Electronic Journal of Qualitative Theory of Differential Equations
2011, No. **19**, 1-15; **http://www.math.u-szeged.hu/ejqtde/**

Necessary and Sufficient Conditions for the Oscillation of Differential Equations Involving Distributed Arguments

Julio G. Dix[*], Başak Karpuz[†‡] and Radhanath Rath[§]

March 9, 2011

Abstract

In this article, we establish necessary and sufficient conditions for the oscillation of both bounded and unbounded solutions of the differential equation

$$\left[x(t)+\int_0^\lambda p(t,v)x(\tau(t,v))\,\mathrm{d}v\right]^{(n)}+\int_0^\lambda q(t,v)x(\sigma(t,v))\,\mathrm{d}v=\varphi(t)\quad\text{for } t\geq t_0,$$

where $n\in\mathbb{N}$, $t_0,\lambda\in\mathbb{R}^+$, $p\in C([t_0,\infty)\times[0,\lambda],\mathbb{R})$, $q\in C([t_0,\infty)\times[0,\lambda],\mathbb{R}^+)$, $\tau\in C([t_0,\infty)\times[0,\lambda],\mathbb{R})$ with $\lim_{t\to\infty}\inf_{v\in[0,\lambda]}\tau(t,v)=\infty$ and $\sup_{v\in[0,\lambda]}\tau(t,v)\leq t$ for all $t\geq t_0$, $\sigma\in C([t_0,\infty)\times[0,\lambda],\mathbb{R})$ with $\lim_{t\to\infty}\inf_{v\in[0,\lambda]}\sigma(t,v)=\infty$, and $\varphi\in C([t_0,\infty),\mathbb{R})$. We also give illustrating examples to show the applicability of these results.

1 Introduction

In the past two decades, there have been important developments in the theory of oscillation for neutral differential equations in which the higher-order derivative of the

[*]**Address**: Texas State University, Department of Mathematics, 601 University Drive, San Marcos, TX 78666, USA. **Email**: jd01@txstate.edu

[†]**Address**: A. N. S. Campus, Afyon Kocatepe University, Faculty of Science and Arts, Department of Mathematics, 03200 Afyonkarahisar, Turkey. **Email**: bkarpuz@gmail.com, **URL**: `http://www2.aku.edu.tr/~bkarpuz`

[‡]Corresponding author

[§]**Address**: Veer Surendra Sai University of Technology, Department of Mathematics, Burla, (Formerly UCE Burla) Sambalpur, 768018 Orissa, India. **Email**: radhanathmath@yahoo.co.in

unknown involves both its present and its past states. This type of equations appear in the mathematical modeling of real life problems; for example in economics, where the demand depends on current price and the supply depends on the price at an earlier time, and in the study of lossless communication channels systems.

In this article, we study the asymptotic behaviour of differential equations of the form

$$\Big[x(t)+\int_0^\lambda p(t,v)x(\tau(t,v))\,\mathrm{d}v\Big]^{(n)}+\int_0^\lambda q(t,v)x(\sigma(t,v))\,\mathrm{d}v=\varphi(t)\quad\text{for } t\geq t_0, \tag{1}$$

where $n\in\mathbb{N}$, $t_0,\lambda\in\mathbb{R}^+$, $p\in C([t_0,\infty)\times[0,\lambda],\mathbb{R})$, $q\in C([t_0,\infty)\times[0,\lambda],\mathbb{R}^+)$, $\tau\in C([t_0,\infty)\times[0,\lambda],\mathbb{R})$ with $\lim_{t\to\infty}\tau(t,v)=\infty$ and $\tau(t,v)\leq t$ for all $t\geq t_0$ and all $v\in[0,\lambda]$, $\sigma\in C([t_0,\infty)\times[0,\lambda],\mathbb{R})$ with $\lim_{t\to\infty}\sigma(t,v)=\infty$ for all $v\in[0,\lambda]$. We will use the following assumptions:

(A1) $\liminf_{t\to\infty}\big[\sigma(t,v)/t\big]>0$ for each $v\in[0,\lambda]$;

(A2) $\int_{t_0}^\infty u^{n-1}\int_0^\lambda q(u,v)\,\mathrm{d}v\mathrm{d}u=\infty$;

(A3) $\int_{t_0}^\infty u^{n-2}\int_0^\lambda q(u,v)\,\mathrm{d}v\mathrm{d}u=\infty$;

(A4) there exists a function $\Phi\in C^n([t_0,\infty),\mathbb{R})$ such that $\Phi^{(n)}=\varphi$ on $t_0,\infty)$ and $\lim_{t\to\infty}\Phi(t)=0$;

(A5) there exists a bounded function $\Phi\in C^n([t_0,\infty),\mathbb{R})$ such that $\Phi^{(n)}=\varphi$ on $t_0,\infty)$;

(C1) $\limsup_{t\to\infty}\int_0^\lambda p^+(t,v)\mathrm{d}v+\limsup_{t\to\infty}\int_0^\lambda p^-(t,v)\,\mathrm{d}v<1$, where $p^+(t,v):=\max\{0,p(t,v)\}$ and $p^-(t,v):=\max\{0,-p(t,v)\}$ for $t\geq t_0$ and $v\in[0,\lambda]$. Note that $p\equiv p^+-p^-$, $|p|\equiv p^++p^-$, and $-p^-\leq p\leq p^+$.

