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Existence and concentration of solutions for nonautomous Schrödinger–Poisson systems with critical growth

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Abstract. In this paper, we study the following Schrödinger–Poisson system

$$\begin{cases} -\Delta u + u + \mu\phi u = \lambda f(x,u) + u^5 & \text{in } \mathbb{R}^3, \\ -\Delta\phi = \mu u^2 & \text{in } \mathbb{R}^3, \end{cases}$$

where μ, $\lambda > 0$ are parameters and $f \in C(\mathbb{R}^3 \times \mathbb{R}, \mathbb{R})$. Under certain general assumptions on $f(x,u)$, we prove the existence and concentration of solutions of the above system for each $\mu > 0$ and λ sufficiently large. Our main result can be viewed as an extension of the results by Zhang [*Nonlinear Anal.* **75**(2012), 6391–6401].

Keywords: Schrödinger–Poisson system; critical growth; variational methods.

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1 Introduction and main results

Consider the following Schrödinger–Poisson system

$$\begin{cases} -\Delta u + u + \mu\phi u = \lambda f(x,u) + u^5 & \text{in } \mathbb{R}^3, \\ -\Delta\phi = \mu u^2 & \text{in } \mathbb{R}^3, \end{cases} \tag{1.1}$$

where μ, $\lambda > 0$ are parameters and $f \in C(\mathbb{R}^3 \times \mathbb{R}, \mathbb{R})$. Equation (1.1) or the more general one

$$\begin{cases} -\Delta u + V(x)u + K(x)\phi u = f(x,u) & \text{in } \mathbb{R}^3, \\ -\Delta\phi = K(x)u^2 & \text{in } \mathbb{R}^3, \end{cases} \tag{1.2}$$

arise from several interesting physical fields, such as in quantum electrodynamics, describing the interaction between a charged particle interacting with the electromagnetic field, and also in semiconductor theory and in plasma physics. For more details in physical background we refer to [5,8] and the references therein.

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There are many papers studying the existence of solutions of system (1.2), see [2–4, 7–10, 12–14, 16–22] and their references. A lot of works focus on the study of problem (1.2) with the very special case $V = K = 1$ and $f(x,u) = |u|^{p-2}u$, and existence and multiplicity of positive solutions as well as radial or nonradial symmetric solutions are obtained, see e.g. [2, 3, 7–10, 13]. The Schrödinger–Poisson system with critical nonlinearity of the form

$$\begin{cases} -\Delta u + u + \phi u = P(x)|u|^4 u + \lambda Q(x)|u|^{q-2}u & \text{in } \mathbb{R}^3, \\ -\Delta \phi = u^2 \quad \text{in } \mathbb{R}^3, \quad 2 < q < 6, \quad \lambda > 0, \end{cases}$$

has been studied in [22]. Besides some other conditions, Zhao et al. assume that $P \in C(\mathbb{R}^3, \mathbb{R})$, $\lim_{|x|\to\infty} P(x) = P_\infty \in (0, +\infty)$ and $P(x) \geq P_\infty$ and prove the existence of one positive solution for $4 < q < 6$ and each $\lambda > 0$. It is also proven the existence of one positive solution for $q = 4$ and λ large enough. Zhang [18] considers the following type of Schrödinger–Poisson system

$$\begin{cases} -\Delta u + u + \mu\phi u = f(u) & \text{in } \mathbb{R}^3, \\ -\Delta \phi = \mu u^2 & \text{in } \mathbb{R}^3, \end{cases} \tag{1.3}$$

where $f \in C(\mathbb{R}^+, \mathbb{R}^+)$ satisfies $\lim_{u\to+\infty} f(u)/u^5 = K > 0$ and $f(u) \geq Ku^5 + Du^{q-1}$ for some $D > 0$, which exhibits a critical growth. Applying a combined technique consisting in a truncation argument and a monotonicity trick, he proves that for $\mu > 0$ small, problem (1.3) admits a positive solution for $q \in (2, 4]$ with D sufficiently large or $q \in (4, 6)$. In [20], the same author studies problem (1.1) when $V = 1$ and $f(x,u) = a(x)|u|^{p-2}u + \lambda b(x)|u|^{q-2}u + u^5$, where p, $q \in (4, 6)$, $\lambda > 0$ is a parameter. Under certain decay rate conditions on $K(x)$, $a(x)$ and $b(x)$, he proves the existence of ground state solution and two nontrivial solutions for $\lambda > 0$ small. Recently, the Schödinger–Poisson system with nonconstant coefficient of the following version

$$\begin{cases} -\Delta u + V(x)u + \varepsilon\phi u = \lambda f(u) & \text{in } \mathbb{R}^3, \\ -\Delta \phi = u^2 \text{ in } \mathbb{R}^3, \quad \lim_{|x|\to\infty} \phi(x) = 0, \end{cases}$$

has been discussed in Mao et al. [12]. Assuming that V is coercive, i.e. $V(x) \to \infty$ as $|x| \to \infty$ and f is local subcritical and 4-superlinear at the origin, the authors prove the existence of nontrivial solution and its asymptotic behavior depending on ε and λ.

Motivated by the works described above, in this paper, we try to prove the existence of solutions of problem (1.1) with a much more general nonlinearity in critical growth. Precisely, we make the following hypotheses.

- (f_1) There exist $c_0 > 0$ and $2 < p_1 < p_2 < 6$ such that $|f(x,s)| \leq c_0(|s|^{p_1-1} + |s|^{p_2-1})$ for all $(x,s) \in \mathbb{R}^3 \times \mathbb{R}$.
- (f_2) $F(x,s) \geq 0$ for all $(x,s) \in \mathbb{R}^3 \times \mathbb{R}$, and there exist c_1, $\rho_0 > 0$ and $q \in (2, 6)$ such that $F(x,s) \geq c_1|s|^q$ for $x \in \mathbb{R}^3$ and $|s| \geq \rho_0$.
- (f_3) There exists $\theta \in (2, 6)$ such that $f(x,s)s - \theta F(x,s) \geq 0$ for all $(x,s) \in \mathbb{R}^3 \times \mathbb{R}$.

