Electronic Journal of Qualitative Theory of Differential Equations
2018, No. **6**, 1–18; https://doi.org/10.14232/ejqtde.2018.1.6 www.math.u-szeged.hu/ejqtde/

Homoclinic orbits for a class of second-order Hamiltonian systems with concave–convex nonlinearities

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Received 8 June 2017, appeared 12 February 2018
Communicated by Gabriele Bonanno

Abstract. In this paper, we study the existence of multiple homoclinic solutions for the following second order Hamiltonian systems

$$\ddot{u}(t) - L(t)u(t) + \nabla W(t, u(t)) = 0,$$

where $L(t)$ satisfies a boundedness assumption which is different from the coercive condition and W is a combination of subquadratic and superquadratic terms.

Keywords: multiple homoclinic solutions, concave–convex nonlinearities, second-order Hamiltonian systems, bounded potential, variational methods.

2010 Mathematics Subject Classification: 35J20, 34C37.

1 Introduction and main results

In this paper, we consider the following second-order Hamiltonian systems

$$\ddot{u}(t) - L(t)u(t) + \nabla W(t, u(t)) = 0, \tag{1.1}$$

where $W : R \times R^N \to R$ is a C^1-map and $L : R \to R^{N^2}$ is a matrix valued function. We say that a solution $u(t)$ of problem (1.1) is nontrivial homoclinic (to 0) if $u \not\equiv 0$, $u(t) \to 0$ as $t \to \pm\infty$.

The dynamical system is a class of classical mathematical model to describe the evolution of natural status, which have been studied by many mathematicians (see [1–41]). It was shown by Poincaré that the homoclinic orbits are very important in study of the behavior of dynamical systems. In the last decades, variational methods and the critical point theorem have been used successfully in studying the existence and multiplicity of homoclinic solutions for differential equations by many mathematicians (see [1,3–5,8–17,19–21,23,24,27–29,32–41] and the references therein).

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In [20], Rabinowitz made use of the periodicity of $L(t)$ and $W(t,x)$ to obtain the existence of nontrivial homoclinic solution for problem (1.1) as the limit of a sequence of periodic solutions. While $L(t)$ and $W(t,x)$ are neither independent of t nor periodic in t, the problem is quite different from the periodic one since the lack of compactness. In order to get the compactness back, Rabinowitz and Tanaka [21] introduced the following coercive condition on $L(t)$.

(L_0) $L \in C(R, R^{N^2})$ is a symmetric and positively definite matrix for all $t \in R$ and there exists a continuous function $l : R \to R$ such that $l(t) > 0$ for all $t \in R$ and

$$(L(t)x, x) \geq l(t)|x|^2 \quad \text{with } l(t) \to \infty \ \text{ as } |t| \to \infty.$$

With condition (L_0), Omana and Willem [16] obtained a new compact embedding theorem and got the existence and multiplicity of homoclinic solutions for problem (1.1). It is obvious that there are many functions which do not satisfy condition (L_0). For instance, let $L(t) = (4 + \arctan t)\, Id_n$, where Id_n is the $n \times n$ identity matrix.

If there is no periodic or coercive assumption, it is difficult to obtain the compactness of the embedding theorem. Therefore, there are only few papers concerning about this kind of situation. In the present paper, we consider the following condition on $L(t)$.

(L) $L \in C(R, R^{N^2})$ is a symmetric and positively definite matrix for all $t \in R$ and there exist constants $0 < \tau_2 < \tau_1$ such that

$$\tau_1 |x|^2 \geq (L(t)x, x) \geq \tau_2 |x|^2 \quad \text{for all } (t, x) \in R \times R^N.$$

Condition (L) was introduced by Zhang, Xiang and Yuan in [41]. With condition (L), the authors obtained a new compact embedding theorem. In this paper, W is assumed to be of the following form

$$W(t,x) = \lambda F(t,x) + K(t,x). \tag{1.2}$$

The existence and multiplicity of homoclinic for problem (1.1) with mixed nonlinearities have been considered in some previous works. In 2011, Yang, Chen and Sun [33] showed the existence of infinitely many homoclinic solutions for problem (1.1). In a recent paper [32], Wu, Tang and Wu obtained the existence and nonuniqueness of homoclinic solutions for problem (1.1) with some nonlinear terms which are more general than those in [33]. However, condition (L_0) is needed in both of above papers. In this paper, we take advantage of condition (L) to study problem (1.1) with concave-convex nonlinearities. Now we state our main results.

Theorem 1.1. *Suppose that (L), (1.2) and the following conditions hold*

(W_1) $K(t,x) = a_1(t)|x|^s$, *where $s > 2$ and $a_1 \in L^\infty(R, R)$;*

(W_2) *there exists an open interval $\Lambda \subset R$ such that $a_1(t) > 0$ for all $t \in \Lambda$;*

(W_3) $a_1(t) \to 0$ *as* $|t| \to +\infty$;

(W_4) $F(t, 0) = 0$ *and* $F(t,x) \in C^1(R \times R^N, R)$;

(W_5) *there exist $\bar{t} \in R$, $r_0 \in (1,2)$ and $b_0 > 0$ such that $F(\bar{t}, x) \geq b_0 |x|^{r_0}$ for all $x \in R^N$;*

(W_6) *for any* $(t,x) \in R \times R^N$, *there exist* $r_1, r_2 \in (1,2)$ *such that*

$$|\nabla F(t,x)| \le b_1(t)|x|^{r_1-1} + b_2(t)|x|^{r_2-1}, \tag{1.3}$$

where $b_1(t) \in L^{\beta_1}(R,R^+)$ *and* $b_2(t) \in L^{\beta_2}(R,R^+)$ *for some* $\beta_1 \in (1, \frac{2}{2-r_1}]$ *and* $\beta_2 \in (1, \frac{2}{2-r_2}]$.

Then there exists $\lambda_1 > 0$ *such that for all* $\lambda \in (0,\lambda_1)$, *problem* (1.1) *possesses at least two homoclinic solutions.*

Remark 1.2. In [9,33,36], the authors also considered the concave-convex nonlinearities, but in [9,33], $L(t)$ was required to satisfy the coercive condition (L_0), which is different from condition (L). In [36], only a class of specific nonlinearities was considered and the concave term was assumed to be positive.

Theorem 1.3. *Suppose that* (L), (1.2), (W_1)–(W_4), (W_6) *and the following condition hold*

(W_7) $F(t,-x) = F(t,x)$ *for all* $(t,x) \in R \times R^N$.

Then problem (1.1) *possesses infinitely many homoclinic solutions.*

Remark 1.4. Note that $F(t,x) \equiv 0$ satisfies the conditions of Theorem 1.3. Moreover, $F(t,x)$ and $W(t,x)$ can change signs, which is different from the results in [9,33,36].

In the following theorems, we consider the case when the convex term is positive.

