 4bd \neq 0$ *and* $c + bd \neq 1$, *then the general solution to system* (1.2) *is given by* (3.66)–(3.69), *where sequence* $(a_m)_{m \geq -1}$ *is given by* (3.71) *and* $(x_m)_{m \geq -1}$ *is given by* (3.73).

(n) *If* $a = 0$, $bd \neq 0$, $c^2 + 4bd \neq 0$ *and* $c + bd = 1$, *then the general solution to system* (1.2) *is given by* (3.66)–(3.69), *where sequence* $(a_m)_{m \geq -1}$ *is given by* (3.71) *with* $\lambda_1 = -bd$ *and* $\lambda_2 = 1$ *and* $(x_m)_{m \geq -1}$ *is given by* (3.74).

(o) *If* $a = 0$, $bd \neq 0$, $c^2 + 4bd = 0$ *and* $c \neq 2$, *then the general solution to system* (1.2) *is given by* (3.66)–(3.69), *where sequence* $(a_m)_{m \geq -1}$ *is given by* (3.76) *and* $(x_m)_{m \geq -1}$ *is given by* (3.78).

(p) *If* $a = 0$, $c^2 + 4bd = 0$ *and* $c = 2$, *then the general solution to system* (1.2) *is given by* (3.66)–(3.69), *where sequence* $(a_m)_{m \geq -1}$ *is given by* (3.76) *with* $c = 2$, *and* $(x_m)_{m \geq -1}$ *is given by* (3.79)

(q) *If* $bd \neq 0$, *then the general solution to system* (1.2) *is given by* (3.92) *and* (3.105), *where the sequence* $(a_k)_{k \geq -3}$ *satisfies difference equation* (3.93) *with initial conditions in* (3.98) *and where* $(y_k)_{k \in \mathbb{N}}$ *is given by* (3.99) *and* (3.100).

### 3.2 Structure of the solutions to system (1.2) in the case $bd \neq 0$

Equation (3.93), in the case $bd \neq 0$, is solvable since the characteristic polynomial

$$p_4(\lambda) = \lambda^4 - a_1\lambda^3 - b_1\lambda^2 - c_1\lambda - d_1, \tag{3.106}$$

associated to the equation is of the fourth order.

In this case polynomial (3.106) has the following zeros:

$$\lambda_1 = \frac{a}{4} - \frac{1}{2}\sqrt{\frac{a^2}{4} + \frac{2c}{3} + s} - \frac{1}{2}\sqrt{\frac{a^2}{2} + \frac{4c}{3} - s + \frac{Q}{4\sqrt{\frac{a^2}{4} + \frac{2c}{3} + s}}}, \tag{3.107}$$

$$\lambda_2 = \frac{a}{4} - \frac{1}{2}\sqrt{\frac{a^2}{4} + \frac{2c}{3} + s} + \frac{1}{2}\sqrt{\frac{a^2}{2} + \frac{4c}{3} - s + \frac{Q}{4\sqrt{\frac{a^2}{4} + \frac{2c}{3} + s}}}, \tag{3.108}$$

$$\lambda_3 = \frac{a}{4} + \frac{1}{2}\sqrt{\frac{a^2}{4} + \frac{2c}{3} + s} - \frac{1}{2}\sqrt{\frac{a^2}{2} + \frac{4c}{3} - s - \frac{Q}{4\sqrt{\frac{a^2}{4} + \frac{2c}{3} + s}}}, \tag{3.109}$$

$$\lambda_4 = \frac{a}{4} + \frac{1}{2}\sqrt{\frac{a^2}{4} + \frac{2c}{3} + s} + \frac{1}{2}\sqrt{\frac{a^2}{2} + \frac{4c}{3} - s - \frac{Q}{4\sqrt{\frac{a^2}{4} + \frac{2c}{3} + s}}}, \tag{3.110}$$

where

$$s = \frac{1}{3\sqrt[3]{2}}\left(\sqrt[3]{\Delta_1 - \sqrt{\Delta_1^2 - 4\Delta_0^3}} + \sqrt[3]{\Delta_1 + \sqrt{\Delta_1^2 - 4\Delta_0^3}}\right), \tag{3.111}$$

$$\Delta_0 := c^2 + 3a^2c - 12bd, \tag{3.112}$$

$$\Delta_1 := 18a^2c^2 - 2c^3 - 27a^2bd - 72bcd, \tag{3.113}$$

$$Q := -a^3 + 4ac. \tag{3.114}$$

**Remark 3.3.** Number $s$ defined in (3.111) is a zero of the following third-order polynomial equation

$$\lambda^3 + c\lambda^2 + (4bd - a^2c)\lambda + 4bcd + a^2bd - a^2c^2 = 0, \tag{3.115}$$

which is a resolvent cubic equation of the quartic one $p_4(\lambda) = 0$. We point out here that a resolvent cubic equation of a quartic is not always the same, since it depends on the way how the quartic one is solved. Zeros (3.107)–(3.110) of polynomial (3.106) are obtained here by writing $p_4(\lambda)$ as follows

$$p_4(\lambda) = \left(\lambda^2 - \frac{a}{2}\lambda + \frac{s}{2}\right)^2 - \left(\left(\frac{a^2}{4} + c + s\right)\lambda^2 - \left(\frac{as}{2} + ac\right)\lambda + bd + \frac{s^2}{4}\right)$$

and then choosing parameter $s$ such that the following condition is satisfied

$$\left(\frac{as}{2} + ac\right)^2 = 4\left(\frac{a^2}{4} + c + s\right)\left(bd + \frac{s^2}{4}\right),$$

from which is obtained equation (3.115) [3].