Theorem 1.1. *Assume that (f_1)–(f_3) are satisfied with $p_1 > 3q - 4$. Then, for any $\mu > 0$, problem (1.1) possesses a nontrivial solution u_λ for $\lambda > 0$ sufficiently large. Moreover, $u_\lambda \to 0$ as $\lambda \to +\infty$.*

Theorem 1.1 can be viewed as an extension of the main results in [18]. Note that, in [18], the existence of solution is obtained by using the radially symmetric Sobolev space $H_r^1(\mathbb{R}^3)$, where the embedding $H_r^1(\mathbb{R}^3) \hookrightarrow L^s(\mathbb{R}^3)$ $(2 < s < 6)$ is compact. However, in our case since

f is nonradially symmetric, we have to deal with (1.1) in $H^1(\mathbb{R}^3)$ and the Sobolev embedding $H^1(\mathbb{R}^3) \hookrightarrow L^s(\mathbb{R}^3)$ $(2 < s < 6)$ is not compact any more. Moreover, the critical exponential growth makes the problem more complicated. To overcome these difficulties, we use a truncation argument (see [11]) together with careful analysis of the $(PS)_{c_\lambda}$ sequence and prove the $(PS)_{c_\lambda}$ condition holds for a suitable range of c_λ indirectly.

Notations

- $L^s(\mathbb{R}^3)$ $(1 \leq s \leq +\infty)$ is a Lebesgue space whose norm is denoted by $\|\cdot\|_s$.
- $H^1(\mathbb{R}^3)$ is the usual Hilbert space endowed with the norm $\|u\|^2 = \int_{\mathbb{R}^3}(|\nabla u|^2 + u^2)dx$.
- $\mathcal{D}^{1,2}(\mathbb{R}^3)$ is the completion of $C_0^\infty(\mathbb{R}^3)$ with respect to the norm $\|u\|^2_{\mathcal{D}^{1,2}} := \int_{\mathbb{R}^3} |\nabla u|^2 dx$.
- S denotes the best Sobolev constant

$$S := \inf_{u \in \mathcal{D}^{1,2}(\mathbb{R}^3)\setminus\{0\}} \frac{\|u\|^2_{\mathcal{D}^{1,2}}}{\|u\|_6^2}.$$

- For every $2 \leq q < 6$, denote

$$S_q := \inf_{u \in H^1(\mathbb{R}^3)\setminus\{0\}} \frac{\|u\|^2}{\|u\|_q^2}.$$

- C and C_i $(i = 1, 2, \dots)$ denotes various positive constants, which may vary from line to line.

2 Proof of Theorem 1.1

For simplicity, we assume $\mu = 1$ and denote $H = H^1(\mathbb{R}^3)$. We first recall the following well-known facts.

Lemma 2.1 (see [4]). *For each $u \in H$, there exists a unique $\phi_u \in \mathcal{D}^{1,2}(\mathbb{R}^3)$ solution of*

$$-\Delta \phi_u = u^2 \quad \text{in } \mathbb{R}^3,$$

Moreover,

(i) $\phi_u \geq 0$;

(ii) $\phi_{tu} = t^2 \phi_u$, $\forall t > 0$;

(iii) there exists $C_0 > 0$ such that

$$\|\phi_u\|_{\mathcal{D}^{1,2}} \leq C_0 \|u\|_\alpha^2 \quad \text{and} \quad \int_{\mathbb{R}^3} \phi_u u^2 dx \leq C_0 \|u\|_\alpha^4,$$

where $\alpha = 12/5$.

Define the functional associated to problem (1.1)

$$I(u) = \frac{1}{2}\|u\|^2 + \frac{1}{4}\int_{\mathbb{R}^3} \phi_u u^2 dx - \int_{\mathbb{R}^3}\left(\lambda F(x,u) + \frac{1}{6}u^6\right) dx,$$

where $u \in H$. It is easy to check that $I \in C^1(H, \mathbb{R})$ and $(u, \phi) \in H \times \mathcal{D}^{1,2}(\mathbb{R}^3)$ is a weak solution of problem (1.1) if and only if $u \in H$ is a critical point of I and $\phi = \phi_u$.

We introduce the cut-off function $\chi \in C^\infty(\mathbb{R}_+, \mathbb{R})$ satisfying $\chi(s) = 1$ for $s \in [0,1]$, $\chi(s) = 0$ for $s \in [2, +\infty)$, $0 \le \chi \le 1$ and $\|\chi'\|_\infty \le 2$. Consider the truncated functional $I_T : H \to \mathbb{R}$

$$I_T(u) = \frac{1}{2}\|u\|^2 + \frac{1}{4}K_T(u)\int_{\mathbb{R}^3}\phi_u u^2 dx - \int_{\mathbb{R}^3}\left(\lambda F(x,u) + \frac{1}{6}u^6\right)dx,$$

where, for each $T > 0$, $K_T(u) = \chi\left(\frac{\|u\|_\alpha^\alpha}{T^\alpha}\right)$. For λ sufficiently large, we will find a critical point u_λ of I_T such that $\|u_\lambda\|_\alpha \le T$ and so we conclude that u_λ is also a critical point of I.

Lemma 2.2. *The functional I_T possesses a mountain pass geometry:*

(i) there exist constants α, $\rho > 0$ such that $I_T(u) \ge \alpha$ for all $\|u\| = \rho$;

(ii) there exists $e \in H$ such that $\|e\| > \rho$ and $I_T(e) < 0$.