Theorem 1.5. *Suppose that* (L), (1.2), (W_4)–(W_6) *and the following conditions hold*

(W_8) $K(t,x) = a_2(t)G(x)$, *where* $a_2(t) \in L^\infty(R,R)$;

(W_9) $a_2(t) > 0$ *for all* $t \in R$ *and* $a_2(t) \to 0$ *as* $t \to \infty$;

(W_{10}) $G \in C^1(R^N,R)$, $G(0) = 0$ *and* $\nabla G(x) = o(|x|)$ *as* $x \to 0$;

(W_{11}) $G(x)/|x|^2 \to +\infty$ *as* $|x| \to \infty$;

(W_{12}) *there exist* $\nu > 2$ *and* $d_1, \rho_\infty > 0$ *such that*

$$(\nabla G(x),x) - \nu G(x) \ge -d_1|x|^2 \quad \textit{for all } |x| \ge \rho_\infty.$$

If $G(x) \ge 0$, *there exists* $\lambda_2 > 0$ *such that for all* $\lambda \in (0,\lambda_2)$, *problem* (1.1) *possesses at least two homoclinic solutions.*

Remark 1.6. In Theorem 1.5, (W_{10})–(W_{12}) are all local conditions. There are functions satisfying the conditions (W_{10})–(W_{12}). For example, let

$$G(x) = \begin{cases} -|x|^4 + |x|^3 & \text{for } |x| \le \frac{4}{5}, \\ \left(|x| - \dfrac{4+4^{\frac{1}{3}}}{5}\right)^4 + \dfrac{64-4^{\frac{4}{3}}}{625} & \text{for } |x| \ge \frac{4}{5}. \end{cases} \tag{1.4}$$

Obviously, with function (1.4), $K(t,x)$ does not satisfy the following global condition

$$(\nabla K(t,x),x) - 2K(t,x) \ge 0 \quad \text{for all } (t,x) \in R \times R^N,$$

which is needed in many papers [1,8,10–12,16,17,20,21,27,29,33–37,41].

With a symmetric condition, we can obtain infinitely many homoclinic solutions for problem (1.1).

Theorem 1.7. *Suppose that* (L), (1.2), (W_4), (W_6)–(W_{12}) *and the following condition hold*

(W_{13}) $G(-x) = G(x) \geq 0$ *for all* $(t, x) \in R \times R^N$.

Then problem (1.1) *possesses infinitely many homoclinic solutions.*

In our proofs, the following critical point theorems are needed.

Lemma 1.8 (Lu [13]). *Let* X *be a real reflexive Banach space and* $\Omega \subset X$ *a closed bounded convex subset of* X. *Suppose that* $\varphi : X \to R$ *is a weakly lower semi-continuous (w.l.s.c. for short) functional. If there exists a point* $x_0 \in \Omega \setminus \partial\Omega$ *such that*

$$\varphi(x) > \varphi(x_0), \qquad \forall\, x \in \partial\Omega$$

then there is a $x^* \in \Omega \setminus \partial\Omega$ *such that*

$$\varphi(x^*) = \inf_{x\in\Omega} \varphi(x).$$

Lemma 1.9 (Chang [7]). *Suppose that* E *is a Hilbert space,* $I \in C^1(E, R)$ *is even with* $I(0) = 0$, *and that*

- (Z_1) *there are constants* ϱ, $\alpha > 0$ *and a finite dimensional linear subspace* X *such that* $I|_{X^\perp \bigcap \partial B_\varrho} \geq \alpha$, *where* $B_\varrho = \{u \in E : \|u\| \leq \varrho\}$;
- (Z_2) *there is a sequence of linear subspaces* $\tilde{X}_m$, $\dim \tilde{X}_m = m$, *and there exists* $r_m > 0$ *such that*

$$I(u) \leq 0 \quad \text{on } \tilde{X}_m \setminus B_{r_m}, \qquad m = 1, 2, \ldots$$

If, further, I *satisfies the* $(PS)^*$ *condition with respect to* $\{\tilde{X}_m \mid m = 1, 2, \ldots\}$, *then* I *possesses infinitely many distinct critical points corresponding to positive critical values.*

We recall that a functional I is said to satisfy the $(PS)^*$ condition with respect to $\{\tilde{X}_m \mid m = 1, 2, \ldots\}$, if any sequence $\{x_m \mid x_m \in \tilde{X}_m\}$, satisfying

$$|I(x_m)| < \infty \quad \text{and} \quad I'|_{\tilde{X}_m}(x_m) \to 0,$$

has a convergent subsequence.

2 Preliminaries

Set

$$E = \left\{u \in H^1(R, R^N) : \int_R |\dot{u}(t)|^2 + (L(t)u(t), u(t))dt < +\infty\right\},$$

with the inner product

$$(u, v)_E := \int_R ((\dot{u}, \dot{v}) + (L(t)u(t), v(t)))\, dt$$

and the norm $\|u\| = (u,u)_E^{1/2}$. Note that the embedding $E \hookrightarrow L^p(R, R^N)$ is continuous for all $p \in [2, +\infty]$, then there exists $C_p > 0$ such that

$$\|u\|_p \le C_p \|u\| \quad \text{for all } u \in E. \tag{2.1}$$

Furthermore, the corresponding functional of (1.1) is defined by

$$I(u) = \frac{1}{2}\|u\|^2 - \int_R W(t, u(t))dt. \tag{2.2}$$

Let $L^2_\varphi(R, R^N)$ be the weighted space of measurable functions $u : R \to R^N$ under the norm

$$\|u\|_{L^2_\varphi} = \left(\int_R \varphi(t)|u(t)|^2 dt\right)^{1/2}, \tag{2.3}$$

where $\varphi(t) \in C(R, R^+)$.

With condition (L), Lv and Tang obtained the following compact embedding theorem.

Lemma 2.1 (Lv and Tang [14])**.** *Suppose that assumption (L) holds. Then the imbedding of E in $L^p_\omega(R, R^N)$ is compact, where $p \in (1, 2)$, $\gamma \in (1, \frac{2}{2-p}]$ and $\omega \in L^\gamma(R, R^+)$.*

The following lemma is a complement to Lemma 2.1 with the case $p = 2$.

Lemma 2.2 (Yuan and Zhang [37])**.** *Under condition (L), the embedding $E \hookrightarrow L^2_h(R, R^N)$ is continuous and compact for any $h(t) \in C(R, R^+)$ with $h(t) \to 0$ as $|t| \to \infty$.*

Then we can prove the following lemma.

Lemma 2.3. *Suppose that the conditions (W_6), (W_8), (W_9), (W_{10}) hold, then we have $\nabla W(t, u_k) \to \nabla W(t, u)$ in $L^2(R, R^N)$ if $u_k \rightharpoonup u$ in E.*

Proof. Assume that $u_k \rightharpoonup u$ in E. By the Banach–Steinhaus theorem and (2.1), there exists $D > 0$ such that

$$\sup_{k\in N} \|u_k\|_\infty \le D \quad \text{and} \quad \|u\|_\infty \le D. \tag{2.4}$$