The nature of these zeros depends on the sign of the discriminant

$$\Delta := \frac{1}{27}(4\Delta_0^3 - \Delta_1^2), \tag{3.116}$$

and signs of the following quantities

$$P := -8c - 3a^2 \tag{3.117}$$

and

$$D := -64bd - 16c^2 - 3a^4. \tag{3.118}$$

The following proposition, which was essentially proved in [16], explains the nature of the zeros of an arbitrary polynomial of the fourth order in terms of the corresponding quantities $\Delta$, $\Delta_0$, $D$, $P$ and $Q$ (the quantities in (3.112), (3.114), (3.116)–(3.118) are special cases of them for the case of polynomial (3.106)).

**Proposition 3.4.** *Let*

$$P_4(t) = t^4 + bt^3 + ct^2 + dt + e,$$

$$\Delta_0 = c^2 - 3bd + 12e, \quad \Delta_1 = 2c^3 - 9bcd + 27b^2e + 27d^2 - 72ce, \quad \Delta = \frac{1}{27}(4\Delta_0^3 - \Delta_1^2),$$

$$P = 8c - 3b^2, \quad Q = b^3 + 8d - 4bc, \quad D = 64e - 16c^2 + 16b^2c - 16bd - 3b^4.$$

*Then the following statements are true.*

1° *If $\Delta < 0$, then two zeros of $P_4$ are real and different, and two are non-real complex conjugate;*

2° *If $\Delta > 0$, then all the zeroes of $P_4$ are real or none is. More precisely*

    2.1° *if $P < 0$ and $D < 0$, then all four zeros of $P_4$ are real and different;*

    2.2° *if $P > 0$ or $D > 0$, then there are two pairs of non-real complex conjugate zeros of $P_4$.*

3° *If $\Delta = 0$, then and only then the polynomial has a multiple zero. The following cases can occur:*

    3.1° *if $P < 0$, $D < 0$ and $\Delta_0 \neq 0$, then two zeros of $P_4$ are real end equal and two are real and simple;*

    3.2° *if $D > 0$ or ($P > 0$ and ($D \neq 0$ or $Q \neq 0$)), then two zeros of $P_4$ are real and equal and two are complex conjugate;*

    3.3° *if $\Delta_0 = 0$ and $D \neq 0$, there is a triple zero of $P_4$ and one simple, all real;*

    3.4° *if $D = 0$ then*

        3.4.1° *if $P < 0$ there are two double real zeros of $P_4$;*

        3.4.2° *if $P > 0$ and $Q = 0$ there are two double complex conjugate zeros of $P_4$;*

        3.4.3° *if $\Delta_0 = 0$, then all four zeros of $P_4$ are real and equal to $-b/4$.*

*Case $\Delta \neq 0$.* In this case all the zeros $\lambda_i$, $i = \overline{1,4}$ of polynomial (3.106) are mutually different, and the general solution to equation (3.93) has the following form

$$a_n = \alpha_1\lambda_1^n + \alpha_2\lambda_2^n + \alpha_3\lambda_3^n + \alpha_4\lambda_4^n, \quad n \in \mathbb{N}, \tag{3.119}$$

where $\alpha_i$, $i = \overline{1,4}$, are arbitrary constants.

If, for example, $a = 1$, $c = 2$ and $bd = 3$, then

$$p_4(\lambda) = \lambda^4 - \lambda^3 - 2\lambda^2 + 2\lambda - 3 \tag{3.120}$$

and $\Delta \neq 0$, which by Proposition 3.4 shows that there are the cases when all the zeros of polynomial (3.106) are different. Moreover, since $p_4(1) = -3$ none of the zeros of polynomial (3.120) is equal to one, and since $\Delta < 0$ two zeros are complex-conjugate, i.e., $\lambda_1 = \overline{\lambda}_2$ and two are real and different, i.e., $\lambda_3, \lambda_4 \in \mathbb{R}$ and $\lambda_3 \neq \lambda_4$.

Since when $d_1 = bd \neq 0$ the solution to equation (3.93) can be prolonged for nonpositive indices, we may assume that (3.119) holds for $n \geq -3$ (or for every $n \geq -s$, for each $s \in \mathbb{N}$).

If we apply Lemma 2.2 to polynomial $p_4$ in (3.106), we have

$$\sum_{j=1}^{4} \frac{\lambda_j^l}{p_4'(\lambda_j)} = 0$$

for $l = \overline{0,2}$, and

$$\sum_{j=1}^{4} \frac{\lambda_j^3}{p_4'(\lambda_j)} = 1,$$

where $\lambda_i$, $i = \overline{1,4}$ are given by (3.107)–(3.110).