Proof. It follows from (f_1) that

$$|F(x,s)| \le c_0(|s|^{p_1} + |s|^{p_2}), \qquad \forall (x,s) \in \mathbb{R}^3 \times \mathbb{R}.$$

Then, by Sobolev's inequality, we have

$$\begin{aligned} I_T(u) &\ge \frac{1}{2}\|u\|^2 - \lambda c_0 \int_{\mathbb{R}^3}(|u|^{p_1} + |u|^{p_2})dx - \frac{1}{6}\int_{\mathbb{R}^3} u^6 dx \\ &\ge \frac{1}{2}\|u\|^2 - C\left(\|u\|^{p_1} + \|u\|^{p_2}\right) - \frac{1}{6}S^{-3}\|u\|^6. \end{aligned}$$

Since p_1, $p_2 > 2$, there exist α, $\rho > 0$ such that $I_T|_{\|u\|=\rho} \ge \alpha$.

Choose $w \in H \setminus \{0\}$ such that $w \ge 0$. By Lemma 2.1 and (f_2), we have

$$I_T(tw) \le \frac{t^2}{2}\|w\|^2 + C_0 t^4 \|w\|_\alpha^4 - \frac{t^6}{6}\int_{\mathbb{R}^3} w^6 dx \to -\infty \quad \text{as } t \to +\infty.$$

Hence there exists $t_0 > 0$ large enough such that $I_T(t_0 w) < 0$ and $\|t_0 w\| \ge \rho$. □

Therefore, according to the mountain pass theorem (see [1]), there exists a $(PS)_{c_\lambda}$ sequence $(u_n) \subset H$ such that

$$I_T(u_n) \xrightarrow{n} c_\lambda, \qquad I'_T(u_n) \xrightarrow{n} 0, \tag{2.1}$$

where

$$c_\lambda = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I_T(\gamma(t))$$

with $\Gamma = \{\gamma \in C([0,1], H) : \gamma(0) = 0, I_T(\gamma(1)) < 0\}$.

For $\varepsilon > 0$, let

$$v_\varepsilon(x) = \frac{\psi(x)\varepsilon^{\frac{1}{4}}}{(\varepsilon + |x|^2)^{\frac{1}{2}}},$$

where $\psi \in C_0^\infty(\mathbb{R}^3, [0,1])$ such that $\psi(x) = 1$ for $|x| \le r$ and $\psi(x) = 0$ for $|x| \ge 2r$. It is well known that S is attained by the function $\frac{\varepsilon^{1/4}}{(\varepsilon+|x|^2)^{1/2}}$. Direct calculation shows that (see [15]):

$$\begin{aligned} \int_{\mathbb{R}^3} |\nabla v_\varepsilon|^2 dx &= \int_{\mathbb{R}^3} \frac{|x|^2}{(1+|x|^2)^3} dx + O(\varepsilon^{\frac{1}{2}}) := K_1 + O(\varepsilon^{\frac{1}{2}}), \qquad (2.2) \\ \int_{\mathbb{R}^3} |v_\varepsilon|^6 dx &= \int_{\mathbb{R}^3} \frac{1}{(1+|x|^2)^3} dx := K_2 + O(\varepsilon^{\frac{3}{2}}) \end{aligned}$$

and

$$\int_{\mathbb{R}^3} |v_\varepsilon|^t dx = \begin{cases} O(\varepsilon^{\frac{6-t}{4}}), & t \in (3,6), \\ O(\varepsilon^{\frac{3}{4}}|\ln \varepsilon|), & t = 3, \\ O(\varepsilon^{\frac{t}{4}}), & t \in [2,3), \end{cases} \tag{2.3}$$

where K_1, K_2 are positive constants and $S = K_1/K_2^{1/3}$. By the definition of c_λ, we have $c_\lambda \leq \sup_{t\geq 0} I_T(tv_\varepsilon)$.

Lemma 2.3. *There is a constant $D_0 > 0$ independent of λ such that $c_\lambda \leq \frac{D_0}{\lambda^{\frac{2}{q-2}}}$.*

Proof. It follows from (2.2) and (2.3) that there exists $\varepsilon_1 > 0$ such that for $\varepsilon \in (0, \varepsilon_1)$,

$$\frac{K_1}{2} \leq \|v_\varepsilon\|^2 \leq \frac{3K_1}{2}, \qquad \frac{K_2}{2} \leq \|v_\varepsilon\|_6^6 \leq \frac{3K_2}{2}. \tag{2.4}$$

Since $F \geq 0$ for all (x,s), one sees that

$$I_T(tv_\varepsilon) \leq \frac{t^2}{2}\|v_\varepsilon\|^2 + \frac{t^4}{4}C_0 S_{12/5}^{-2}\|v_\varepsilon\|^4 - \frac{t^6}{6}\|v_\varepsilon\|_6^6.$$

Thus, using (2.4), there exist $t' > 0$ small and $t'' > 0$ large (independent of $\varepsilon \in (0, \varepsilon_1)$) such that

$$\sup_{t\in[0,t']\cup[t'',+\infty)} I_T(tv_\varepsilon) \leq \frac{q-2}{2q}\left(\frac{3K_1}{2}\right)^{\frac{q}{q-2}}\left(\frac{1}{q\tilde{a}}\right)^{\frac{2}{q-2}}\frac{1}{\lambda^{\frac{2}{q-2}}}, \tag{2.5}$$

where $\tilde{a} = \frac{c_1}{2^{q/2}}\int_{|x|\leq 1} dx$.