We can deduce from (W_{10}) and (2.4) that there exists $d_2 > 0$ such that

$$|\nabla G(u_k)| \le d_2|u_k(t)| \quad \text{for all } t \in R. \tag{2.5}$$

It follows from (1.2), (W_6) and (2.5) that

$$\begin{aligned}
&|\nabla W(t, u_k(t)) - \nabla W(t, u(t))|^2 \\
&\quad \le 8\lambda b_1^2(t)(|u_k(t)|^{2r_1-2} + |u(t)|^{2r_1-2}) + 8\lambda b_2^2(t)(|u_k(t)|^{2r_2-2} + |u(t)|^{2r_2-2}) \\
&\qquad + 4d_2 a_2^2(t)(|u_k(t)|^2 + |u(t)|^2) \\
&\quad \le 8\lambda b_1^2(t)(|u_k(t) - u(t)|^{2r_1-2} + 2|u(t)|^{2r_1-2}) + 8\lambda b_2^2(t)(|u_k(t) - u(t)|^{2r_2-2} + 2|u(t)|^{2r_2-2}) \\
&\qquad + 4d_2 a_2^2(t)(|u_k(t)|^2 + |u(t)|^2) \\
&\quad \le 8\lambda b_1^2(t)((2D)^{\eta_1}|u_k(t) - u(t)|^{2r_1-2-\eta_1} + 2D^{\eta_1}|u(t)|^{2r_1-2-\eta_1}) \\
&\qquad + 8\lambda b_2^2(t)((2D)^{\eta_2}|u_k(t) - u(t)|^{2r_2-2-\eta_2} + 2D^{\eta_2}|u(t)|^{2r_2-2-\eta_2}) \\
&\qquad + 4d_2 a_2^2(t)(|u_k(t) - u(t)|^2 + 2|u(t)|^2),
\end{aligned}$$

where $\eta_i = r_i - 2 + \frac{r_i}{\beta_i - 1} (i = 1, 2)$. By Lemma 2.1, $u_k(t) \to u(t)$ in $L^p_\omega(R, R^N)$, for any $p \in (1, 2)$, $\gamma \in (1, \frac{2}{p-2}]$ and $\omega \in L^\gamma(R, R^+)$. Passing to a subsequence if necessary, it can be assumed that

$$\sum_{k=1}^{\infty} \|u_k - u\|_{L^p_\omega} < \infty,$$

which implies that $u_k(t) \to u(t)$ for a.e. $t \in R$. Set

$$\psi = \sum_{k=1}^{\infty} |u_k(t) - u(t)|.$$

Then we can get that $\psi \in L^{r_i}_{b_i}(R, R^N)$, for any $i = 1, 2$. By (W_6) and the definition of η_i, we have

$$\begin{aligned}
\int_R b_i^2(t)|u_k(t) - u(t)|^{2r_i - 2 - \eta_i} dt &\le \int_R b_i^2(t) \psi^{2r_i - 2 - \eta_i} dt \\
&= \int_R \left(|b_i(t)|^{\frac{2+\eta_i}{r_i}} \right) \left(|b_i(t)|^{\frac{2r_i - 2 - \eta_i}{r_i}} \psi^{2r_i - 2 - \eta_i} \right) dt \\
&\le \left(\int_R |b_i(t)|^{\frac{2+\eta_i}{2 - r_i + \eta_i}} dt \right)^{\frac{2 - r_i + \eta_i}{r_i}} \left(\int_R |b_i(t)| \psi^{r_i} dt \right)^{\frac{2(r_i - 1) - \eta_i}{r_i}} \\
&= \left(\int_R |b_i(t)|^{\beta_i} dt \right)^{\frac{2 - r_i + \eta_i}{r_i}} \left(\int_R |b_i(t)| \psi^{r_i} dt \right)^{\frac{2(r_i - 1) - \eta_i}{r_i}} \\
&< \infty
\end{aligned}$$

for any $i = 1, 2$. Similarly, we can obtain

$$\int_R b_i^2(t)|u(t)|^{2r_i - 2 - \eta_i} dt < \infty.$$

Furthermore, (W_9) and Lemma 2.2 show that

$$\int_R a_2^2(t)|u_k(t) - u(t)|^2 dt < \infty \quad \text{and} \quad \int_R a_2^2(t)|u(t)|^2 dt < \infty.$$

Using Lebesgue's dominated convergence theorem, the lemma is proved. □

Remark 2.4. Obviously, the result of Lemma 2.3 still holds under the conditions (W_1), (W_3), (W_6).

Similar to the proof of Lemma 2.3 in [9], we can see that $I \in C^1(E, R)$ is w.l.s.c. and

$$\begin{aligned}
\langle I'(u), v \rangle &= \int_R ((\dot{u}(t), \dot{v}(t)) + (L(t)u(t), v(t))) dt - \int_R (\nabla W(t, u(t)), u(t)) dt \\
&= \int_R ((\dot{u}(t), \dot{v}(t)) + (L(t)u(t), v(t))) dt \\
&\quad - \lambda \int_R (\nabla F(t, u(t)), v(t)) dt - \int_R (\nabla K(t, u(t)), v(t)) dt
\end{aligned}$$

for any $v \in E$, which implies that

$$\langle I'(u), u \rangle = \|u\|^2 - \lambda \int_R (\nabla F(t, u(t)), u(t)) dt - \int_R (\nabla K(t, u(t)), u(t)) dt. \tag{2.6}$$

Remark 2.5. Similar to Lemma 3.1 in [41], under condition (L), all the critical points of I are homoclinic solutions for problem (1.1).

3 Proof of Theorem 1.1

The existence of homoclinic solution is obtained by the Mountain Pass Theorem with (C) condition which is stated as follows.

Lemma 3.1 (See [2]). *Let E be a real Banach space and $I : R \to R^N$ be a C^1-smooth functional and satisfy the (C) condition that is, (u_j) has a convergent subsequence in $W^{1,2}(R, R^N)$ whenever $\{I(u_j)\}$ is bounded and $\|I'(u_j)\|(1 + \|u_j\|) \to 0$ as $j \to \infty$. If I satisfies the following conditions:*

(i) $I(0) = 0$,

(ii) *there exist constants* $\varrho, \alpha > 0$ *such that* $I_{|\partial B_\varrho(0)} \geq \alpha$,

(iii) *there exists* $e \in E \setminus \bar{B}_\varrho(0)$ *such that* $I(e) \leq 0$,

where $B_\varrho(0)$ is an open ball in E of radius ϱ centred at 0, then I possesses a critical value $c \geq \alpha$ given by

$$c = \inf_{g \in \Gamma} \max_{s \in [0,1]} I(g(s)),$$

where

$$\Gamma = \{g \in C([0,1], E) : g(0) = 0, \ \ g(1) = e\}.$$

Lemma 3.2. *Suppose the conditions of Theorem 1.1 hold, then there exist λ_1, ϱ_1, $\alpha_1 > 0$, such that $I|_{\partial B_{\varrho_1}} \geq \alpha_1$ for all $\lambda \in (0, \lambda_1)$, where $B_{\varrho_1} = \{u \in E : \|u\| \leq \varrho_1\}$.*