From this, since from (3.98) we have $a_{-3} = a_{-2} = a_{-1} = 0$ and $a_0 = 1$, and general solution to equation (3.93) has the form in (3.119), we obtain

$$a_n = \sum_{j=1}^{4} \frac{\lambda_j^{n+3}}{p_4'(\lambda_j)} = \frac{\lambda_1^{n+3}}{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)(\lambda_1 - \lambda_4)} + \frac{\lambda_2^{n+3}}{(\lambda_2 - \lambda_1)(\lambda_2 - \lambda_3)(\lambda_2 - \lambda_4)}$$

$$+ \frac{\lambda_3^{n+3}}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_3 - \lambda_4)} + \frac{\lambda_4^{n+3}}{(\lambda_4 - \lambda_1)(\lambda_4 - \lambda_2)(\lambda_4 - \lambda_3)}, \tag{3.121}$$

for $n \geq -3$.

On the other hand, from (3.88) we get

$$b_n = a_{n+1} - a_1 a_n, \quad c_n = c_1 a_{n-1} + d_1 a_{n-2}, \quad d_n = d_1 a_{n-1}, \tag{3.122}$$

for $n \geq -3$.

By using (3.121) into (3.122), we get

$$b_n = \sum_{j=1}^{4} \frac{\lambda_j - a}{p_4'(\lambda_j)} \lambda_j^{n+3} \tag{3.123}$$

$$c_n = \sum_{j=1}^{4} \frac{-ac\lambda_j + bd}{p_4'(\lambda_j)} \lambda_j^{n+1} \tag{3.124}$$

$$d_n = \sum_{j=1}^{4} \frac{bd}{p_4'(\lambda_j)} \lambda_j^{n+2}, \tag{3.125}$$

for $n \geq -3$.

By using (3.121) into (3.100) it follows that

$$y_n = \sum_{j=0}^{n-1} \sum_{i=1}^{4} \frac{\lambda_i^{j+3}}{p_4'(\lambda_i)} = \sum_{i=1}^{4} \frac{\lambda_i^3(\lambda_i^n - 1)}{p_4'(\lambda_i)(\lambda_i - 1)}, \quad n \in \mathbb{N}, \tag{3.126}$$

when $p_4(1) \neq 1$, i.e., when $\lambda_i \neq 1$, $i = \overline{1,4}$. Moreover, a direct calculation along with Lemma 2.2 shows that formula (3.126) also holds for $n = -j$, $j = \overline{0,3}$.

*Case $\Delta \neq 0$ and one of the zeros is equal to one.* The characteristic polynomial (3.106) will have a zero equal to one if

$$p_4(1) = 1 - a - c + ac - bd = 0,$$

that is, if

$$(a-1)(c-1) = bd, \tag{3.127}$$

so that

$$p_4(\lambda) = \lambda^4 - a\lambda^3 - c\lambda^2 + ac\lambda - (a-1)(c-1). \tag{3.128}$$

If $a = 2$ and $c = 2$, then $bd = 1 \neq 0$ and consequently

$$p_4(\lambda) = \lambda^4 - 2\lambda^3 - 2\lambda^2 + 4\lambda - 1 = (\lambda - 1)(\lambda^3 - \lambda^2 - 3\lambda + 1).$$

All the zeros of the polynomial are mutually different and exactly one of them is equal to one, say $\lambda_1$.

In this case the general solution has the following form

$$a_n = \widehat{\alpha}_1 + \widehat{\alpha}_2 \lambda_2^n + \widehat{\alpha}_3 \lambda_3^n + \widehat{\alpha}_4 \lambda_4^n, \quad n \in \mathbb{N}, \tag{3.129}$$

where $\widehat{\alpha}_i$, $i = \overline{1,4}$, are arbitrary constants.

In this case formulas (3.121), (3.123), (3.124) and (3.125) also holds but with $\lambda_1 = 1$. On the other hand, we have that

$$
\begin{aligned}
y_n &= \sum_{j=0}^{n-1} \frac{1}{p_4'(1)} + \sum_{j=0}^{n-1} \sum_{i=2}^{4} \frac{\lambda_i^{j+3}}{p_4'(\lambda_i)} \\
&= \frac{n}{4 - 3a - 2c + ac} + \sum_{i=2}^{4} \frac{\lambda_i^3(\lambda_i^n - 1)}{p_4'(\lambda_i)(\lambda_i - 1)},
\end{aligned}
\tag{3.130}
$$

since $p_4'(1) = 4 - 3a - 2c + ac$. Moreover, a direct calculation along with Lemma 2.2 shows that formula (3.130) also holds for $n = -j$, $j = \overline{0,3}$.

From the above consideration and Corollary 3.2 *(q)* we have that the following result holds.

**Corollary 3.5.** *Consider system (1.2) with $a, b, c, d \in \mathbb{Z}$ and $bd \neq 0$. Assume that $z_{-2}, z_{-1}, z_0, w_{-1}, w_0 \in \mathbb{C} \setminus \{0\}$ and $\Delta \neq 0$. Then the following statements are true.*

*(a) If none of the zeros of characteristic polynomial (3.106) is equal to one, i.e., if $(a-1)(c-1) \neq bd$, then the general solution to system (1.2) is given by formulas (3.92) and (3.105), where sequence $(a_n)_{n \geq -3}$ is given by (3.121), while $(y_n)_{n \geq -3}$ is given by (3.126).*

*(b) If (exactly) one of the zeros of characteristic polynomial (3.106) is equal to one, say $\lambda_1$, i.e., if $(a-1)(c-1) = bd$ and $4 - 3a - 2c + ac \neq 0$, then the general solution to system (1.2) is given by formulas (3.92) and (3.105), where sequence $(a_n)_{n \geq -3}$ is given by (3.121) with $\lambda_1 = 1$, while $(y_n)_{n \geq -3}$ is given by (3.130).*

*Case when there is only one double zero.* For $a = 4$, $c = 0$ and $bd = -27$ is obtained

$$p_4(\lambda) = \lambda^4 - 4\lambda^3 + 27 = (\lambda - 3)^2(\lambda + 1 + i\sqrt{2})(\lambda + 1 - i\sqrt{2}), \qquad (3.131)$$

(it is easy to check that $\Delta = 0$, $\Delta_0 \neq 0$ and $D > 0$). So, polynomial (3.131) has two (real) equal zeros and two are complex-conjugate, but none of them is equal to one.