Choose $\varepsilon_0 \in (0, \min\{1, \varepsilon_1, r^2\})$ such that

$$\frac{t'\varepsilon_0^{-\frac{1}{4}}}{\sqrt{2}} \geq \rho_0, \qquad \frac{t''^4}{4}C_0\|v_{\varepsilon_0}\|_\alpha^4 \leq \frac{K_2}{12}t'^6. \tag{2.6}$$

By the definition of $v_{\varepsilon_0}(x)$, we get

$$v_{\varepsilon_0}(x) \geq \frac{\varepsilon_0^{-\frac{1}{4}}}{\sqrt{2}}, \qquad \forall |x| \leq \varepsilon_0^{1/2},$$

and then

$$tv_{\varepsilon_0}(x) \geq \frac{t'\varepsilon_0^{-\frac{1}{4}}}{\sqrt{2}} \geq \rho_0, \qquad \forall t \geq t', \ \ \forall |x| \leq \varepsilon_0^{1/2}.$$

so that, by (f_2),

$$\int_{\mathbb{R}^3} F(x, tv_{\varepsilon_0})dx \geq c_1 \int_{|x|\leq\varepsilon_0{}^{1/2}} |tv_{\varepsilon_0}|^q dx \geq c_1 \int_{|x|\leq\varepsilon_0{}^{1/2}} \frac{\varepsilon_0^{-\frac{q}{4}}}{2^{\frac{q}{2}}} t^q dx = \tilde{a}\varepsilon_0^{\frac{(6-q)}{4}} t^q \tag{2.7}$$

for all $t \geq t'$, where $\tilde{a}$ is the same constant as in (2.5). Hence, by (2.7), (2.6) and (2.4), we deduce that

$$\begin{aligned}
\sup_{t\in[t',t'']} I_T(tv_{\varepsilon_0}) &\le \sup_{t\in[t',t'']}\left(\frac{t^2}{2}\|v_{\varepsilon_0}\|^2-\lambda\int_{\mathbb{R}^3}F(x,tv_{\varepsilon_0})dx\right)+\left(\frac{t''^4}{4}C_0\|v_{\varepsilon_0}\|_\alpha^4-\frac{K_2t'^6}{12}\right)\\
&\le \sup_{t\ge t'}\left(\frac{3K_1}{4}t^2-\lambda\tilde{a}\varepsilon_0^{\frac{6-q}{4}}t^q\right)\\
&\le \sup_{t\ge 0}\left(\frac{3K_1}{4}t^2-\lambda\tilde{a}\varepsilon_0^{\frac{6-q}{4}}t^q\right)\\
&=\frac{q-2}{2q}\left(\frac{3K_1}{2}\right)^{\frac{q}{q-2}}\left(\frac{1}{q\tilde{a}\varepsilon_0^{\frac{6-q}{4}}}\right)^{\frac{2}{q-2}}\frac{1}{\lambda^{\frac{2}{q-2}}}.
\end{aligned}$$

Combining this with (2.5) shows that

$$c_\lambda\le\sup_{t\ge0}I_T(tv_{\varepsilon_0})\le\frac{q-2}{2q}\left(\frac{3K_1}{2}\right)^{\frac{q}{q-2}}\left(\frac{1}{q\tilde{a}\varepsilon_0^{\frac{6-q}{4}}}\right)^{\frac{2}{q-2}}\frac{1}{\lambda^{\frac{2}{q-2}}}=:\frac{D_0}{\lambda^{\frac{2}{q-2}}}.$$ □

Lemma 2.4. *There is a constant $D_1>0$ independent of λ such that, for any $(PS)_{c_\lambda}$-sequence (u_n) with*

$$c_\lambda\in\left(0,\frac{D_1}{\lambda^{\frac{6}{p_1-2}}}\right),$$

(u_n) has a strongly convergent subsequence.

Proof. It follows from (2.1) and (f_3) that

$$\begin{aligned}
c_\lambda+o(1)\|u_n\| &= I_T(u_n)-\frac{1}{\theta}\langle I_T'(u_n),u_n\rangle\\
&\ge\left(\frac12-\frac1\theta\right)\|u_n\|^2+\left(\frac14-\frac1\theta\right)K_T(u_n)\int_{\mathbb{R}^3}\phi_{u_n}u_n^2dx\\
&\quad-\frac{\alpha}{4\theta T^\alpha}\chi'\left(\frac{\|u_n\|_\alpha^\alpha}{T^\alpha}\right)\|u_n\|_\alpha^\alpha\int_{\mathbb{R}^3}\phi_{u_n}u_n^2dx\\
&\ge\left(\frac12-\frac1\theta\right)\|u_n\|^2-\frac{|4-\theta|}{4\theta}C_02^{\frac4\alpha}T^4-\frac{\alpha}{\theta}C_02^{\frac4\alpha}T^4,
\end{aligned}$$

which implies that $(u_n)_{n\in\mathbb{N}}$ is bounded in H. Thus, going if necessary to a subsequence, we may assume for each bounded domain $\Omega\subset\mathbb{R}^3$,

$$\begin{aligned}
&u_n\rightharpoonup u_\lambda\quad\text{in } H,\qquad u_n(x)\to u_\lambda(x)\quad\text{a.e. } x\in\mathbb{R}^3,\\
&u_n\to u_\lambda\quad\text{in } L^t(\Omega)\ (2\le t<6),\\
&|u_n(x)|\le w(x)\quad\text{for some } w\in L^t(\Omega).
\end{aligned}\tag{2.8}$$