When $\tau(t,v)$ or $\sigma(t,v)$ are independent of v, Equation (1) includes the following two cases

$$\big[x(t)+p(t)x(\tau(t))\big]^{(n)}+q(t)x(\sigma(t))=\varphi(t)\quad\text{for } t\geq t_0, \tag{2}$$

and

$$\big[x(t)+p(t)x(\tau(t))\big]^{(n)}+\int_0^\lambda q(t,v)x(\sigma(t,v))\,\mathrm{d}v=\varphi(t)\quad\text{for } t\geq t_0; \tag{3}$$

that have been studied by many authors; see for example [2, 11, 12, 13, 15] and their references. However, in all the mentioned papers above except for [2], the authors study (2) only when p is eventually of fixed sign. The reference [2] is one of the rare studies that allow p to oscillate. Therein it is assumed that $\lim_{t\to\infty}p(t)=0$ (this indicates that every bounded nonoscillatory solution of (2) has a finite limit at infinity) and results only cover bounded solutions. Our objective is to generalize and to improve the results of these references. In particular, we allow p to alternate in sign infinitely many times without assuming that $\lim_{t\to\infty} p(t) = 0$. Note that if $\lim_{t\to\infty} p(t) = 0$, then p satisfies (C1); therefore our results also include those in [2].

Set $t_{-1} := \inf_{t\in[t_0,\infty)} \big\{ \inf_{v\in[0,\lambda]}\{\tau(t,v), \sigma(t,v)\}\big\}$. By a *solution* to (1), we mean a function $x \in C([t_{-1},\infty),\mathbb{R})$ such that $x + \int_0^\lambda p(\cdot,v)x(\tau(\cdot,v))\,\mathrm{d}v \in C^n([t_0,\infty),\mathbb{R})$, and (1) is satisfied on $[t_0,\infty)$. A solution to (1) is called *oscillatory* if it has arbitrary large zeros, otherwise it is called *nonoscillatory*.

Throughout this paper, we shall restrict our attention to those solutions of (1) which do not vanish on any sub half-line of $[t_0,\infty)$. In the proofs, we consider only eventually positive solutions as non-oscillatory solutions; because if x is an eventually negative solution of (1), then $-x$ is eventually positive and satisfies (1) with $-\varphi$ instead of φ. It is clear that if φ satisfies (A4) or (A5), then so does $-\varphi$.

This paper is organized as follows: in § 2, we state two important theorems, which will be needed in the sequel; in § 3, our main results on the asymptotic behaviour of bounded and unbounded solutions to (1) are given together with some corollaries, remarks and simple examples. Finally, in § 4, we compare our results with the existing results in the literature, and give a short account on how to extend them to nonlinear equations.

2 Auxiliary Lemmas

Below, we state the well-known Kiguradze's lemma, which is one of the most useful tools in the study of asymptotic behaviour of solutions.

Kiguradze's lemma ([9, Lemma 2]). *Let $f \in C^n([t_0,\infty),\mathbb{R})$ be a function of fixed sign such that $f^{(n)}$ is of fixed sign and not identically zero on any sub half-line of $[t_0,\infty)$. Then, there exists $m \in \mathbb{Z}$ and $t_1 \in [t_0,\infty)$ such that $0 \le m \le n-1$, and $(-1)^{n+m} f(t)f^{(n)}(t) \ge 0$,*

$$f(t)f^{(j)}(t) > 0 \quad \text{for } j = 0,1,\dots,m-1 \text{ when } m \ge 1$$

and

$$(-1)^{m+j} f(t)f^{(j)}(t) > 0 \quad \text{for } j = m,m+1,\dots,n-1 \text{ when } m \le n-1$$

for all $t \in [t_1,\infty)$.

Now, we state the well-known Krasnoselskii's fixed point theorem, which will be employed in proving the existence of a nonoscillatory solution.

Krasnoselskii's fixed point theorem ([7, Theorem 1.4.5]). *Let A be a bounded, convex and closed subset of the Banach space B. Suppose further that there exist two operators $\Gamma, \Psi : A \to B$ such that*

(i) $\Gamma x + \Psi y \in A$ *holds for all* $x, y \in A$,

(ii) Γ *is a contraction mapping,*

(iii) Ψ *is completely continuous.*

Then, $\Gamma + \Psi$ *has a fixed point in* A*; i.e., there exists* $x \in A$ *such that* $(\Gamma + \Psi)x = x$.

3 Main Results

We start this section with a result on bounded solutions of (1).

Theorem 3.1. *Under the assumptions (A2), (A4) and (C1), every bounded solution of* (1) *oscillates or tends to zero asymptotically.*

Proof. Let x be an eventually positive solution of (1). To prove this theorem, we show that x tends to zero at infinity. We pick $t_1 \geq t_0$ such that $x(t), x(\tau(t,v)), x(\sigma(t,v)) > 0$ for all $t \geq t_1$ and all $v \in [0, \lambda]$. Set

$$y(t) := x(t) + \int_0^\lambda p(t,v)x(\tau(t,v))\,\mathrm{d}v \quad \text{and} \quad z(t) := y(t) - \Phi(t) \tag{4}$$

for $t \geq t_1$. Then

$$z^{(n)}(t) = -\int_0^\lambda q(t,v)x(\sigma(t,v))\,\mathrm{d}v \leq 0 \quad \text{for } t \geq t_1. \tag{5}$$