In the case when only two zeros are equal, say $\lambda_1$ and $\lambda_2$, then the general solution has the following form

$$a_n = (\gamma_1 + \gamma_2 n)\lambda_2^n + \gamma_3\lambda_3^n + \gamma_4\lambda_4^n, \quad n \in \mathbb{N}, \qquad (3.132)$$

where $\gamma_i$, $i = \overline{1,4}$, are arbitrary constants.

To find the solution such that $a_{-3} = a_{-2} = a_{-1} = 0$ and $a_0 = 1$ we will let $\lambda_1 \to \lambda_2$ in formula (3.121).

We have

$$\begin{aligned}
a_n &= \lim_{\lambda_1 \to \lambda_2} \left( \frac{\lambda_1^{n+3}}{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)(\lambda_1 - \lambda_4)} + \frac{\lambda_2^{n+3}}{(\lambda_2 - \lambda_1)(\lambda_2 - \lambda_3)(\lambda_2 - \lambda_4)} \right. \\
&\quad \left. + \frac{\lambda_3^{n+3}}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)(\lambda_3 - \lambda_4)} + \frac{\lambda_4^{n+3}}{(\lambda_4 - \lambda_1)(\lambda_4 - \lambda_2)(\lambda_4 - \lambda_3)} \right) \\
&= \lim_{\lambda_1 \to \lambda_2} \left( \frac{\lambda_1^{n+3}(\lambda_2 - \lambda_3)(\lambda_2 - \lambda_4) - \lambda_2^{n+3}(\lambda_1 - \lambda_3)(\lambda_1 - \lambda_4)}{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)(\lambda_1 - \lambda_4)(\lambda_2 - \lambda_3)(\lambda_2 - \lambda_4)} \right. \\
&\quad \left. + \frac{\lambda_3^{n+3}}{(\lambda_3 - \lambda_2)^2(\lambda_3 - \lambda_4)} + \frac{\lambda_4^{n+3}}{(\lambda_4 - \lambda_2)^2(\lambda_4 - \lambda_3)} \right) \\
&= \frac{\lambda_2^{n+2}((n+3)(\lambda_2 - \lambda_3)(\lambda_2 - \lambda_4) - \lambda_2(2\lambda_2 - \lambda_3 - \lambda_4))}{(\lambda_2 - \lambda_3)^2(\lambda_2 - \lambda_4)^2} \\
&\quad + \frac{\lambda_3^{n+3}}{(\lambda_3 - \lambda_2)^2(\lambda_3 - \lambda_4)} + \frac{\lambda_4^{n+3}}{(\lambda_4 - \lambda_2)^2(\lambda_4 - \lambda_3)}.
\end{aligned} \qquad (3.133)$$

From this it follows that in this case

$$\begin{aligned}
y_n &= \sum_{j=0}^{n-1} \left( \frac{\lambda_2^{j+2}((j+3)(\lambda_2 - \lambda_3)(\lambda_2 - \lambda_4) - \lambda_2(2\lambda_2 - \lambda_3 - \lambda_4))}{(\lambda_2 - \lambda_3)^2(\lambda_2 - \lambda_4)^2} \right. \\
&\quad \left. + \frac{\lambda_3^{j+3}}{(\lambda_3 - \lambda_2)^2(\lambda_3 - \lambda_4)} + \frac{\lambda_4^{j+3}}{(\lambda_4 - \lambda_2)^2(\lambda_4 - \lambda_3)} \right) \\
&= \lambda_2^3 \sum_{j=1}^{n-1} \frac{j\lambda_2^{j-1}}{(\lambda_2 - \lambda_3)(\lambda_2 - \lambda_4)} + \lambda_2^2 \sum_{j=0}^{n-1} \lambda_2^j \frac{\lambda_2^2 - 2\lambda_2\lambda_3 - 2\lambda_2\lambda_4 + 3\lambda_3\lambda_4}{(\lambda_2 - \lambda_3)^2(\lambda_2 - \lambda_4)^2} \\
&\quad + \sum_{j=0}^{n-1} \left( \frac{\lambda_3^{j+3}}{(\lambda_3 - \lambda_2)^2(\lambda_3 - \lambda_4)} + \frac{\lambda_4^{j+3}}{(\lambda_4 - \lambda_2)^2(\lambda_4 - \lambda_3)} \right) \\
&= \frac{\lambda_2^3 - n\lambda_2^{n+2} + (n-1)\lambda_2^{n+3}}{(\lambda_2 - \lambda_3)(\lambda_2 - \lambda_4)(1 - \lambda_2)^2} + \frac{(\lambda_2^4 - 2\lambda_2^3\lambda_3 - 2\lambda_2^3\lambda_4 + 3\lambda_2^2\lambda_3\lambda_4)(\lambda_2^n - 1)}{(\lambda_2 - \lambda_3)^2(\lambda_2 - \lambda_4)^2(\lambda_2 - 1)} \\
&\quad + \frac{\lambda_3^3(\lambda_3^n - 1)}{(\lambda_3 - \lambda_2)^2(\lambda_3 - \lambda_4)(\lambda_3 - 1)} + \frac{\lambda_4^3(\lambda_4^n - 1)}{(\lambda_4 - \lambda_2)^2(\lambda_4 - \lambda_3)(\lambda_4 - 1)}.
\end{aligned} \qquad (3.134)$$