We claim that $u_n\to u_\lambda$ in H. Take

$$\int_{\mathbb{R}^3}\phi_{u_n}u_n^2dx\xrightarrow{n}A,\qquad K_T(u_n)\xrightarrow{n}B,\qquad \chi'\left(\frac{\|u_n\|_\alpha^\alpha}{T^\alpha}\right)\xrightarrow{n}D,\tag{2.9}$$

where A, B, D are nonnegative constants, and define the functionals J_T, Ψ_T on H by

$$\begin{aligned}
J_T(u)&=\frac12\|u\|^2+\frac B4\int_{\mathbb{R}^3}\phi_uu^2dx+\frac{AD}{4T^\alpha}\int_{\mathbb{R}^3}|u|^\alpha dx-\int_{\mathbb{R}^3}\left(\lambda F(x,u)+\frac16u^6\right)dx,\\
\Psi_T(u)&=\frac12\|u\|^2+\frac B4\int_{\mathbb{R}^3}\phi_uu^2dx-\int_{\mathbb{R}^3}\left(\lambda F(x,u)+\frac16u^6\right)dx.
\end{aligned}$$

By (2.8), we see that, for any $\psi \in C_0^\infty(\mathbb{R}^3)$,

$$\int_{\mathbb{R}^3} \nabla u_n \cdot \nabla \psi dx \to \int_{\mathbb{R}^3} \nabla u_\lambda \cdot \nabla \psi dx, \qquad \int_{\mathbb{R}^3} u_n \psi dx \to \int_{\mathbb{R}^3} u_\lambda \psi dx, \tag{2.10}$$

and

$$\int_{\mathbb{R}^3} f(x,u_n)\psi dx = \int_{\operatorname{supp}\psi} f(x,u_n)\psi dx \to \int_{\mathbb{R}^3} f(x,u_\lambda)\psi dx, \tag{2.11}$$

where we have used Lebesgue dominated convergent theorem in the last limit. From $u_n \to u_\lambda$ a.e. in $\mathbb{R}^3$ and $\phi_{u_n}(x) \to \phi_{u_\lambda}(x)$ a.e. in $\mathbb{R}^3$, we know that $\phi_{u_n}(x)u_n(x) \to \phi_{u_\lambda}(x)u_\lambda(x)$ a.e. in $\mathbb{R}^3$. Using the fact

$$\|\phi_{u_n}u_n\|_2 \le \|\phi_{u_n}\|_6\|u_n\|_3 \le C_0 S^{-\frac{1}{2}} S_{12/5}^{-1}\|u_n\|^2\|u_n\|_3 \le C,$$

we get that $\phi_{u_n}u_n \in L^2(\mathbb{R}^3)$ and $(\phi_{u_n}u_n)_{n\in\mathbb{N}}$ is bounded in $L^2(\mathbb{R}^3)$. Therefore, up to a subsequence, $\phi_{u_n}u_n \rightharpoonup \phi_{u_\lambda}u_\lambda$ in $L^2(\mathbb{R}^3)$ and

$$\int_{\mathbb{R}^3} \phi_{u_n}u_n\psi dx \overset{n}{\longrightarrow} \int_{\mathbb{R}^3} \phi_{u_\lambda}u_\lambda\psi dx. \tag{2.12}$$

Moreover, observe that $\left\{|u_n|^{\alpha-2}u_n\right\} \subset L^{\alpha/(\alpha-1)}(\mathbb{R}^3)$ is bounded. This and the fact

$$|u_n(x)|^{\alpha-2}u_n(x) \to |u_\lambda(x)|^{\alpha-2}u_\lambda(x) \quad \text{a.e. } x \in \mathbb{R}^3$$

implies that $|u_n|^{\alpha-2}u_n \rightharpoonup |u_\lambda|^{\alpha-2}u_\lambda$ in $L^{\alpha/(\alpha-1)}(\mathbb{R}^3)$. So

$$\int_{\mathbb{R}^3} |u_n|^{\alpha-2}u_n\psi dx \overset{n}{\longrightarrow} \int_{\mathbb{R}^3} |u_\lambda|^{\alpha-2}u_\lambda\psi dx. \tag{2.13}$$

Similarly, we deduce that as $n \to \infty$,

$$\int_{\mathbb{R}^3} u_n^5\psi dx \to \int_{\mathbb{R}^3} u_\lambda^5\psi dx. \tag{2.14}$$

Combining (2.10)–(2.14), we achieve that

$$\begin{aligned} o(1) &= \langle I_T'(u_n), \psi\rangle \\ &= (u_n,\psi) + \left[K_T(u_n)\int_{\mathbb{R}^3}\phi_{u_n}u_n\psi dx + \frac{\alpha}{4T^\alpha}\chi'\left(\frac{\|u_n\|_\alpha^\alpha}{T^\alpha}\right)\int_{\mathbb{R}^3}|u_n|^{\alpha-2}u_n\psi dx\int_{\mathbb{R}^3}\phi_{u_n}u_n^2 dx\right] \\ &\quad - \int_{\mathbb{R}^3}\left(\lambda f(x,u_n)\psi + u_n^5\psi\right)dx \\ &= (u_\lambda,\psi) + B\int_{\mathbb{R}^3}\phi_{u_\lambda}u_\lambda\psi dx + \frac{\alpha AD}{4T^\alpha}\int_{\mathbb{R}^3}|u_\lambda|^{\alpha-2}u_\lambda\psi dx \\ &\quad - \int_{\mathbb{R}^3}\left(\lambda f(x,u_\lambda)\psi + u_\lambda^5\psi\right)dx + o(1) \\ &= J_T'(u_\lambda)\psi + o(1), \qquad \forall \psi \in C_0^\infty(\mathbb{R}^3), \end{aligned}$$

which implies that $J_T'(u_\lambda) = 0$.