This ensures the existence of $t_2 \geq t_1$ such that $z^{(i)}$ is monotonic on $[t_2, \infty)$ for $i = 1, 2, \ldots, n-1$. Note that the property that the solution x is bounded, (C1) and (A4) imply that the functions y and z are bounded. Therefore, z is monotonic and bounded which implies the existence of $\lim_{t\to\infty} z(t)$ as finite constant, and that $\lim_{t\to\infty} z^{(i)}(t) = 0$ for $i = 1, 2, \ldots, n-1$. ¿From (4) and (A4), we see that $\lim_{t\to\infty} y(t)$ exists and is a finite constant. Integrating (5) from t to ∞ repeatedly for a total of $(n-1)$ times, we obtain

$$z'(t) = \frac{(-1)^{n-2}}{(n-2)!}\int_t^\infty (u-t)^{n-2}\int_0^\lambda q(u,v)x(\sigma(u,v))\,\mathrm{d}v\mathrm{d}u$$

for $t \geq t_2$. Then integrating the resulting equation from t_2 to ∞, yields

$$\infty > z(t_2) - \lim_{t\to\infty} z(t) = \frac{(-1)^{n-1}}{(n-1)!}\int_{t_2}^\infty (u-t_2)^{n-1}\int_0^\lambda q(u,v)x(\sigma(u,v))\,\mathrm{d}v\mathrm{d}u,$$

which indicates that

$$\liminf_{t\to\infty} x(t) = 0.$$

Using the property that the solution x is bounded, define $\ell := \limsup_{t\to\infty} x(t)$. We pick two increasing divergent sequences $\{\xi_k\}_{k\in\mathbb{N}}, \{\zeta_k\}_{k\in\mathbb{N}} \subset [t_2, \infty)$ such that $x(\xi_k) \to 0$ and $x(\zeta_k) \to \ell$ as $k \to \infty$. Clearly, we have $\limsup_{k\to\infty} x(\tau(\xi_k, v)) \le \ell$ and $\limsup_{k\to\infty} x(\tau(\zeta_k, v)) \le \ell$ for any $v \in [0, \lambda]$. Thus, for any $k \in \mathbb{N}$, we obtain

$$\begin{aligned} y(\xi_k) - y(\zeta_k) =& x(\xi_k) + \int_0^\lambda p(\xi_k, v)x(\tau(\xi_k, v))\,\mathrm{d}v - \Big(x(\zeta_k) + \int_0^\lambda p(\zeta_k, v)x(\tau(\zeta_k, v))\mathrm{d}v\Big) \\ \le& x(\xi_k) + \int_0^\lambda p^+(\xi_k, v)x(\tau(\xi_k, v))\,\mathrm{d}v - x(\zeta_k) + \int_0^\lambda p^-(\zeta_k, v)x(\tau(\zeta_k, v))\,\mathrm{d}v. \end{aligned}$$

Letting k tend to infinity, we have

$$\begin{aligned} 0 \le& \left(\limsup_{k\to\infty} \int_0^\lambda p^+(\xi_k, v)\,\mathrm{d}v + \limsup_{k\to\infty} \int_0^\lambda p^-(\xi_k, v)\mathrm{d}v - 1 \right)\ell \\ \le& \left(\limsup_{t\to\infty} \int_0^\lambda p^+(t, v)\,\mathrm{d}v + \limsup_{t\to\infty} \int_0^\lambda p^-(t, v)\,\mathrm{d}v - 1 \right)\ell. \end{aligned}$$

By (C1) the expression in parentheses is negative, so ℓ must be zero, and $\lim_{t\to\infty} x(t) = 0$. The proof is complete. □

Now, we have the following examples.

Example 3.1. Consider the sixth-order equation

$$\left[x(t) + \int_0^{2\pi} \frac{\sin(t-v)}{6} x(t-v)\,\mathrm{d}v\right]^{(6)} + \int_0^{2\pi} \frac{1}{4} x(t - (v + 3\pi)/2)\,\mathrm{d}v = 0 \quad \text{for } t \ge 0. \tag{6}$$

We see that $\lambda = 2\pi$, $n = 6$, $p(t, v) = \sin(t - v)/6$, $\tau(t, v) = t - v$, $q(t, v) = 1/4$, $\sigma(t, v) = t - (v + 3\pi)/2$ and $\varphi(t) = 0$ for $t \ge 0$ and $v \in [0, \pi]$. Let $\Phi(t) \equiv 0$ for $t \ge 0$. It is easy to see that (A2) and (A4) hold. On the other hand,

$$\limsup_{t\to\infty} \int_0^{2\pi} \left[\frac{\sin(t-v)}{6}\right]^- \mathrm{d}v = \limsup_{t\to\infty} \left[-\frac{\cos(t)}{3}\right]^- = \frac{1}{3},$$

and

$$\limsup_{t\to\infty} \int_0^{2\pi} \left[\frac{\sin(t-v)}{6}\right]^+ \mathrm{d}v = \frac{1}{3}.$$

Therefore, (C1) holds. Theorem 3.1 ensures that every bounded solution of (6) oscillates or tends to zero at infinity. It can be easily shown that $x(t) = \cos(t)$ for $t \ge 0$ is an oscillating bounded solution of (6).

Example 3.2. Consider the third-order equation:

$$\left[x(t)+\int_0^1 \frac{vt^2}{2t^2+1}x(t(v+1)/2)\,\mathrm{d}v\right]^{(3)}+\int_0^1 \frac{v}{t^3}x(t(v+1)/2)\,\mathrm{d}v=\frac{1}{t^5}\quad \text{for } t\geq 1. \tag{7}$$

We see that $\lambda = 1$, $n = 3$, $p(t,v) = vt^2/(2t^2+1)$, $\tau(t,v) = t(v+1)/2$, $q(t,v) = v/t^3$, $\sigma(t,v) = t(v+1)/2$ and $\varphi(t) = 1/t^5$ for $t \geq 1$ and $v \in [0,1]$. All the conditions of Theorem 3.1 hold, and thus every bounded solution of (7) oscillates or tends to zero at infinity.

The following corollary drops the assumption that the solutions in Theorem 3.1 are bounded, when $n = 1$.