*Case one is a double zero.* We have already mentioned that polynomial (3.106) has a zero equal to one if and only if $(a-1)(c-1) = bd$. Now, if $\lambda = 1$ is a double zero then it must be $p_4'(1) = 4 - 3a - 2c + ac = 0$ or $(a-2)(c-3) = 2$. Since $a, b, c, d \in \mathbb{Z}$, and $bd \neq 0$ this is possible only when $a = 3$, $c = 5$, $bd = 8$ or $a = 4$, $c = 4$, $bd = 9$ or $a = 0$, $c = 2$, $bd = -1$ (case $a = c = 1$ is not possible, since it implies that $bd = 0$).

If $a = 3$, $c = 5$, then

$$p_4(\lambda) = \lambda^4 - 3\lambda^3 - 5\lambda^2 + 15\lambda - 8 = (\lambda - 1)^2(\lambda^2 - \lambda - 8)$$

and it has a real double zero equal to one and two (real) simple zeros.

If $a = 4$, $c = 4$, then

$$p_4(\lambda) = \lambda^4 - 4\lambda^3 - 4\lambda^2 + 16\lambda - 9 = (\lambda - 1)^2(\lambda^2 - 2\lambda - 9)$$

and it also has a real double zero equal to one and two other (real) simple zeros.

Case $a = 0$ and $c = 2$ has been treated in the proof of Theorem 3.1, so it is omitted here.

In this case we have that

$$a_n = \frac{n(1 - \lambda_3)(1 - \lambda_4) + 3\lambda_3\lambda_4 - 2\lambda_3 - 2\lambda_4 + 1}{(1 - \lambda_3)^2(1 - \lambda_4)^2}$$
$$+ \frac{\lambda_3^{n+3}}{(\lambda_3 - 1)^2(\lambda_3 - \lambda_4)} + \frac{\lambda_4^{n+3}}{(\lambda_4 - 1)^2(\lambda_4 - \lambda_3)}, \tag{3.135}$$

and

$$y_n = \sum_{j=0}^{n-1} \left( \frac{j(1 - \lambda_3)(1 - \lambda_4) + 3\lambda_3\lambda_4 - 2\lambda_3 - 2\lambda_4 + 1}{(1 - \lambda_3)^2(1 - \lambda_4)^2} \right.$$
$$\left. + \frac{\lambda_3^{j+3}}{(\lambda_3 - 1)^2(\lambda_3 - \lambda_4)} + \frac{\lambda_4^{j+3}}{(\lambda_4 - 1)^2(\lambda_4 - \lambda_3)} \right)$$
$$= \frac{(n-1)n}{2(1 - \lambda_3)(1 - \lambda_4)} + \frac{n(3\lambda_3\lambda_4 - 2\lambda_3 - 2\lambda_4 + 1)}{(1 - \lambda_3)^2(1 - \lambda_4)^2}$$
$$+ \frac{\lambda_3^3(\lambda_3^n - 1)}{(\lambda_3 - 1)^3(\lambda_3 - \lambda_4)} + \frac{\lambda_4^3(\lambda_4^n - 1)}{(\lambda_4 - 1)^3(\lambda_4 - \lambda_3)}. \tag{3.136}$$

**Corollary 3.6.** *Consider system* (1.2) *with* $a, b, c, d \in \mathbb{Z}$ *and* $bd \neq 0$. *Assume that* $z_{-2}$, $z_{-1}$, $z_0$, $w_{-1}$, $w_0 \in \mathbb{C} \setminus \{0\}$. *Then the following statements are true.*

(a) *If only one of the zeros of characteristic polynomial* (3.106) *is double and different from one, then the general solution to system* (1.2) *is given by formulas* (3.92) *and* (3.105), *where sequence* $(a_n)_{n \geq -3}$ *is given by* (3.133), *while* $(y_n)_{n \geq -3}$ *is given by* (3.134).

(b) *If only double zero of characteristic polynomial* (3.106) *is equal to one, say* $\lambda_1 = \lambda_2 = 1$, *then the general solution to system* (1.2) *is given by formulas* (3.92) *and* (3.105), *where sequence* $(a_n)_{n \geq -3}$ *is given by* (3.135), *while* $(y_n)_{n \geq -3}$ *is given by* (3.136).