Proof. By (W_4) and (W_6), we can deduce that

$$|(\nabla F(t,x), x)| \leq b_1(t)|x|^{r_1} + b_2(t)|x|^{r_2} \tag{3.1}$$

and

$$|F(t,x)| \leq \frac{1}{r_1} b_1(t)|x|^{r_1} + \frac{1}{r_2} b_2(t)|x|^{r_2} \tag{3.2}$$

for all $(t,x) \in R \times R^N$. It follows from (2.2), (1.2), (W_1), (W_6), (3.2) and (2.1) that

$$\begin{aligned}
I(u) &= \frac{1}{2}\|u\|^2 - \lambda \int_R F(t,u(t))dt - \int_R a_1(t)|u(t)|^s dt \\
&\geq \frac{1}{2}\|u\|^2 - \lambda \left(\frac{1}{r_1} \int_R b_1(t)|u(t)|^{r_1} dt + \frac{1}{r_2} \int_R b_2(t)|u(t)|^{r_2} dt \right) - \|a_1\|_\infty \|u\|_\infty^{s-2} \int_R |u(t)|^2 dt \\
&\geq \frac{1}{2}\|u\|^2 - \lambda \left(\frac{1}{r_1} C_{r_1\beta_1^*}^{r_1} \|b_1\|_{\beta_1} \|u\|^{r_1} + \frac{1}{r_2} C_{r_2\beta_2^*}^{r_2} \|b_2\|_{\beta_2} \|u\|^{r_2} \right) - C_2^2 C_\infty^{s-2} \|a_1\|_\infty \|u\|^s \\
&\geq \left(\frac{1}{8} + \left(\frac{1}{8} - \frac{\lambda}{r_1} C_{r_1\beta_1^*}^{r_1} \|b_1\|_{\beta_1} \|u\|^{r_1} \right) + \left(\frac{1}{8} - \frac{\lambda}{r_2} C_{r_2\beta_2^*}^{r_2} \|b_2\|_{\beta_2} \|u\|^{r_2} \right) \right. \\
&\quad \left. + \left(\frac{1}{8} - C_2^2 C_\infty^{s-2} \|a_1\|_\infty \|u\|^{s-2} \right) \right) \|u\|^2,
\end{aligned}$$

where $\frac{1}{\beta_i} + \frac{1}{\beta_i^*} = 1$ $(i = 1, 2)$. Choose $\varrho_1 = \left(\frac{1}{8C_2^2 C_\infty^{s-2} \|a_1\|_\infty} \right)^{\frac{1}{s-2}}$, then we can set

$$\lambda_1 = \min \left\{ \frac{r_1}{8C_{r_1\beta_1^*}^{r_1} \|b_1\|_{\beta_1} \varrho_1^{r_1}}, \frac{r_2}{8C_{r_2\beta_2^*}^{r_2} \|b_2\|_{\beta_2} \varrho_1^{r_2}} \right\}.$$

Hence for every $\lambda \in (0, \lambda_1)$ there exist $\varrho_1 > 0$ and $\alpha_1 > 0$ such that $I|_{\partial B_{\varrho_1}} \geq \alpha_1$. $\square$

Lemma 3.3. *Suppose the conditions of Theorem 1.1 hold, then there exists $e_1 \in E$ such that $\|e_1\| > \varrho_1$ and $I(e_1) \le 0$, where ϱ_1 is defined in Lemma 3.2.*

Proof. Choose $\varphi_1 \in C_0^\infty(\Lambda, R^N) \setminus \{0\}$, where Λ is the interval considered in (W_2). Then by (2.2), (W_1), (W_6) and (3.2), for any $\xi \in R^+$, we obtain

$$\begin{aligned}
I(\xi\varphi_1) &= \frac{\xi^2}{2}\|\varphi_1\|^2 - \lambda\int_\Lambda F(t,\xi\varphi_1(t))dt - \xi^s\int_\Lambda a_1(t)|\varphi_1(t)|^s dt \\
&\le \frac{\xi^2}{2}\|\varphi_1\|^2 + \lambda\left(\frac{1}{r_1}\int_R b_1(t)|\xi\varphi_1(t)|^{r_1}dt + \frac{1}{r_2}\int_R b_2(t)|\xi\varphi_1(t)|^{r_2}dt\right) - \xi^s\int_\Lambda a_1(t)|\varphi_1(t)|^s dt \\
&\le \frac{\xi^2}{2}\|\varphi_1\|^2 + \lambda\left(\frac{\xi^{r_1}}{r_1}C_{r_1\beta_1^*}^{r_1}\|b_1\|_{\beta_1}\|\varphi_1\|^{r_1} + \frac{\xi^{r_2}}{r_2}C_{r_2\beta_2^*}^{r_2}\|b_2\|_{\beta_2}\|\varphi_1\|^{r_2}\right) - \xi^s\int_\Lambda a_1(t)|\varphi_1(t)|^s dt,
\end{aligned}$$

which implies that

$$I(\xi\varphi_1) \to -\infty \quad \text{as } \xi \to +\infty.$$

Therefore, there exists $\xi_1 > 0$ such that $I(\xi_1\varphi_1) < 0$ and $\|\xi_1\varphi_1\| > \varrho_1$. Let $e_1 = \xi_1\varphi_1$, we can see $I(e_1) < 0$, which proves this lemma. □

Lemma 3.4. *Suppose the conditions of Theorem 1.1 hold, then I satisfies condition (C).*

Proof. Assume that $\{u_n\}_{n\in N} \subset E$ is a sequence such that $\{I(u_n)\}$ is bounded and $\|I'(u_n)\|(1+\|u_n\|) \to 0$ as $n \to \infty$. Then there exists a constant $M_1 > 0$ such that

$$|I(u_n)| \le M_1, \qquad \|I'(u_n)\|(1+\|u_n\|) \le M_1. \tag{3.3}$$

Subsequently, we show that $\{u_n\}$ is bounded in E. Arguing in an indirect way, we assume that $\|u_n\| \to \infty$ as $n \to \infty$. It follows from (3.3), (2.2), (2.6), (3.1), (3.2) and (2.1) that there exist $M_2, M_3 > 0$ such that

$$\begin{aligned}
o(1) &= \frac{(s+1)M_1}{\|u_n\|^2} \\
&\ge \frac{sI(u_n) + \|I'(u_n)\|(1+\|u_n\|)}{\|u_n\|^2} \\
&\ge \frac{sI(u_n) - \langle I'(u_n), u_n\rangle}{\|u_n\|^2} \\
&= \left(\frac{s}{2}-1\right) - \frac{\lambda\int_R sF(t,u_n(t)) - (\nabla F(t,u_n(t)), u_n(t))dt}{\|u_n\|^2} \\
&\ge \left(\frac{s}{2}-1\right) - \frac{\lambda M_2\int_R b_1(t)|u_n(t)|^{r_1} + b_2(t)|u_n(t)|^{r_2}dt}{\|u_n\|^2} \\
&\ge \left(\frac{s}{2}-1\right) - \lambda M_3\left(\|u_n\|^{r_1-2} + \|u_n\|^{r_2-2}\right) \\
&\to \left(\frac{s}{2}-1\right) \quad \text{as } n \to \infty,
\end{aligned}$$

which is a contradiction. Hence $\{u_n\}$ is bounded in E. Consequently, there exists a subsequence, still denoted by $\{u_n\}$, such that $u_n \rightharpoonup u$ in E. Therefore

$$\langle I'(u_n) - I'(u), u_n - u\rangle \to 0 \quad \text{as } n \to +\infty.$$

By Remark 2.4, we have

$$\int_R (\nabla W(t,u_n) - \nabla W(t,u), u_n - u)dt \to 0 \quad \text{as } n \to +\infty.$$

It follows from (2.6) that

$$\langle I'(u_n) - I'(u), u_n - u\rangle = \|u_n - u\|^2 - \int_R (\nabla W(t,u_n) - \nabla W(t,u), u_n - u)dt,$$

which implies that $\|u_n - u\| \to 0$ as $n \to +\infty$. Hence I satisfies condition (C).