Denote $v_n := u_n - u_\lambda$. By (f_1) and [23, Lemma 2.2], one obtains that

$$\int_{\mathbb{R}^3}\left(F(x,u_n) - F(x,u_\lambda) - F(x,v_n)\right)dx = o(1) \tag{2.15}$$

and

$$\int_{\mathbb{R}^3} (f(x,u_n)u_n - f(x,u_\lambda)u_\lambda - f(x,v_n)v_n)\, dx = o(1). \tag{2.16}$$

From the Brezis–Lieb lemma (see [6]), we have

$$\int_{\mathbb{R}^3} (|u_n|^\alpha - |u_\lambda|^\alpha - |v_n|^\alpha)\, dx = o(1), \qquad \int_{\mathbb{R}^3} (|u_n|^6 - |u_\lambda|^6 - |v_n|^6)\, dx = o(1). \tag{2.17}$$

Furthermore, by [21, Lemma 2.2], we get

$$\int_{\mathbb{R}^3} (\phi_{u_n}u_n^2 - \phi_{u_\lambda}u_\lambda^2 - \phi_{v_n}v_n^2)\, dx = o(1). \tag{2.18}$$

Hence, using (2.15)–(2.18) and the fact $J_T'(u_\lambda) = 0$, we deduce that

$$\begin{aligned} o(1) &= \langle J_T'(u_n), u_n\rangle - \langle J_T'(u_\lambda), u_\lambda\rangle \\ &= \|v_n\|^2 + B\int_{\mathbb{R}^3} \phi_{v_n}v_n^2 dx + \frac{\alpha AD}{4T^\alpha}\int_{\mathbb{R}^3} |v_n|^\alpha dx - \int_{\mathbb{R}^3} (\lambda f(x,v_n)v_n + v_n^6)\, dx + o(1) \\ &= \langle J_T'(v_n), v_n\rangle + o(1) \end{aligned} \tag{2.19}$$

and

$$\begin{aligned} c_\lambda + o(1) &= I_T(u_n) \\ &= \frac{1}{2}(\|u_\lambda\|^2 + \|v_n\|^2) + \frac{B}{4}\int_{\mathbb{R}^3} (\phi_{u_\lambda}u_\lambda^2 + \phi_{v_n}v_n^2)\, dx \\ &\quad - \int_{\mathbb{R}^3} \lambda\,(F(x,u_\lambda) + F(x,v_n))\, dx - \frac{1}{6}\int_{\mathbb{R}^3} (u_\lambda^6 + v_n^6) dx + o(1) \\ &= \Psi_T(u_\lambda) + \Psi_T(v_n) + o(1). \end{aligned} \tag{2.20}$$

It follows from (2.19) that

$$\|v_n\|^2 \le \lambda \int_{\mathbb{R}^3} f(x,v_n)v_n dx + \int_{\mathbb{R}^3} v_n^6 dx + o(1). \tag{2.21}$$

Now we estimate the right-hand side of the above inequality. By (f_1) and Young's inequality, we have that

$$\begin{aligned} |f(x,u)u| &\le c_0\left(|u|^{\frac{6-p_1}{2}}|u|^{\frac{3(p_1-2)}{2}} + |u|^{\frac{6-p_2}{2}}|u|^{\frac{3(p_2-2)}{2}}\right) \\ &\le C_1\left(\frac{6-p_1}{4}\varepsilon^{\frac{4}{6-p_1}} + \frac{6-p_2}{4}\varepsilon^{\frac{4}{6-p_2}}\right)|u|^2 + C_1\left(\frac{p_1-2}{4}\frac{1}{\varepsilon^{\frac{4}{p_1-2}}} + \frac{p_2-2}{4}\frac{1}{\varepsilon^{\frac{4}{p_2-2}}}\right)|u|^6 \\ &\le C_2\varepsilon^{\frac{4}{6-p_1}}|u|^2 + C_2\frac{1}{\varepsilon^{\frac{4}{p_1-2}}}|u|^6 \end{aligned}$$

for $\varepsilon > 0$ small. Hence, substituting this equality into (2.21) and taking $\varepsilon = \frac{1}{(2\lambda C_2)^{\frac{6-p_1}{4}}}$, we deduce that for $\lambda > 0$ large

$$\begin{aligned} \frac{S}{2}\left(\int_{\mathbb{R}^3} v_n^6 dx\right)^{1/3} &\le \frac{1}{2}\|v_n\|^2 \\ &\le \left(\frac{C_2\lambda}{\varepsilon^{\frac{4}{p_1-2}}} + 1\right)\int_{\mathbb{R}^3} |v_n|^6 dx + o(1) \\ &\le C_3\lambda^{\frac{4}{p_1-2}}\int_{\mathbb{R}^3} |v_n|^6 dx + o(1). \end{aligned} \tag{2.22}$$

Let $\int_{\mathbb{R}^3} |v_n|^6 dx \longrightarrow l \geq 0$. If $l > 0$, then (2.22) implies that $l \geq \left(\frac{S}{2C_3}\right)^{\frac{3}{2}} \frac{1}{\lambda^{\frac{6}{p_1-2}}}$. Choose $T > 0$ such that

$$\left(\frac{|4-\theta|}{4\theta} 2^{\frac{2}{\alpha}} C_0 + \frac{\alpha C_0}{\theta} 2^{\frac{2}{\alpha}}\right) S_{12/5}^{-1} T^2 \leq \frac{1}{2}\left(\frac{1}{2} - \frac{1}{\theta}\right). \tag{2.23}$$