Corollary 3.1. *Assume (A2), (A4), (C1) and $n = 1$. Then, every solution of* (1) *oscillates or tends to zero at infinity.*

Proof. It suffices to show that every eventually positive solution of (1) is bounded. Suppose that x is an eventually positive unbounded solution. Then for some $t_1 \geq t_0$, we have $x(t), x(\tau(t,v)), x(\sigma(t,v)) > 0$ for all $t \geq t_1$ and all $v \in [0,\lambda]$. Set y, z as in (4), then we have (5) on $[t_1,\infty)$. Since $n = 1$, $z'(t) \leq 0$ and z is non-increasing on $[t_1,\infty)$. Since x is unbounded, there exists an increasing divergent sequence $\{\varsigma_k\}_{k\in\mathbb{N}} \subset [t_1,\infty)$ such that $\{x(\varsigma_k)\}_{k\in\mathbb{N}}$ is unbounded and $x(\varsigma_k) = \max\{x(t) : t \in [t_1,\varsigma_k]\}$ for all $k \in \mathbb{N}$. From (4) and using that $\tau(\varsigma_k,v) \leq \varsigma_k$, we obtain

$$\begin{aligned} z(\varsigma_k) =& x(\varsigma_k)+\int_0^\lambda p(\varsigma_k,v)x(\tau(\varsigma_k,v))\,\mathrm{d}v-\Phi(\varsigma_k)\\ \geq& \left(1-\int_0^\lambda p^-(\varsigma_k,v)\,\mathrm{d}v\right)x(\varsigma_k)-\Phi(\varsigma_k). \end{aligned}$$

In view of (A4) and (C1), we obtain that $\lim_{t\to\infty} z(t) = \infty$. This contradicts the nonincreasing nature of z; thus every eventually positive solution of (1) (with $n = 1$) is bounded. Hence, the proof is complete. □

Remark 3.1. The claim of Corollary 3.1 can be stated for unbounded solutions by removing (A2) and replacing (A4) with (A5); i.e., every unbounded solution of (1) with $n = 1$ oscillates provided that (A5) and (C1) hold.

The following example illustrates the result in the above remark.

Example 3.3. Consider the first-order equation

$$\begin{aligned} \left[x(t)+\int_0^\pi \frac{\cos(t-v)}{6}x(t-v)\,\mathrm{d}v\right]'+\int_0^\pi \frac{1}{2}x(t-v)\,\mathrm{d}v \\ = -2\sin(t)-\frac{\pi}{3}\sin(2t)-\frac{\pi}{2}\cos(t)\quad \text{for } t\geq 0. \end{aligned} \tag{8}$$

We see that $\lambda = \pi$, $n = 6$, $p(t, v) = \cos(t - v)/6$, $\tau(t, v) = t - v$, $q(t, v) = 1/2$, $\sigma(t, v) = t - v$ and $\varphi(t) = -2\sin(t) - \pi\sin(2t)/3 - \pi\cos(t)/2$ for $t \geq 0$ and $v \in [0, \pi]$. Then, we may let $\Phi(t) = 2\cos(t) + \pi\cos(2t)/12 - \pi\sin(t)/2$ for $t \geq 0$. It is clear that (A5) and (C1) hold. Due to Remark 3.1, every unbounded solution of (8) oscillates, and $x(t) = t\sin(t)$ for $t \geq 0$ is an oscillating unbounded solution of (8).

For even-order homogeneous equations, we can give the following oscillation result.

Corollary 3.2. *Assume (A2), (A4), (C1), n is even, $\phi \equiv 0$, and p is eventually nonnegative. Then, every bounded solution of* (1) *oscillates.*

Proof. Let x be an eventually positive solution of (1). Set y, z as in (4), so that $y \equiv z$ which are positive functions. Since n is even, it follows from Kiguradze's lemma that the eventually positive function y is increasing, which indicates $\lim_{t\to\infty} y(t) > 0$. Using the result of Theorem 3.1, we see that $\lim_{t\to\infty} y(t) = 0$ because of $\lim_{t\to\infty} x(t) = 0$ and (C1). This contradiction completes the proof. □

Next, we state an important oscillation result on unbounded solutions of (1).

Theorem 3.2. *Assume $n \geq 2$, (A1), (A3), (A5) and (C1). Then every unbounded solution of* (1) *oscillates.*

Proof. Assume on the contrary that x is an unbounded solution which is eventually positive. There exists $t_1 \geq t_0$ such that $x(t), x(\tau(t, v)), x(\sigma(t, v)) > 0$ for all $t \geq t_1$ and all $v \in [0, \lambda]$. Set y, z as in (4), thus we have (5) on $[t_1, \infty)$. Now, we prove that $\lim_{t\to\infty} z(t) = \infty$. Clearly, since x is unbounded, we can find an increasing divergent sequence $\{\varsigma_k\}_{k\in\mathbb{N}} \subset [t_1, \infty)$ such that $\{x(\varsigma_k)\}_{k\in\mathbb{N}}$ is unbounded and $x(\varsigma_k) = \max\{x(t) : t \in [t_1, \varsigma_k]\}$ for all $k \in \mathbb{N}$. By a reasoning as in the proof of Corollary 3.1, we see that $z(\varsigma_k) \to \infty$ as $k \to \infty$ since x is unbounded, (A5) and (C1). Hence, $\lim_{t\to\infty} z(t) = \infty$, and from (4) and (A5), we get

$$\lim_{t\to\infty} \frac{y(t)}{z(t)} = 1. \tag{9}$$

Since z is increasing, from (C1), we have

$$\limsup_{t\to\infty} \int_0^\lambda \frac{p^+(t, v)z(\tau(t, v))}{z(t)}\mathrm{d}v + \limsup_{t\to\infty} \int_0^\lambda \frac{p^-(t, v)z(\tau(t, v))}{z(t)}\mathrm{d}v < 1. \tag{10}$$