By Lemma 3.1, I possesses a critical value $c \geq \alpha_1 > 0$ given by

$$c = \inf_{g\in\Gamma} \max_{s\in[0,1]} I(g(s)),$$

where

$$\Gamma = \{g \in C([0,1], E) : g(0) = 0,\ g(1) = e_1\}.$$

Hence, there exists $u_0 \in E$ such that

$$I(u_0) = c > 0, \qquad I'(u_0) = 0.$$

Then the function u_0 is a desired homoclinic solution of problem (1.1). Subsequently, we search for the second critical point of I corresponding to negative critical value. □

Lemma 3.5. *Suppose that the conditions of Theorem 1.1 hold, then there exists a critical point of I corresponding to a negative critical value.*

Proof. By (W_4) and (W_5), there exists $\sigma > 0$ such that

$$F(\bar{t}, x) > \frac{1}{2} b_0 |x|^{r_0} \tag{3.4}$$

for all $t \in (\bar{t} - \sigma, \bar{t} + \sigma)$ and $x \in R^N$. Choose $\varphi_2 \in C_0^\infty((\bar{t} - \sigma, \bar{t} + \sigma), R^N) \setminus \{0\}$, then it follows from (2.2), (W_1), (3.4) and $1 < r_0 < 2 < s$ that

$$\begin{aligned} I(\theta\varphi_2) &= \frac{\theta^2}{2}\|\varphi_2\|^2 - \lambda \int_R F(t, \theta\varphi_2(t))dt - \theta^s \int_R a_1(t)|\varphi_2(t)|^s dt \\ &\leq \frac{\theta^2}{2}\|\varphi_2\|^2 - \frac{\theta^{r_0}}{2}\lambda b_0 \int_{\bar{t}-\sigma}^{\bar{t}+\sigma} |\varphi_2(t)|^{r_0} dt + \theta^s \int_{\bar{t}-\sigma}^{\bar{t}+\sigma} |a_1(t)||\varphi_2(t)|^s dt \\ &< 0 \end{aligned}$$

for $\theta > 0$ small enough. By Lemma 3.2 and Lemma 1.8, this lemma is proved. □

By Lemma 3.2–Lemma 3.5, we can see that I possesses at least two distinct nontrivial critical points. By Remark 2.5, problem (1.1) possesses at least two homoclinic solutions.

4 Proof of Theorem 1.3

In this section, we will use Lemma 1.9 to prove the existence of infinitely many homoclinic solutions for problem (1.1).

Lemma 4.1. *Suppose the conditions of Theorem 1.3 hold, then I satisfies (Z_1).*

Proof. Let $\{x_j\}_{j=1}^{\infty}$ be a complete orthonormal basis of E and $X_k = \bigoplus_{j=1}^{k} Z_j$, where $Z_j = \operatorname{span}\{x_j\}$. For any $q \in [2, +\infty]$, we set

$$h_k(q) = \sup_{u \in X_k^{\perp}, \|u\|=1} \|u\|_q. \tag{4.1}$$

It is easy to see that $h_k(q) \to 0$ as $k \to \infty$ for any $q \in [2, +\infty]$. Let $\varrho_2 = 1$, we can deduce from (2.2), (3.2), (W_6) and (4.1) that for any $u \in X_k^{\perp} \bigcap \partial B_{\varrho_2}$

$$\begin{aligned} I(u) &= \frac{1}{2} - \lambda \int_R F(t, u(t))dt - \int_R a_1(t)|u(t)|^s dt \\ &\geq \frac{1}{2} - \frac{\lambda}{r_1} \int_R b_1(t)|u(t)|^{r_1} dt - \frac{\lambda}{r_2} \int_R b_2(t)|u(t)|^{r_2} dt - \|a_1\|_\infty \|u\|_\infty^{s-2} \int_R |u(t)|^2 dt \\ &\geq \frac{1}{2} - \left(\frac{\lambda}{r_1} h_{k_0}^{r_1}(r_1\beta_1^*)\|b_1\|_{\beta_1} + \frac{\lambda}{r_2} h_{k_0}^{r_2}(r_2\beta_2^*)\|b_2\|_{\beta_2} + h_{k_0}^2(2) h_{k_0}^{s-2}(\infty)\|a_1\|_\infty \right), \end{aligned} \tag{4.2}$$

which implies that there exists a $k_0 > 0$ such that $I(u) > \frac{1}{4}$ for all $u \in X_{k_0}^{\perp} \bigcap \partial B_{\varrho_2}$. Hence there exist $\varrho_2, \alpha_2 > 0$ such that $I|_{X_{k_0}^{\perp} \bigcap \partial B_{\varrho_2}} \geq \alpha_2$. □

Lemma 4.2. *Suppose the conditions of Theorem 1.3 hold, then for any $m \in N$, there exist a linear subspace $\tilde{X}_m$ and $r_m > 0$ such that $\dim \tilde{X}_m = m$ and*

$$I(u) \leq 0 \quad \text{on } \tilde{X}_m \setminus B_{r_m}.$$

Proof. By (W_2), there exist $a_0 > 0$ and $\Lambda_0 \subset \Lambda$ such that $a_1(t) > a_0$ for all $t \in \Lambda_0$ with $\operatorname{meas}(\Lambda_0) > 0$. Choose a complete orthonormal basis $\{e_j(t)\}_{j=1}^{\infty}$ of $W_0^{1,2}(\Lambda_0, R^N)$. Subsequently, set $E_j = \operatorname{span}\{e_j(t)\}$ and $\tilde{X}_m = \bigoplus_{j=1}^{m} E_j$. Then there exists a constant $\sigma_m > 0$, such that

$$\|u\|_s \geq \sigma_m \|u\| \tag{4.3}$$

for all $u \in \tilde{X}_m$. For any $u_m \in \tilde{X}_m$, we can see $\operatorname{supp} u_m \subset \Lambda_0$. It follows from (2.2), (2.1), (3.2), (W_1), (W_6) and (4.3) that

$$\begin{aligned} I(u_m) &= \frac{1}{2}\|u_m\|^2 - \lambda \int_R F(t, u_m(t))dt - \int_R a_1(t)|u_m(t)|^s dt \\ &= \frac{1}{2}\|u_m\|^2 - \lambda \int_{t_0-\delta}^{t_0+\delta} F(t, u_m(t))dt - \int_{t_0-\delta}^{t_0+\delta} a_1(t)|u_m(t)|^s dt \\ &\leq \frac{1}{2}\|u_m\|^2 + \frac{\lambda}{r_1} C_{r_1\beta_1^*}^{r_1} \|b_1\|_{\beta_1} \|u_m\|^{r_1} + \frac{\lambda}{r_2} C_{r_2\beta_2^*}^{r_2} \|b_2\|_{\beta_2} \|u_m\|^{r_2} - a_0 \|u_m\|_s^s \\ &\leq \frac{1}{2}\|u_m\|^2 + \frac{\lambda}{r_1} C_{r_1\beta_1^*}^{r_1} \|b_1\|_{\beta_1} \|u_m\|^{r_1} + \frac{\lambda}{r_2} C_{r_2\beta_2^*}^{r_2} \|b_2\|_{\beta_2} \|u_m\|^{r_2} - \sigma_m^s a_0 \|u_m\|^s. \end{aligned}$$