**Remark 3.7.** Case one zero is equal to one and there is a double zero different from one seems to be not simple. From above consideration we see that

$$p_4(\lambda) = (\lambda - 1)(\lambda^3 + (1 - a)\lambda^2 + (1 - a - c)\lambda + (a - 1)(c - 1)), \tag{3.137}$$

holds and we should see if the polynomial

$$p_3(\lambda) = \lambda^3 + (1-a)\lambda^2 + (1-a-c)\lambda + (a-1)(c-1)$$

can have a double zero, which is equivalent to the fact that the discriminant $\Delta_3 = 4A^3 + B^2$ is equal to zero and that $A \neq 0 \neq B$, where

$$A = 2 - a - a^2 - 3c \quad \text{and} \quad B = -20 + 15a + 3a^2 + 2a^3 + 18c - 18ac.$$

This is equivalent to

$$(a-1)^2(2a^2 + 5a + 20 - 18c)^2 = 4((a+2)(a-1) + 3c)^3. \tag{3.138}$$

An investigation that we have done along with using computers suggests that equation (3.138) does not have integer solutions such that $a \neq 1 \neq c$. However, we are not able at the moment to show this, so we leave this problem for the further study.

*Case two pairs of double zeroes both different from one.* From Proposition 3.4, we see that in this case it must be $\Delta = D = 0$. The characteristic polynomial (3.106), in this case, has two double zeros, say, $\lambda_1 = \lambda_2$ and $\lambda_3 = \lambda_4$, so the general solution to equation (3.93) has the following form

$$a_n = (\gamma_1 + \gamma_2 n)\lambda_1^n + (\gamma_3 + \gamma_4 n)\lambda_3^n, \quad n \in \mathbb{N}, \tag{3.139}$$

where $\gamma_i$, $i = \overline{1,4}$, are arbitrary constants.

From $D = 0$ we get

$$bd = -\frac{16c^2 + 3a^4}{64}. \tag{3.140}$$

Employing (3.140) in the expressions for $\Delta_0$ and $\Delta_1$, we get

$$\Delta_0 = 4c^2 + 3a^2c + \frac{9}{16}a^4 = \left(2c + \frac{3}{4}a^2\right)^2$$

$$\Delta_1 = \frac{2^{10}c^3 + 11 \cdot 2^4 3^2 a^2 c^2 + 3^3 2^3 ca^4 + 3^4 a^6}{64}.$$

Hence, $\Delta = 0$ is equivalent to the relation

$$(2^{10}c^3 + 11 \cdot 2^4 3^2 a^2 c^2 + 3^3 2^3 ca^4 + 3^4 a^6)^2 = 4(2^3 c + 3a^2)^6,$$

from which it follows that

$$2^{10}c^3 + 11 \cdot 2^4 3^2 a^2 c^2 + 3^3 2^3 ca^4 + 3^4 a^6 = \pm 2(2^3 c + 3a^2)^3. \tag{3.141}$$

By some calculation from (3.141) we get that it must be

$$a^2(4c - a^2)^2 = 0, \tag{3.142}$$

or

$$2^{11}c^3 + 3^2 2^4 19 a^2 c^2 + 3^4 2^3 ca^4 + 5 \cdot 3^3 a^6 = 0. \tag{3.143}$$

*Subcase when (3.142) holds.* If $a = 0$ and $c \neq 0$, then $bd = -c^2/4$. Hence

$$p_4(\lambda) = \lambda^4 - c\lambda^2 + \frac{c^2}{4} = \left(\lambda^2 - \frac{c}{2}\right)^2.$$

If $c > 0$, then clearly

$$\lambda_{1,2} = \sqrt{c/2} \quad \text{and} \quad \lambda_{3,4} = -\sqrt{c/2},$$

while if $c < 0$, then

$$\lambda_{1,2} = i\sqrt{-c/2} \quad \text{and} \quad \lambda_{3,4} = -i\sqrt{-c/2}.$$

If $4c = a^2 \neq 0$, then if $t = \lambda/a$, we have

$$
\begin{aligned}
p_4(\lambda) &= \lambda^4 - a\lambda^3 - \frac{a^2}{4}\lambda^2 + \frac{a^3}{4}\lambda + \frac{a^4}{16} \\
&= a^4\left(t^4 - t^3 - \frac{t^2}{4} + \frac{t}{4} + \frac{1}{16}\right) \\
&= a^4 t^2\left(\left(t - \frac{1}{4t}\right)^2 - \left(t - \frac{1}{4t}\right) + \frac{1}{4}\right) \\
&= a^4 t^2\left(t - \frac{1}{4t} - \frac{1}{2}\right)^2 \\
&= a^4\left(t^2 - \frac{1}{2}t - \frac{1}{4}\right)^2 \\
&= \left(\lambda^2 - \frac{a\lambda}{2} - \frac{a^2}{4}\right)^2.
\end{aligned}
\tag{3.144}
$$

From (3.144) we get that

$$
\lambda_{1,2} = a\frac{1 + \sqrt{5}}{4} \quad \text{and} \quad \lambda_{3,4} = a\frac{1 - \sqrt{5}}{4}.
$$