We now prove that $\ell := \limsup_{t\to\infty} \left[x(t)/z(t)\right]$ is a finite constant. Otherwise, there exists an increasing divergent sequence $\{\varsigma_k\}_{k\in\mathbb{N}} \subset [t_1, \infty)$ such that $\{x(\varsigma_k)/z(\varsigma_k)\}_{k\in\mathbb{N}}$ is unbounded and $x(\varsigma_k)/z(\varsigma_k) = \max\{x(t)/z(t) : t \in [t_1, \varsigma_k]\}$ for all $k \in \mathbb{N}$. It follows from (4), (9) and (10) that

$$\begin{aligned} \frac{y(\varsigma_k)}{z(\varsigma_k)} \geq & \frac{x(\varsigma_k)}{z(\varsigma_k)} - \int_0^\lambda p^-(\varsigma_k, v) \frac{x(\tau(\varsigma_k, v))}{z(\varsigma_k)} \mathrm{d}v \\ = & \frac{x(\varsigma_k)}{z(\varsigma_k)} - \int_0^\lambda \frac{p^-(\varsigma_k, v) z(\tau(\varsigma_k, v))}{z(\varsigma_k)} \frac{x(\tau(\varsigma_k, v))}{z(\tau(\varsigma_k, v))} \mathrm{d}v \\ \geq & \left(1 - \int_0^\lambda \frac{p^-(\varsigma_k, v) z(\tau(\varsigma_k, v))}{z(\varsigma_k)} \mathrm{d}v\right) \frac{x(\varsigma_k)}{z(\varsigma_k)}. \end{aligned}$$

In the limit as $k \to \infty$, the left-hand side approaches 1 while the right-hand side approaches $+\infty$. This contradiction implies that ℓ is a finite constant.

Recalling that z is eventually positive and increasing, and applying Kiguradze's lemma, we see that there exists $m \geq 1$ such that

$$z^{(j)}(t) > 0 \quad \text{for } j = 0, 1, \dots, m-1 \tag{11}$$

and

$$(-1)^{m+j} z^{(j)}(t) > 0 \quad \text{for } j = m, m+1, \dots, n-1, \text{ when } m \leq n-1 \tag{12}$$

for some $t_2 \geq t_1$ and all $t \geq t_2$. Therefore, $z^{(m)}$ is positive and decreasing on $[t_2, \infty)$ by (12). This implies that $\lim_{t\to\infty} z^{(m)}(t)$ exists and is finite and $\lim_{t\to\infty} z^{(i)}(t) = 0$ for $i = m+1, m+2, \dots, n-1$. Integrating (5) from t to ∞ for a total of $(n-m-2)$ times and then integrating the resulting from t_2 to ∞, we obtain

$$\infty > z^{(m)}(t_2) - \lim_{t\to\infty} z^{(m)}(t) = \frac{(-1)^{n-m-1}}{(n-m-1)!} \int_{t_2}^\infty (u - t_2)^{n-m-1} \int_0^\lambda q(u, v) x(\sigma(u, v)) \, \mathrm{d}v \mathrm{d}u.$$

Since $n \geq 2$, by (A3) the above inequality implies

$$\liminf_{t\to\infty} \frac{x(\sigma(t, v))}{t^{m-1}} = 0 \quad \text{for any } v \in [0, \lambda].$$

For $v \in [0, \lambda]$, we have

$$\begin{aligned} 0 = & \liminf_{t\to\infty} \frac{x(\sigma(t, v))}{t^{m-1}} \\ = & \liminf_{t\to\infty} \left(\frac{x(\sigma(t, v))}{[\sigma(t, v)]^{m-1}} \left(\frac{\sigma(t, v)}{t}\right)^{m-1} \right) \\ \geq & \left(\liminf_{t\to\infty} \frac{x(\sigma(t, v))}{[\sigma(t, v)]^{m-1}} \right) \left(\liminf_{t\to\infty} \frac{\sigma(t, v)}{t} \right)^{m-1}. \end{aligned}$$

From (A1) and (A3), the inequality above gives

$$\liminf_{t\to\infty} \frac{x(t)}{t^{m-1}} = 0. \tag{13}$$

On the other hand, from Taylor's formula and (11), we have

$$\begin{aligned} z(t) &= \sum_{i=0}^{m-1} \frac{z^{(i)}(t_2)}{i!}(t-t_2)^i + \frac{1}{(m-1)!}\int_{t_2}^{t}(t-u)^{m-1}z^{(m)}(u)\,\mathrm{d}u \\ &\geq \frac{z^{(m-1)}(t_2)}{(m-1)!}(t-t_2)^{m-1} \end{aligned}$$

for all $t \geq t_2$, which implies

$$\liminf_{t\to\infty} \frac{z(t)}{t^{m-1}} = \frac{z^{(m-1)}(t_2)}{(m-1)!} > 0. \tag{14}$$

Using (13) and (14), we learn that

$$\liminf_{t\to\infty} \frac{x(t)}{z(t)} = 0.$$

Following similar steps to those in the proof of Theorem 3.1 (by replacing x with x/z), we deduce $\limsup_{t\to\infty}[x(t)/z(t)] = 0$. Using this, we see that

$$\lim_{t\to\infty} \frac{y(t)}{z(t)} = \lim_{t\to\infty} \left(\frac{x(t)}{z(t)} + \int_0^{\lambda} \frac{p(t,v)z(\tau(t,u))}{z(t)} \frac{x(\tau(t,u))}{z(\tau(t,u))}\mathrm{d}v \right) = 0,$$

which is in a contradiction with (9). Thus the proof is complete. □

From Theorem 3.1 and Theorem 3.2, the following remark can be inferred.