Since $s > 2 > \max\{r_1, r_2\}$, there exists $r_m > 0$ such that $I(u_m) \leq 0$ for all $u_m \in \tilde{X}_m \setminus B_{r_m}$, which proves this lemma. □

Lemma 4.3. *Suppose the conditions of Theorem 1.3 hold, then I satisfies the $(PS)^*$ condition.*

Proof. The proof of this lemma is similar to Lemma 3.4. □

Proof of Theorem 1.3. By Lemmas 4.1–4.3 and Lemma 1.9, I possesses infinitely many distinct critical points corresponding to positive critical values. The proof of Theorem 1.3 is finished. □

5 Proof of Theorem 1.5

Lemma 5.1. *Suppose the conditions of Theorem 1.5 hold, then there exist λ_2, ϱ_3, $\alpha_3 > 0$ such that $I|_{\partial B_{\varrho_3}} \geq \alpha_3$ for all $\lambda \in (0, \lambda_2)$.*

Proof. It follows from (W_{10}) that there exists $\rho_1 > 0$ such that

$$|\nabla G(x)| \leq \frac{|x|}{4C_2^2 \|a_2\|_\infty}, \qquad \forall\ |x| \leq \rho_1.$$

By $G(0) = 0$, we can deduce that

$$\begin{aligned}
|G(x)| &= |G(x) - G(0)| \\
&= \left| \int_0^1 (\nabla G(\phi x), x) d\phi \right| \\
&\leq \int_0^1 |\nabla G(\phi x)||x| d\phi \\
&\leq \int_0^1 \frac{1}{4C_2^2 \|a_2\|_\infty} |\phi x||x| d\phi \\
&\leq \frac{|x|^2}{4C_2^2 \|a_2\|_\infty}
\end{aligned} \tag{5.1}$$

for all $|x| \leq \rho_1$. By (2.2), (5.1), (3.2), (W_5), (W_8) and (2.1), for any $u \in B_{\frac{\rho_1}{C_\infty}}$, we have

$$\begin{aligned}
I(u) &= \frac{1}{2}\|u\|^2 - \lambda \int_R F(t, u(t)) dt - \int_R a_2(t) G(u(t)) dt \\
&\geq \frac{1}{2}\|u\|^2 - \lambda \left(\frac{1}{r_1} \int_R b_1(t)|u(t)|^{r_1} dt + \frac{1}{r_2} \int_R b_2(t)|u(t)|^{r_2} dt \right) - \frac{1}{4C_2^2} \int_R |u(t)|^2 dt \\
&\geq \frac{1}{2}\|u\|^2 - \lambda \left(\frac{1}{r_1} C^{r_1}_{r_1\beta_1^*} \|b_1\|_{\beta_1} \|u\|^{r_1} + \frac{1}{r_2} C^{r_2}_{r_2\beta_2^*} \|b_2\|_{\beta_2} \|u\|^{r_2} \right) - \frac{1}{4}\|u\|^2 \\
&= \left(\frac{1}{12} + \left(\frac{1}{12} - \frac{\lambda}{r_1} C^{r_1}_{r_1\beta_1^*} \|b_1\|_{\beta_1} \|u\|^{r_1-2} \right) + \left(\frac{1}{12} - \frac{1}{r_2} C^{r_2}_{r_2\beta_2^*} \|b_2\|_{\beta_2} \|u\|^{r_2-2} \right) \right) \|u\|^2.
\end{aligned} \tag{5.2}$$

Let $\varrho_3 = \frac{\rho_1}{C_\infty}$. From (5.2), we set

$$\lambda_2 = \min \left\{ \frac{r_1}{12 C^{r_1}_{r_1\beta_1^*} \|b_1\|_{\beta_1} \varrho_3^{r_1-2}}, \frac{r_2}{12 C^{r_2}_{r_2\beta_2^*} \|b_2\|_{\beta_2} \varrho_3^{r_2-2}} \right\},$$

which implies that $I|_{\partial B_{\varrho_3}} > \alpha_3$ for some $\alpha_3 > 0$ and all $\lambda \in (0, \lambda_2)$. Then we finish the proof of this lemma. □

Lemma 5.2. *Suppose the conditions of Theorem 1.5 hold, then there exists $e_2 \in E$ such that $\|e_2\| > \varrho_3$ and $I(e_2) \leq 0$, where ϱ_3 is defined in Lemma 5.1.*

Proof. Choose $e_3 \in C_0^\infty(-1, 1)$ such that $\|e_3\| = 1$. It follows from (W_9) that there exists $\tilde{a} > 0$ such that $a_2(t) \geq \tilde{a}$ for all $t \in (-1, 1)$. We can see that there exist $\tilde{e} > 0$ and $\Upsilon \subset (-1, 1)$ such that $|e_3(t)| \geq \tilde{e}$ for all $t \in \Upsilon$ with meas$(\Upsilon) > 0$. By (W_{11}), for any $A > 0$ there exists $Q > 0$ such that

$$\frac{G(x)}{|x|^2} \geq A$$

for all $|x| \geq Q$, which implies that

$$\int_{\Upsilon} \frac{G(\eta e_3(t))}{|\eta e_3(t)|^2} dt \geq A \,\mathrm{meas}(\Upsilon),$$

for all $\eta \geq Q/\tilde{e}$. Then by the arbitrariness of A, we obtain

$$\int_{\Upsilon} \frac{G(\eta e_3(t))}{|\eta e_3(t)|^2} dt \to \infty \quad \text{as } |\eta| \to \infty. \tag{5.3}$$

By (2.2), (5.3), (2.1), (3.2) and (W_6), we have

$$\begin{aligned}
\frac{I(\eta e_3)}{\eta^2} &= \frac{1}{2} - \lambda \int_R \frac{F(t, \eta e_3(t))}{\eta^2} dt - \int_R \frac{a_2(t) G(\eta e_3(t))}{\eta^2} dt \\
&\leq \frac{1}{2} + \lambda \left(\frac{1}{\eta^2 r_1} \int_R b_1(t) |\eta e_3(t)|^{r_1} dt + \frac{1}{\eta^2 r_2} \int_R b_2(t) |\eta e_3(t)|^{r_2} dt \right) - \tilde{a} \int_{-1}^{1} \frac{G(\eta e_3(t))}{\eta^2} dt \\
&\leq \frac{1}{2} + \lambda \left(\frac{1}{r_1} C_{r_1 \beta_1^*}^{r_1} \|b_1\|_{\beta_1} |\eta|^{r_1 - 2} + \frac{1}{r_2} C_{r_2 \beta_2^*}^{r_2} \|b_2\|_{\beta_2} |\eta|^{r_2 - 2} \right) - \tilde{a} \int_{\Upsilon} \frac{G(\eta e_3(t))}{|\eta e_3(t)|^2} |e_3(t)|^2 dt \\
&\leq \frac{1}{2} + \lambda \left(\frac{1}{r_1} C_{r_1 \beta_1^*}^{r_1} \|b_1\|_{\beta_1} |\eta|^{r_1 - 2} + \frac{1}{r_2} C_{r_2 \beta_2^*}^{r_2} \|b_2\|_{\beta_2} |\eta|^{r_2 - 2} \right) - \tilde{a} \tilde{e}^2 \int_{\Upsilon} \frac{G(\eta e_3(t))}{|\eta e_3(t)|^2} dt \\
&\to -\infty \quad \text{as } |\eta| \to \infty.
\end{aligned}$$