In these cases we have that

$$
\begin{aligned}
a_n &= \frac{\lambda_2^{n+2}(n(\lambda_2 - \lambda_4)^2 + \lambda_2^2 - 4\lambda_2\lambda_4 + 3\lambda_4^2)}{(\lambda_2 - \lambda_4)^4} \\
&\quad + \frac{\lambda_4^{n+2}(n(\lambda_4 - \lambda_2)^2 + \lambda_4^2 - 4\lambda_2\lambda_4 + 3\lambda_2^2)}{(\lambda_4 - \lambda_2)^4}
\end{aligned}
\tag{3.145}
$$

and

$$
\begin{aligned}
y_n &= \sum_{j=0}^{n-1}\left(\frac{\lambda_2^{j+2}(j(\lambda_2 - \lambda_4)^2 + \lambda_2^2 - 4\lambda_2\lambda_4 + 3\lambda_4^2)}{(\lambda_2 - \lambda_4)^4}\right. \\
&\quad \left. + \frac{\lambda_4^{j+2}(j(\lambda_4 - \lambda_2)^2 + \lambda_4^2 - 4\lambda_2\lambda_4 + 3\lambda_2^2)}{(\lambda_4 - \lambda_2)^4}\right) \\
&= \lambda_2^3\sum_{j=1}^{n-1}\frac{j\lambda_2^{j-1}}{(\lambda_2 - \lambda_4)^2} + \lambda_2^2\sum_{j=0}^{n-1}\lambda_2^j\frac{\lambda_2^2 - 4\lambda_2\lambda_4 + 3\lambda_4^2}{(\lambda_2 - \lambda_4)^4} \\
&\quad + \lambda_4^3\sum_{j=1}^{n-1}\frac{j\lambda_4^{j-1}}{(\lambda_4 - \lambda_2)^2} + \lambda_4^2\sum_{j=0}^{n-1}\lambda_4^j\frac{\lambda_4^2 - 4\lambda_2\lambda_4 + 3\lambda_2^2}{(\lambda_4 - \lambda_2)^4} \\
&= \frac{\lambda_2^3 - n\lambda_2^{n+2} + (n-1)\lambda_2^{n+3}}{(\lambda_2 - \lambda_4)^2(1 - \lambda_2)^2} + \frac{(\lambda_2^4 - 4\lambda_2^3\lambda_4 + 3\lambda_2^2\lambda_4^2)(\lambda_2^n - 1)}{(\lambda_2 - \lambda_4)^4(\lambda_2 - 1)} \\
&\quad + \frac{\lambda_4^3 - n\lambda_4^{n+2} + (n-1)\lambda_4^{n+3}}{(\lambda_4 - \lambda_2)^2(1 - \lambda_4)^2} + \frac{(\lambda_4^4 - 4\lambda_2\lambda_4^3 + 3\lambda_2^2\lambda_4^2)(\lambda_4^n - 1)}{(\lambda_4 - \lambda_2)^4(\lambda_4 - 1)}.
\end{aligned}
\tag{3.146}
$$

*Subcase when* (3.143) *holds.* There are two possibilities that the relation in (3.143) holds. First, if it were $a = 0$ or $c = 0$ in equation (3.143), then we would get $a = c = 0$ and

consequently by (3.140) it would follow that $bd = 0$, which would be a contradiction with the assumption $bd \neq 0$. Hence, the subcase is not possible.

If $a \neq 0 \neq c$ in (3.143), then since the polynomial

$$q_3(t) = 2^{11}t^3 + 3^2 2^4 19 t^2 + 3^4 2^3 t + 5 \cdot 3^3,$$

has obviously a real zero, say $t_0$ (it is shown that $t_0 \approx -1, 10331$), we get that $c = t_0 a^2$, and consequently $P = -(8t_0 + 3)a^2 > 0$, $bd = -(16t_0^2 + 3)a^4/64 \neq 0$, $Q = a^3(4t_0 - 1) \neq 0$, and

$$p_4(\lambda) = \lambda^4 - a\lambda^3 - t_0 a^2 \lambda^2 + t_0 a^3 \lambda - \frac{16t_0^2 + 3}{64}a^4. \tag{3.147}$$

From Proposition 3.4 we see that conditions $P > 0$ and $Q \neq 0$ cannot hold simultaneously with conditions $\Delta = 0$ and $D = 0$, which guarantee the existence of a double zero. Hence, it is not possible in this case that polynomial (3.147) has two double zeros.

*Case two pairs od double zeroes, one of them equal to one.* The characteristic polynomial (3.106), in this case, has two double zeros, say, $\lambda_1 = \lambda_2 \neq 1$ and $\lambda_3 = \lambda_4 = 1$, so the general solution to equation (3.93) has the following form

$$a_n = (\widehat{\gamma}_1 + \widehat{\gamma}_2 n)\lambda_2^n + (\widehat{\gamma}_3 + \widehat{\gamma}_4 n), \quad n \in \mathbb{N}, \tag{3.148}$$

where $\widehat{\gamma}_i$, $i = \overline{1,4}$, are arbitrary constants.

If $a = 0$, $c = 2$, $bd = -1$, then

$$p_4(\lambda) = \lambda^4 - 2\lambda^2 + 1 = (\lambda - 1)^2(\lambda + 1)^2,$$

from which it follows that the polynomial has a real double zero equal to 1 and another real double zero equal to $-1$.