Remark 3.2. The property that the solutions Theorem 3.1 are bounded can be removed by replacing (A2) with the stronger assumption (A3). Recall that (A3) implies (A2), and (A4) implies (A5). Therefore, if a solution of (1) is bounded, then we know that it oscillates or tends to zero by Theorem 3.1, and if it is unbounded, by Theorem 3.2, then it oscillates.

We have the following illustrating example.

Example 3.4. Consider the fourth-order equation:

$$\left[x(t) - \int_0^1 \frac{v}{t^{\kappa}} x(t-v)\,\mathrm{d}v\right]^{(4)} + \int_0^1 \frac{v}{t^{\kappa}} x(t-v)\,\mathrm{d}v = 0 \quad \text{for } t \geq 1, \tag{15}$$

where κ is a positive constant. We see that $\lambda = 1$, $n = 4$, $p(t, v) = -v/t^\kappa$, $\tau(t, v) = t - v$, $q(t, v) = v/t^\kappa$, $\sigma(t, v) = t - v$ and $\varphi(t) \equiv 0$ for $t \geq 1$ and $v \in [0, 1]$. One can easily show that the assumptions of Theorem 3.2 holds when $\kappa \geq 3$, thus in this case every unbounded solution of (15) oscillates. And, when $\kappa \geq 4$, we know that every bounded solution of (15) oscillates or tends to zero asymptotically by Theorem 3.1.

The following theorem shows that (A2) is a necessary condition for solutions of (1) to be oscillatory or convergent to zero at infinity.

Theorem 3.3. *Assume (A5) and (C1). If every bounded solution of* (1) *oscillates or tends to zero asymptotically, then (A2) holds.*

Proof. By the contrapositive statement, we assume that (A2) does not hold and show that there exists a bounded solution of (1), which does not oscillate and does not tend to zero asymptotically.

From (A5), there exists $t_1 \geq t_0$ and $M > 0$ such that $|\Phi(t)| \leq M$ for all $t \geq t_1$ From (C1), we have $\alpha, \beta \geq 0$ with $\alpha + \beta < 1$ such that for some fixed $t_2 \geq t_1$, and all $t \geq t_2$, $-\alpha \leq \int_0^\lambda p(t, v)\,\mathrm{d}v \leq \beta$. Now, select $b > a > 0$ such that $M = [-a + (1 - \beta - \alpha)b]/4$. Since (H2) does not hold and $q \geq 0$, there exists $t_3 \geq t_2$ such that for all $t \geq t_3$,

$$\int_t^\infty (u - t)^{n-1} \int_0^\lambda q(u, v)\,\mathrm{d}v\mathrm{d}u \leq \frac{(n-1)!}{b} M. \tag{16}$$

Let $BC([t_3, \infty), \mathbb{R})$ be the Banach space of real-valued, bounded and continuous functions on $[t_3, \infty)$ endowed with the supremum norm $\|x\| := \sup\{|x(t)| : t \in [t_3, \infty)\}$. Also let

$$\Omega := \big\{x \in BC([t_3, \infty), \mathbb{R}) : a \leq x(t) \leq b \quad \text{for all } t \in [t_3, \infty)\big\}. \tag{17}$$

Then, Ω is a closed, bounded and convex subset of $BC([t_3, \infty), \mathbb{R})$. Pick $t_4 \geq t_3$ such that $\min_{v\in[0,\lambda]} \tau(t_4, v) \geq t_3$ and $\min_{v\in[0,\lambda]} \sigma(t_4, v) \geq t_3$. Let $N := [a + (1 - \alpha + \beta)b]/2$ and define two mappings $\Gamma, \Psi : \Omega \to \Omega$ as follows:

$$\Gamma x(t) := \begin{cases} \Gamma x(t_4), & t \in [t_3, t_4) \\ N - \displaystyle\int_0^\lambda p(t, v) x(\tau(t, v))\,\mathrm{d}v + \Phi(t), & t \in [t_4, \infty), \end{cases}$$

and

$$\Psi x(t) := \begin{cases} \Psi x(t_4), & t \in [t_3, t_4) \\ \displaystyle\int_t^\infty \frac{(t-u)^{n-1}}{(n-1)!} \int_0^\lambda q(u, v) x(\sigma(u, v))\,\mathrm{d}v\mathrm{d}u, & t \in [t_4, \infty). \end{cases}$$

We assert that $\Gamma x + \Psi x$ has a fixed point in Ω, by means of the Krasnoselskii's fixed point theorem. Note that this fixed point is a solution of (1).

First, we show that $\Gamma x+\Psi y\in\Omega$ for all $x,y\in\Omega$. In fact, by (16), for all $x,y\in\Omega$ and all $t\ge t_4$, we have

$$\begin{aligned}\Gamma x(t)+\Psi y(t)\le& N+\int_0^\lambda p^-(t,v)x(\tau(t,v))\,\mathrm{d}v+M\\&+\left|\int_t^\infty\frac{(u-t)^{n-1}}{(n-1)!}\int_0^\lambda q(u,v)y(\sigma(u,v))\,\mathrm{d}v\mathrm{d}u\right|\\\le& N+\alpha b+2M=b\end{aligned}$$

and

$$\begin{aligned}\Gamma x(t)+\Psi y(t)\ge& N-\int_0^\lambda p^+(t,v)x(\tau(t,v))\,\mathrm{d}v-M\\&-\left|\int_t^\infty\frac{(u-t)^{n-1}}{(n-1)!}\int_0^\lambda q(u,v)y(\sigma(u,v))\,\mathrm{d}v\mathrm{d}u\right|\\\ge& N-\beta b-2M=a.\end{aligned}$$

Hence, $\Gamma x+\Psi y\in\Omega$ for any $x,y\in\Omega$.