Therefore, there exists $\eta_1 > 0$ such that $I(\eta_1 e_3) < 0$ and $\|\eta_1 e_3\| > \varrho_3$. Let $e_2 = \eta_1 e_3$, we can see $I(e_2) < 0$, which proves this lemma. □

Lemma 5.3. *Suppose the conditions of Theorem 1.5 hold, then I satisfies the (PS) condition.*

Proof. Assume that $\{u_n\}_{n \in N} \subset E$ is a sequence such that

$$|I(u_n)| < \infty \quad \text{and} \quad I'(u_n) \to 0.$$

Then there exists a constant $M_4 > 0$ such that

$$|I(u_n)| \leq M_4, \qquad \|I'(u_n)\|_{E^*} \leq M_4. \tag{5.4}$$

Subsequently, we show that $\{u_n\}$ is bounded in E. Set

$$\widetilde{G}(x) = (\nabla G(x), x) - \nu G(x),$$

where ν is defined in (W_{12}). From (W_{10}), we can deduce that $\widetilde{G}(x) = o(|x|^2)$ as $|x| \to 0$, then there exists $\rho_2 \in (0, \rho_\infty)$ such that

$$|\widetilde{G}(x)| \leq |x|^2 \tag{5.5}$$

for all $|x| \leq \rho_2$. Arguing by contradiction, we assume that $\|u_n\| \to +\infty$ as $n \to \infty$. Set $z_n = \frac{u_n}{\|u_n\|}$, then $\|z_n\| = 1$, which implies that there exists a subsequence of $\{z_n\}$, still denoted by $\{z_n\}$, such that $z_n \rightharpoonup z_0$ in E and $z_n \to z_0$ uniformly on R as $n \to \infty$. The following discussion is divided into two cases.

Case 1: $z_0 \not\equiv 0$. Let $\Omega = \{t \in R \mid |z_0(t)| > 0\}$. Then we can see that $\mathrm{meas}(\Omega) > 0$. It is easy to see that there exists $\Omega_0 \subset \Omega$ such that $\mathrm{meas}(\Omega_0) > 0$ and $\sup_{t \in \Omega_0} |t| < \infty$. Otherwise, for any $n \in N$, we have $\mathrm{meas}\,(B_n \bigcap \Omega) = 0$, where $B_n = \{t \in R \mid |t| \leq n\}$. Then we can deduce that $\lim_{n \to \infty} \mathrm{meas}\,(B_n \bigcap \Omega) = 0$, which implies that $\mathrm{meas}(\Omega) = 0$, which is a contradiction. Since

$\|u_n\| \to +\infty$ as $n \to \infty$ and $|u_n(t)| = |z_n(t)| \cdot \|u_n\|$, then we have $|u_n(t)| \to +\infty$ as $n \to \infty$ for a.e. $t \in \Omega_0$. On one hand, it follows from (2.2), (3.2) and (W_6) that

$$\begin{aligned}\left|\int_R \frac{K(t,u_n)}{\|u_n\|^2}dt - \frac{1}{2}\right| &= \left|\frac{I(u_n)}{\|u_n\|^2} + \lambda\int_R \frac{F(t,u_n)}{\|u_n\|^2}dt\right| \\ &\le \frac{M_4}{\|u_n\|^2} + \frac{\frac{\lambda}{r_1}C^{r_1}_{r_1\beta_1^*}\|b_1\|_{\beta_1}\|u_n\|^{r_1} + \frac{\lambda}{r_2}C^{r_2}_{r_2\beta_2^*}\|b_2\|_{\beta_2}\|u_n\|^{r_2}}{\|u_n\|^2} \\ &\to 0 \quad \text{as } n \to \infty, \end{aligned} \tag{5.6}$$

which implies that

$$\lim_{n\to\infty}\int_R \frac{K(t,u_n)}{\|u_n\|^2}dt = \frac{1}{2}. \tag{5.7}$$

On the other hand, by the property of Ω_0 and (W_9), there exists $\bar{a} > 0$ such that $a_2(t) \ge \bar{a}$ for all $t \in \Omega_0$. It follows from (W_8), (W_9), $G(x) \ge 0$ and Fatou's lemma that

$$\begin{aligned}\lim_{n\to\infty}\int_R \frac{K(t,u_n)}{\|u_n\|^2}dt &\ge \lim_{n\to\infty}\int_{\Omega_0} \frac{K(t,u_n)}{\|u_n\|^2}dt \\ &= \lim_{n\to\infty}\int_{\Omega_0} \frac{a_2(t)G(u_n)}{|u_n(t)|^2}|z_n(t)|^2dt \\ &\ge \bar{a}\lim_{n\to\infty}\int_{\Omega_0} \frac{G(u_n)}{|u_n(t)|^2}|z_n(t)|^2dt \\ &= +\infty, \end{aligned}$$

which contradicts (5.7).

Case 2: $z_0 \equiv 0$. It follows from (5.4), (2.2), (2.6), (W_9), (5.5), (3.1), (3.2) and Lemma 2.2 that that

$$\begin{aligned} o(1) &= \frac{\nu M_4 + M_4\|u_n\|}{\|u_n\|^2} \\ &\ge \frac{\nu I(u_n) - \langle I'(u_n), u_n\rangle}{\|u_n\|^2} \\ &\ge \left(\frac{\nu}{2}-1\right) - \frac{\lambda(\nu+1)}{\|u_n\|^2}\left(\frac{1}{r_1}C^{r_1}_{r_1\beta_1^*}\|b_1\|_{\beta_1}\|u_n\|^{r_1} + \frac{1}{r_2}C^{r_2}_{r_2\beta_2^*}\|b_2\|_{\beta_2}\|u_n\|^{r_2}\right) \\ &\quad + \frac{1}{\|u_n\|^2}\int_R a_2(t)\widetilde{G}(u_n(t))dt \\ &\ge \left(\frac{\nu}{2}-1\right) + \frac{1}{\|u_n\|^2}\left(\int_{\{t\in R||u_n|\le\rho_2\}} a_2(t)\widetilde{G}(u_n(t))dt + \int_{\{t\in R||u_n|>\rho_\infty\}} a_2(t)\widetilde{G}(u_n(t))dt\right) \\ &\quad + \frac{1}{\|u_n\|^2}\int_{\{t\in R|\rho_2<|u_n|\le\rho_\infty\}} a_2(t)\widetilde{G}(u_n(t))dt + o(1) \\ &\ge \left(\frac{\nu}{2}-1\right) - \frac{1}{\|u_n\|^2}\left(\int_{\{t\in R||u_n|\le\rho_2\}} a_2(t)|u_n(t)|^2dt + \int_{\{t\in R||u_n|>\rho_\infty\}} d_1a_2(t)|u_n(t)|^2dt\right) \\ &\quad - \frac{\max_{\rho_2<|x|\le\rho_\infty}|\widetilde{G}(x)|}{\rho_2^2}\int_{\{t\in R|\rho_2<|u_n|\le\rho_\infty\}} a_2(t)\frac{|u_n(t)|^2}{\|u_n\|^2}dt + o(1) \\ &\ge \left(\frac{\nu}{2}-1\right) - \left(1 + d_1 + \frac{\max_{\rho_2<|x|\le\rho_\infty}|\widetilde{G}(x)|}{\rho_2^2}\right)\int_R a_2(t)|z_n(t)|^2dt + o(1) \\ &\to \left(\frac{\nu}{2}-1\right) \quad \text{as } n \to \infty, \end{aligned}$$

which is a contradiction. The rest proof is similar to Lemma 3.4. Thus I satisfies the (PS) condition. □

Lemma 5.4. *Suppose that the conditions of Theorem 1.5 hold, then there exists a critical point of I corresponding to negative critical value.*

Proof. The proof is similar to Lemma 3.5. □

By Lemmas 5.1–5.4, we can deduce that I possesses at least two critical points. Consequently, problem (1.1) possesses at least two homoclinic solutions.