Then, by $J'_T(u_\lambda) = 0$, we obtain that

$$\begin{aligned}
\Psi_T(u_\lambda) &= \Psi_T(u_\lambda) - \frac{1}{\theta}\langle J'_T(u_\lambda), u_\lambda\rangle \\
&\geq \left(\frac{1}{2} - \frac{1}{\theta}\right)\|u_\lambda\|^2 + \left(\frac{1}{4} - \frac{1}{\theta}\right) B \int \phi_{u_\lambda} u_\lambda^2 dx - \frac{\alpha AD}{4\theta T^\alpha} \int |u_\lambda|^\alpha dx \\
&\geq \left[\left(\frac{1}{2} - \frac{1}{\theta}\right) - \left(\frac{|4-\theta|}{4\theta} 2^{\frac{2}{\alpha}} C_0 + \frac{\alpha C_0}{\theta} 2^{\frac{2}{\alpha}}\right) S_{12/5}^{-1} T^2\right] \|u_\lambda\|^2 \\
&\geq 0.
\end{aligned} \tag{2.24}$$

Hence, using (2.24), (2.20) and (2.19), we deduce that

$$\begin{aligned}
c_\lambda + o(1) &\geq \Psi(v_n) + o(1) \\
&= \Psi(v_n) - \frac{1}{\theta}\langle J'_T(v_n), v_n\rangle + o(1) \\
&\geq \left(\frac{1}{2} - \frac{1}{\theta}\right)\|v_n\|^2 + \left(\frac{1}{4} - \frac{1}{\theta}\right) B \int_{\mathbb{R}^3} \phi_{v_n} v_n^2 dx - \frac{\alpha AC}{4\theta T^\alpha} \int_{\mathbb{R}^3} |v_n|^\alpha dx \\
&\quad + \left(\frac{1}{\theta} - \frac{1}{6}\right) \int_{\mathbb{R}^3} v_n^6 dx + o(1) \\
&\geq \left[\left(\frac{1}{2} - \frac{1}{\theta}\right) - \left(\frac{|4-\theta|}{4\theta} 2^{\frac{2}{\alpha}} C_0 + \frac{\alpha C_0}{\theta} 2^{\frac{2}{\alpha}}\right) S_{12/5}^{-1} T^2\right] \|v_n\|^2 \\
&\quad + \left(\frac{1}{\theta} - \frac{1}{6}\right) \int_{\mathbb{R}^3} v_n^6 dx + o(1) \\
&\geq \left(\frac{1}{\theta} - \frac{1}{6}\right) \int_{\mathbb{R}^3} v_n^6 dx + o(1),
\end{aligned}$$

which implies that

$$c_\lambda \geq \left(\frac{1}{\theta} - \frac{1}{6}\right) l \geq \left(\frac{1}{\theta} - \frac{1}{6}\right)\left(\frac{S}{2C_3}\right)^{\frac{3}{2}} \frac{1}{\lambda^{\frac{6}{p_1-2}}} =: \frac{D_1}{\lambda^{\frac{6}{p_1-2}}},$$

a contradiction. Therefore $l = 0$ and $u_n \to u$ in H. □

Proof of Theorem 1.1. In view of Lemmas 2.2 and 2.3, there is a sequence $(u_n) \subset H$ such that

$$I_T(u_n) \to c_\lambda \in \left(0, \frac{D_0}{\lambda^{\frac{2}{q-2}}}\right] \quad \text{and} \quad I'_T(u_n) \to 0.$$

Since $p_1 > 3q - 4$, we find $\lambda_1 \geq 1$ large enough such that

$$c_\lambda \leq \frac{D_0}{\lambda^{\frac{2}{q-2}}} < \frac{D_1}{\lambda^{\frac{6}{p_1-2}}} \quad \text{for } \lambda > \lambda_1.$$

Thus, by Lemma 2.4, one sees that $u_n \to u_\lambda$ in H, $I_T(u_\lambda) = c_\lambda$ and $I'_T(u_\lambda) = 0$. Next we show that $u_\lambda \to 0$ as $\lambda \to +\infty$. It follows from the properties of χ and (2.23) that

$$\begin{aligned}
\frac{D_0}{\lambda^{\frac{2}{q-2}}} \geq c_\lambda &= I_T(u_\lambda) - \frac{1}{\theta}\langle I'_T(u_\lambda), u_\lambda\rangle \\
&\geq \left(\frac{1}{2} - \frac{1}{\theta}\right)\|u_\lambda\|^2 + \left(\frac{1}{4} - \frac{1}{\theta}\right) K_T(u_\lambda) \int_{\mathbb{R}^3} \phi_{u_\lambda} u_\lambda^2 dx \\
&\quad - \frac{\alpha}{4\theta T^\alpha}\chi'\left(\frac{\|u_\lambda\|_\alpha^\alpha}{T^\alpha}\right)\|u_\lambda\|_\alpha^\alpha \int_{\mathbb{R}^3} \phi_{u_\lambda} u_\lambda^2 dx \\
&\geq \left[\left(\frac{1}{2} - \frac{1}{\theta}\right) - \left(\frac{|4-\theta|}{4\theta} C_0 2^{\frac{2}{\alpha}} + \frac{\alpha C_0}{\theta} 2^{\frac{2}{\alpha}}\right) S_{12/5}^{-1} T^2\right]\|u_\lambda\|^2 \\
&\geq \frac{1}{2}\left(\frac{1}{2} - \frac{1}{\theta}\right)\|u_\lambda\|^2.
\end{aligned}$$

Since $c_\lambda \to 0$ as $\lambda \to +\infty$, the above inequality implies that $u_\lambda \to 0$ as $\lambda \to +\infty$. Hence there exists $\lambda^* \geq \lambda_1$ such that $\|u_\lambda\|_\alpha \leq S_{12/5}^{-\frac{1}{2}}\|u_\lambda\| \leq T$ for $\lambda \geq \lambda^*$. So we also get that $I(u_\lambda) = c_\lambda$ and $I'(u_\lambda) = 0$, i.e., u_λ is a nontrivial solution of original problem (1.1). This completes the proof. □

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References

[1] A. Ambrosetti, P. H. Rabinowitz, Dual variational methods in critical point theory and applications, *J. Functional Analysis* **14**(1973), 349–381. MR0370183