In this case we have that

$$a_n = \frac{\lambda_2^{n+2}(n(\lambda_2 - 1)^2 + \lambda_2^2 - 4\lambda_2 + 3)}{(\lambda_2 - 1)^4} + \frac{(n(\lambda_2 - 1)^2 + 1 - 4\lambda_2 + 3\lambda_2^2)}{(\lambda_2 - 1)^4} \tag{3.149}$$

and

$$
\begin{aligned}
y_n &= \sum_{j=0}^{n-1}\left(\frac{\lambda_2^{j+2}(j(\lambda_2 - 1)^2 + \lambda_2^2 - 4\lambda_2 + 3)}{(\lambda_2 - 1)^4} + \frac{(j(\lambda_2 - 1)^2 + 1 - 4\lambda_2 + 3\lambda_2^2)}{(\lambda_2 - 1)^4}\right) \\
&= \frac{\lambda_2^3}{(\lambda_2 - 1)^2}\sum_{j=1}^{n-1} j\lambda_2^{j-1} + \frac{(\lambda_2^3 - 3\lambda_2^2)}{(\lambda_2 - 1)^3}\sum_{j=0}^{n-1}\lambda_2^j + \frac{1}{(\lambda_2 - 1)^2}\sum_{j=0}^{n-1}j + \sum_{j=0}^{n-1}\frac{3\lambda_2 - 1}{(\lambda_2 - 1)^3} \\
&= \frac{\lambda_2^3 - n\lambda_2^{n+2} + (n-1)\lambda_2^{n+3}}{(\lambda_2 - 1)^2} + \frac{(\lambda_2^3 - 3\lambda_2^2)(\lambda_2^n - 1)}{(\lambda_2 - 1)^4} + \frac{(n-1)n}{2(\lambda_2 - 1)^2} + n\frac{3\lambda_2 - 1}{(\lambda_2 - 1)^3} \tag{3.150}
\end{aligned}
$$

**Corollary 3.8.** *Consider system* (1.2) *with* $a, b, c, d \in \mathbb{Z}$ *and* $bd \neq 0$. *Assume* $z_{-2}, z_{-1}, z_0, w_{-1}, w_0 \in \mathbb{C} \setminus \{0\}$. *Then the following statements are true.*

(a) *If characteristic polynomial* (3.106) *has two pairs of double zeros both different from one, then the general solution to system* (1.2) *is given by formulas* (3.92) *and* (3.105), *where sequence* $(a_n)_{n \geq -3}$ *is given by* (3.145), *while* $(y_n)_{n \geq -3}$ *is given by* (3.146).

(b) *If characteristic polynomial* (3.106) *has two pairs of double zeros one of them equal to one, say* $\lambda_1$ *and* $\lambda_2$, *then the general solution to system* (1.2) *is given by formulas* (3.92) *and* (3.105), *where sequence* $(a_n)_{n \geq -3}$ *is given by* (3.149), *while* $(y_n)_{n \geq -3}$ *is given by* (3.150).

*Case at least three zeros are equal.* If three zeros of polynomial (3.106) are equal, then it must be $\Delta = \Delta_0 = 0$, which implies $\Delta_1 = 0$. The characteristic polynomial in (3.106) would have four equal zeros if $p_4(\lambda) = p_4'(\lambda) = p_4''(\lambda) = p_4'''(\lambda) = 0$. Since $p_4'''(\lambda) = 24\lambda - 6a$, we would get $\lambda = a/4$. From $p_4'(\frac{a}{4}) = p_4''(\frac{a}{4}) = 0$ it is obtained

$$p_4'\left(\frac{a}{4}\right) = \frac{a(4c - a^2)}{8} = 0 \quad \text{and} \quad p_4''\left(\frac{a}{4}\right) = -\frac{8c + 3a^2}{4} = 0,$$

from which it follows that if $a = 0$ then $c = 0$, while if $4c = a^2$ then $5a^2 = 0$, which implies $a = 0$ and consequently $c = 0$. Hence, in both cases we have that $a = c = 0$, which implies that

$$p_4(\lambda) = \lambda^4 - bd.$$

However, since $bd \neq 0$ polynomial $p_4$ would have four different zeros, which would be a contradiction. Thus, the polynomial (3.106) has at most three equal zeros.

Since $\Delta_0 = 0$ we have that

$$bd = \frac{c^2 + 3a^2c}{12}. \tag{3.151}$$

Employing (3.151) in $\Delta_1 = 0$ we get

$$\begin{aligned}\Delta_1 &= -2c^3 + 18a^2c^2 - \frac{(27a^2 + 72c)(c^2 + 3a^2c)}{12} \\ &= -\frac{c}{4}(32c^2 + 9ca^2 + 27a^4) = 0.\end{aligned} \tag{3.152}$$

If it were $c = 0$, then from (3.151) we would get $bd = 0$, which would be a contradiction. If $c \neq 0$ and $32c^2 + 9ca^2 + 27a^4 = 0$, then since the polynomial $32 + 9t + 27t^2$ is always positive on $\mathbb{R}$ we obtain that the last equation does not have a real solution. So, the case $\Delta = \Delta_0 = 0$ is not possible, which implies that polynomial (3.106) cannot have a triple zero.

Hence, the general solution to equation (3.93) cannot have the following forms

$$\begin{aligned} a_n &= (\delta_1 + \delta_2 n + \delta_3 n^2 + \delta_3 n^2)\lambda_1^n, \\ a_n &= \widehat{\delta}_1\lambda_1^n + (\widehat{\delta}_2 + \widehat{\delta}_3 n + \widehat{\delta}_4 n^2)\lambda_2^n, \quad n \in \mathbb{N}, \end{aligned} \tag{3.153}$$

where $\delta_i$ and $\widehat{\delta}_i$, $i = \overline{1,4}$, are arbitrary constants.

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