6 Proof of Theorem 1.7

Lemma 6.1. *Suppose the conditions of Theorem 1.7 hold, then I satisfies (Z_1).*

Proof. Let X_k and $h_k(q)$ be as defined in Lemma 4.1. For any $u \in X_k^{\perp} \bigcap \partial B_{\varrho_4}$ with $\varrho_4 \leq \min\{1, \frac{\rho_1}{C_\infty}\}$, it follows from (2.2), (3.2), (5.1), (W_6) and (4.1) that

$$\begin{aligned} I(u) &= \frac{1}{2}\varrho_4^2 - \lambda \int_R F(t,u(t))dt - \int_R a_2(t)G(u(t))dt \\ &\geq \frac{1}{2}\varrho_4^2 - \frac{\lambda}{r_1}\int_R b_1(t)|u(t)|^{r_1}dt - \frac{\lambda}{r_2}\int_R b_2(t)|u(t)|^{r_2}dt - \frac{1}{4C_2^2}\int_R |u(t)|^2 dt \\ &\geq \frac{1}{2}\varrho_4^2 - \frac{\lambda}{r_1}h_k^{r_1}(r_1\beta_1^*)\|b_1\|_{\beta_1}\varrho_4^{r_1} - \frac{\lambda}{r_2}h_k^{r_2}(r_2\beta_2^*)\|b_2\|_{\beta_2}\varrho_4^{r_2} - \frac{1}{4C_2^2}h_k^2(2)\varrho_4^2 \\ &\geq \frac{1}{2}\varrho_4^2 - \left(\frac{\lambda}{r_1}h_k^{r_1}(r_1\beta_1^*)\|b_1\|_{\beta_1} + \frac{\lambda}{r_2}h_k^{r_2}(r_2\beta_2^*)\|b_2\|_{\beta_2} + \frac{1}{4C_2^2}h_k^2(2)\right)\varrho_4. \end{aligned}$$

Therefore there exists $k_1 > 0$ such that $I(u) > \frac{1}{4}\varrho_4^2$ for all $u \in X_{k_1}^{\perp} \bigcap \partial B_{\varrho_4}$. Hence there exist ϱ_4, $\alpha_4 > 0$ such that $I|_{X_{k_1}^{\perp} \bigcap \partial B_{\varrho_4}} \geq \alpha_4$. □

Lemma 6.2. *Suppose the conditions of Theorem 1.7 hold, then I satisfies (Z_2).*

Proof. Set $\tilde{X}_m = \bigoplus_{j=1}^m Z_j$, where Z_j is defined in Lemma 4.1. For any $u \in \tilde{X}_m \setminus \{0\}$ and $\vartheta > 0$, set

$$\Gamma_\vartheta(u) = \{t \in R : |u(t)| \geq \vartheta \|u\|\}.$$

Similar to [17], there exists $\vartheta_0 > 0$ such that

$$\text{meas}\,(\Gamma_{\vartheta_0}(u)) \geq \vartheta_0 \tag{6.1}$$

for all $u \in \tilde{X}_m \setminus \{0\}$. Then there exists $\kappa > 0$ such that

$$\text{meas}\,(\Pi_{\vartheta_0}(u)) \geq \frac{1}{2}\vartheta_0, \tag{6.2}$$

for all $u \in \tilde{X}_m \setminus \{0\}$, where $\Pi_\vartheta(u) = \Gamma_\vartheta(u) \bigcap \{R : t \leq \kappa\}$. Letting $a_2 = \min_{t\leq\kappa} a_2(t) > 0$, it follows from (W_{11}) that there exists $\xi > 0$ such that

$$G(u(t)) \geq \frac{1}{a_2\vartheta_0^3}|u(t)|^2 \geq \frac{1}{a_2\vartheta_0}\|u\|^2$$

for all $u \in \tilde{X}_m$ and $t \in \Gamma_{\vartheta_0}(u)$ with $\|u\| \geq \xi$. We can choose $\varsigma_m > \xi$, then for any $u \in \tilde{X}_m \setminus B_{\varsigma_m}$, it follows from (2.2), (2.1), (6.1) and (3.2) that

$$\begin{aligned} I(u) &= \frac{1}{2}\|u\|^2 - \lambda \int_R F(t,u(t))dt - \int_R a_2(t)G(u(t))dt \\ &\leq \frac{1}{2}\|u\|^2 - \lambda \int_R F(t,u(t))dt - \int_{\Gamma_{\vartheta_0}(u)} a_2(t)G(u(t))dt \\ &\leq \frac{1}{2}\|u\|^2 + \frac{\lambda}{r_1} C_{r_1\beta_1^*}^{r_1} \|b_1\|_{\beta_1} \|u\|^{r_1} + \frac{\lambda}{r_2} C_{r_2\beta_2^*}^{r_2} \|b_2\|_{\beta_2} \|u\|^{r_2} - \frac{1}{\vartheta_0} \operatorname{meas}\left(\Gamma_{\vartheta_0}(u)\right) \|u\|^2 \\ &\leq -\frac{1}{2}\|u\|^2 + \frac{\lambda}{r_1} C_{r_1\beta_1^*}^{r_1} \|b_1\|_{\beta_1} \|u\|^{r_1} + \frac{\lambda}{r_2} C_{r_2\beta_2^*}^{r_2} \|b_2\|_{\beta_2} \|u\|^{r_2}. \end{aligned}$$

Then there exists $r_m > \xi$ such that $I(u_m) \leq 0$ for all $u \in \tilde{X}_m \setminus B_{r_m}$, which proves this lemma. □

Lemma 6.3. *Suppose the conditions of Theorem 1.7 hold, then I satisfies the $(PS)^*$ condition.*

Proof. The proof is similar to Lemma 5.3. □

Proof of Theorem 1.7. By Lemmas 6.1–6.3 and Lemma 1.9, I possesses infinitely many distinct critical points corresponding to positive critical values. The proof of Theorem 1.7 is finished. □

Acknowledgements

The authors would like to thank the referees for their valuable comments which helped to improve the manuscript. The first author is supported by the NSFC (No. 11626198), The Youth Science and Technology Innovation Team of Southwest Petroleum University for Nonlinear Systems (No. 2017CXTD02) and The Science and Technology Innovation Team of Education Department of Sichuan for Dynamical System and its Applications (No. 18TD0013). The second and the third authors are supported by the NSFC (No. 11471267).

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