Next, we show that Γ is a contraction mapping on Ω. In fact, for any $x,y\in\Omega$ and for all $t\ge t_4$, we obtain

$$\begin{aligned}\left|\Gamma x(t)-\Gamma y(t)\right|=&\left|\int_0^\lambda p(t,v)\big[x(\tau(t,v))-y(\tau(t,v))\big]\mathrm{d}v\right|\\\le&\left|\int_0^\lambda p^+(t,v)\big[x(\tau(t,v))-y(\tau(t,v))\big]\mathrm{d}v\right|\\&+\left|\int_0^\lambda p^-(t,v)\big[x(\tau(t,v))-y(\tau(t,v))\big]\mathrm{d}v\right|\end{aligned}$$

from which we deduce $\|\Gamma x-\Gamma y\|\le(\beta+\alpha)\|x-y\|$, thus we conclude that Γ is a contraction mapping on Ω since $\beta+\alpha<1$.

Next, we show that Ψ is continuous on Ω. Let $\{x_k\}_{k=1}^\infty$ be a sequence in Ω which converges to $x\in\Omega$. Note that Ω is closed. For all $t\ge t_4$ and $k\in\mathbb{N}$, we have

$$\left|\Psi x_k(t)-\Psi x(t)\right|\le\left|\int_t^\infty\frac{(t-u)^{n-1}}{(n-1)!}\int_0^\lambda p(u,v)\big[x_k(\tau(u,v))-x(\tau(u,v))\big]\,\mathrm{d}v\mathrm{d}u\right|.$$

Since for $t\ge t_4$, $x_k(t)$ converges uniformly to $x(t)$ as k tends to infinity, we have $\lim_{k\to\infty}\|\Psi x_k-\Psi x\|=0$. Therefore, Ψ is continuous on Ω.

Now, we show that $\Psi\Omega$ is relatively compact. It suffices to show that the family of functions $\Psi\Omega$ is uniformly bounded and equicontinuous on $[t_4,\infty)$. The property of being uniformly bounded is obvious. For the equicontinuity, we only need to show that

$\Psi\Omega$ has a uniformly bounded derivative on $[t_4, \infty)$ (see [8, § 1.7]). From (16), we can find a constant $K > 0$ such that

$$\int_t^{\infty} (u-t)^{n-2} \int_0^{\lambda} q(u,v)\, \mathrm{d}v\mathrm{d}u \le \frac{(n-2)!}{b} K \tag{18}$$

holds for all $t \ge t_4$ and all $v \in [0, \lambda]$. From (18), we have

$$\left| \frac{\mathrm{d}}{\mathrm{d}t} \Psi x(t) \right| = \left| \int_t^{\infty} \frac{(u-t)^{n-2}}{(n-2)!} \int_0^{\lambda} p(u,v) x(\sigma(u,v))\, \mathrm{d}v\mathrm{d}u \right| \le K$$

for all $t \ge t_4$, which proves $\Psi\Omega$ is relatively compact.

Therefore, $\Psi\Omega$ is uniformly bounded and equicontinuous on $[t_4, \infty)$ and hence $\Psi\Omega$ is relatively compact. By Krasnoselskii's fixed point theorem, there exists $x \in \Omega$ such that $\Gamma x + \Psi x = x$, which is the desired bounded solution of (1) bounded below by the positive constant a. The proof is complete. □

Now, we have the final example that illustrates Theorem 3.3.

Example 3.5. Consider the second-order equation:

$$\begin{aligned} \left[x(t) + \int_0^{1/2} \frac{\cos(t - \pi(v+1/2))}{8} x(t^{v+1/2}) \mathrm{d}v \right]'' + \int_0^{1/2} \frac{v}{t^3} x(t^{v+1/2}) \mathrm{d}v \\ = \frac{\sin(t) - \cos(t)}{\pi} - \frac{1}{t^3} \quad \text{for } t \ge 0. \end{aligned} \tag{19}$$

for $t \ge 1$. We see that $\lambda = 1/2$, $n = 2$, $p(t,v) = \cos(t - \pi(v+1/2))/8$, $\tau(t,v) = t^{v+1/2}$, $q(t,v) = v/t^3$, $\sigma(t,v) = t^{v+1/2}$ and $\varphi(t) = (\sin(t) - \cos(t))/\pi - 1/t^3$ for $t \ge 1$ and $v \in [0, 1/2]$. Then, we may let $\Phi(t) = (\cos(t) - \sin(t))/\pi - 1/(2t)$ for $t \ge 1$. It is clear that (A5) and (C1) hold, but (A2) does not. Therefore, by Theorem 3.3, (19) has a bounded nonoscillatory solution which does not tend to zero at infinity, and for $t \ge 1$, $x(t) \equiv 8$ is such a nonoscillating solution of (19).

4 Final Discussion and Extension of the Results

In this paper, we give oscillation and nonoscillation conditions for both bounded and unbounded solutions of (1). From Theorem 3.1, Corollary 3.1 and Theorem 3.3, we can give the following necessary and sufficient condition on bounded solutions of (1).