[2] A. Ambrosetti, D. Ruiz, Multiple bound states for the Schrödinger–Poisson problem, *Commun. Contemp. Math.* **10**(2008), No. 3, 391–404. MR2417922; https://doi.org/10.1142/S021919970800282X

[3] A. Azzollini, A. Pomponio, Ground state solutions for the nonlinear Schrödinger–Maxwell equations, *J. Math. Anal. Appl.* **345**(2008), No. 1, 90–108. MR2422637; https://doi.org/10.1016/j.jmaa.2008.03.057

[4] A. Azzollini, P. d'Avenia, A. Pomponio, On the Schrödinger-Maxwell equations under the effect of a general nonlinear term, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **27**(2010), No. 2, 779–791. MR2595202; https://doi.org/10.1016/j.anihpc.2009.11.012

[5] V. Benci, D. Fortunato, An eigenvalue problem for the Schrödinger–Maxwell equations, *Topol. Methods Nonlinear Anal.* **11**(1998), No. 2, 283–293. MR1659454; https://doi.org/10.12775/TMNA.1998.019

[6] H. Brézis, E. Lieb, A relation between pointwise convergence of functions and convergence of functionals, *Proc. Amer. Math. Soc.* **88**(1983), No. 3, 486–490. MR0699419; https://doi.org/10.2307/2044999

[7] J. Chen, Multiple positive solutions of a class of non autonomous Schrödinger–Poisson systems, *Nonlinear Anal. Real World Appl.* **21**(2015), 13–26. MR3261575; https://doi.org/10.1016/j.nonrwa.2014.06.002

[8] T. D'Aprile, D. Mugnai, Solitary waves for nonlinear Klein–Gordon–Maxwell and Schrödinger–Maxwell equations, *Proc. Roy. Soc. Edinburgh Sect. A* **134**(2004), No. 5, 893–906. MR2099569; https://doi.org/10.1017/S030821050000353X

[9] T. D'Aprile, D. Mugnai, Non-existence results for the coupled Klein–Gordon–Maxwell equations, *Adv. Nonlinear Stud.* **4**(2004), No. 3, 307–322. MR2079817; https://doi.org/10.1515/ans-2004-0305

[10] P. d'Avenia, Non-radially symmetric solutions of nonlinear Schrödinger equation coupled with Maxwell equations, *Adv. Nonlinear Stud.* **2**(2002), No. 2, 177–192. MR1896096

[11] L. Jeanjean, S. Le Coz, An existence and stability result for standing waves of nonlinear Schrödinger equations, *Adv. Differential Equations* **11**(2006), No. 7, 813–840. MR2236583

[12] A. Mao, L. Yang, A. Qian, S. Luan, Existence and concentration of solutions of Schrödinger–Poisson system, *Appl. Math. Lett.* **68**(2017), 8–12. MR3614271; https://doi.org/10.1016/j.aml.2016.12.014

[13] D. Ruiz, The Schrödinger–Poisson equation under the effect of a nonlinear local term, *J. Funct. Anal.* **237**(2006), No. 2, 655–674. MR2230354; https://doi.org/10.1016/j.jfa.2006.04.005

[14] J. Sun, S. Ma, Ground state solutions for some Schrödinger–Poisson systems with periodic potentials, *J. Differential Equations* **260**(2016), No. 3, 2119–2149. MR3427661; https://doi.org/10.1016/j.jde.2015.09.057

[15] M. Willem, *Minimax theorems*, Progress in Nonlinear Differential Equations and their Applications, Vol. 24, Birkhäuser, Boston, 1996. MR1400007; https://doi.org/10.1007/978-1-4612-4146-1

[16] Y. Ye, C.-L. Tang, Existence and multiplicity of solutions for Schrödinger–Poisson equations with sign-changing potential, *Calc. Var. Partial Differential Equations* **53**(2015), No. 1–2, 383–411. MR3336325; https://doi.org/10.1007/s00526-014-0753-6

[17] Y. Ye, C. Tang, Existence and multiplicity results for the Schrödinger–Poisson system with superlinear or sublinear terms (in Chinese), *Acta Math. Sci. Ser. A Chin. Ed.* **35**(2015), No. 4, 668–682. MR3393048

[18] J. Zhang, On the Schrödinger–Poisson equations with a general nonlinearity in the critical growth, *Nonlinear Anal.* **75**(2012), No. 18, 6391–6401. MR2965225; https://doi.org/10.1016/j.na.2012.07.008

[19] J. Zhang, On ground state and nodal solutions of Schrödinger–Poisson equations with critical growth, *J. Math. Anal. Appl.* **428**(2015), No. 1, 387–404. MR3326993; https://doi.org/10.1016/j.jmaa.2015.03.032

[20] J. Zhang, Ground state and multiple solutions for Schrödinger–Poisson equations with critical nonlinearity, *J. Math. Anal. Appl.* **440**(2016), No. 2, 466–482. MR3484979; https://doi.org/10.1016/j.jmaa.2016.03.062

[21] L. Zhao, F. Zhao, On the existence of solutions for the Schrödinger–Poisson equations, *J. Math. Anal. Appl.* **346**(2008), No. 1, 155–169. MR2428280; https://doi.org/10.1016/j.jmaa.2008.04.053

[22] L. Zhao, F. Zhao, Positive solutions for Schrödinger–Poisson equations with a critical exponent, *Nonlinear Anal.* **70**(2009), No. 6, 2150–2164. MR2498302; https://doi.org/10.1016/j.na.2008.02.116

[23] X. P. Zhu, D. M. Cao, The concentration-compactness principle in nonlinear elliptic equations, *Acta Math. Sci. (English Ed.)* **9**(1989), No. 3, 307–328. MR1043058