Corollary 4.1. *Assume (A4) and (C1). If $n \ge 2$, then every bounded solution of* (1) *oscillates or tends to zero if and only if (A2) holds. And, if $n = 1$, then every solution of* (1) *oscillates or tends to zero if and only if (A2) holds.*

We would like to point out that the technique employed in this paper can be easily adapted for solutions of the equation

$$\left[x(t)+\sum_{i=1}^{m}\int_0^{\lambda}p_i(t,v)x(\tau_i(t,v))\mathrm{d}v\right]^{(n)}+\sum_{i=1}^{k}\int_0^{\lambda}q_i(t,v)F_i\big(x(\sigma_i(t,v))\big)\mathrm{d}v=\varphi(t),\quad (20)$$

where $m,k\geq 1$ are integers, p_i,τ_i,q_i,σ_i have similar properties to those of (1), and $F_i\in C(\mathbb{R},\mathbb{R})$ satisfies the usual sign condition $F_i(s)/s>0$ for all $s\neq 0$. In this case the following assumptions are used:

(A6) F_{i_0} is nondecreasing with $\liminf_{s\to\infty}\left[F_{i_0}(s)/s\right]>0$ for some $i_0\in\{1,2,\dots,m\}$;

(A7) $\liminf_{t\to\infty}\left[\sigma_{i_0}(t,v)/t\right]>0$ for any $v\in[0,\lambda]$, where i_0 satisfies (A6);

(A8) $\int_{t_0}^{\infty}u^{n-1}\int_0^{\lambda}q_{i_0}(u,v)\,\mathrm{d}v\mathrm{d}u=\infty$ for some $i_0\in\{1,2,\dots,m\}$;

(A9) $\int_{t_0}^{\infty}u^{n-2}\int_0^{\lambda}q_{i_0}(u,v)\,\mathrm{d}v\mathrm{d}u=\infty$, where i_0 satisfies (A6);

(C2) $\limsup_{t\to\infty}\sum_{i=1}^{m}\int_0^{\lambda}p_i^{+}(t,v)\,\mathrm{d}v+\limsup_{t\to\infty}\sum_{i=1}^{m}\int_0^{\lambda}p_i^{-}(t,v)\,\mathrm{d}v<1$.

For bounded solutions of (20), we need the assumptions (A8) and (C2) instead of (A2) and (C1), respectively. On the other hand, for unbounded solutions, we need (A6) together with (A7) and (A9) instead of (A1) and (A3), respectively.

In the literature, there are very few papers that study (2) or (3) with an oscillatory coefficient in the neutral part, and almost all of these results except for [12, Theorem 2.4] are focused on bounded solutions. However, in [12], the authors proved results for both bounded and unbounded solutions by assuming the coefficient p to be periodic. In a recent paper Zhou [15] studied bounded solutions of (3) (with a nonlinear term and several coefficients), and improved the results in [2]. Considering the discussion above, one can infer that the corresponding result of Corollary 4.1 for (20) includes [15, Theorem 2.1, Theorem 2.2].

References

[1] T. M. Abu-Kaf and R. S. Dahiya, Oscillation of solutions of arbitrary order functional-differential equations, *J. Math. Anal. Appl.*, vol. 142, no. 2, pp. 521–541, (1989).

[2] Y. Bolat and Ö. Akın, Oscillatory behaviour of higher order neutral type nonlinear forced differential equation with oscillating coefficients, *J. Math. Anal. Appl.*, vol. 290, pp. 302–309, (2004).

[3] T. Candan and R. S. Dahiya, Oscillation of nth order neutral differential equations with continuous delay, *J. Math. Anal. Appl.*, vol. 290, no. 1, pp. 105–112, (2004).

[4] T. Candan, Oscillation of first-order neutral differential equations with distributed deviating arguments, *Comp. Math. Appl.*, vol. 55, no. 3, pp. 510–515, (2008).

[5] R.S. Dahiya and O. Akinyele, Oscillation theorems of nth-order functional-differential equations with forcing terms, *J. Math. Anal. Appl.*, vol. 42, no. 2, pp. 325–332, (1985).

[6] R. S. Dahiya and Z. Zafer, Oscillation of higher-order neutral type functional differential equations with distributed arguments, *Differential & Difference Equations and Applications*, pp. 315–323, Hindawi Publ. Corp., New York, 2006.

[7] L. H. Erbe, Q. K. Kong and B. G. Zhang, *Oscillation Theory for Functional Differential Equations*, Marcel Dekker, New York, (1995).

[8] I. Győri and G. Ladas, *Oscillation Theory of Delay Differential Equations: With Applications*, Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1991.

[9] I. T. Kiguradze, On the oscillatory character of solutions of the equation $d^m u/dt^m + a(t)|u|^n \operatorname{sign} u = 0$, *Mat. Sb. (N.S.)*, vol. 65, pp. 107–187, (1964).

[10] W. N. Li, Oscillation of higher order delay differential equations of neutral type, *Georgian Math. J.*, vol. 7, no. 2, pp. 347–353, (2000).

[11] N. Parhi and R. N. Rath, On oscillation of solutions of forced nonlinear neutral differential equations of higher order, *Czechoslovak Math. J.*, vol. 53(128), no. 4, pp. 805–825, (2003).

[12] N. Parhi and R. N. Rath, On oscillation of solutions of forced nonlinear neutral differential equations of higher order II, *Ann. Polon. Math.*, vol. 81, no. 2, pp. 101–110, (2003).

[13] Y. Şahiner and A. Zafer, Bounded oscillation of nonlinear neutral differential equations of arbitrary order, *Czechoslovak Math. J.*, vol. 51, no. 126, pp. 185–195, (2001).

[14] A. Zafer and R. S. Dahiya, Oscillation of solutions of arbitrary order neutral functional differential equations with forcing terms, *World Congress of Nonlinear Analysis '92 (Tampa, FL. 1992)*, de Gruyter, Berlin, pp. 1977–19987, (1996).

[15] X. L. Zhou and R. Yu, Oscillatory behavior of higher order nonlinear neutral forced differential equations with oscillating coefficients. *Comp. Math. Appl.*, vol. 56, vol. 6, pp. 1562–1568, (2008).

(Received July 22, 2010)