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Stability and attractivity for Nicholson systems with time-dependent delays

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Abstract. We analyse the stability and attractivity of a class of n-dimensional Nicholson systems with constant coefficients and multiple time-varying delays. Delay-independent sufficient conditions on the coefficients are given, for the existence and absolute global exponential stability of a unique positive equilibrium N^*, generalizing and improving known results for autonomous systems. We further establish delay-dependent criteria for N^* to be a global attractor of all positive solutions. In the latter case, upper bounds on the size of the delays which do not require an a priori explicit knowledge of the equilibrium N^* are also derived.

Keywords: Nicholson system, time-dependent delays, exponential stability, global attractivity, delay-dependent criteria.

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1 Introduction

This paper deals with a Nicholson system with autonomous coefficients and multiple time-varying discrete delays of the form

$$N_i'(t) = -d_i N_i(t) + \sum_{j=1, j\neq i}^{n} a_{ij} N_j(t) + \sum_{k=1}^{m} \beta_{ik} N_i(t-\tau_{ik}(t)) e^{-c_i N_i(t-\tau_{ik}(t))}, \; i = 1, \dots, n, \; t \geq 0, \quad (1.1)$$

where $d_i > 0, c_i > 0, a_{ij} \geq 0, \beta_{ik} \geq 0$ with $\beta_i := \sum_{k=1}^m \beta_{ik} > 0$, $\tau_{ik} : [0,\infty) \to [0,\infty)$ are continuous and bounded, for $i, j = 1, \dots, n$ and $k = 1, \dots, m$. For a biological interpretation of model (1.1) and some applications, see Section 2 and e.g. [1,2,7,11,13].

Multi-dimensional Nicholson systems are a natural extension of the famous Nicholson's blowflies equation $N'(t) = -dN(t) + \beta N(t-\tau)e^{-N(t-\tau)}$ $(d, \beta, \tau > 0)$, for which it is well known that the positive equilibrium $N^* = \log(\beta/d)$ exists and is globally attractive if $1 < \beta/d \leq e^2$. Moreover, for this scalar Nicholson's equation with $\beta/d > e^2$, several criteria on the

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size of the delay τ have been established for the global attractivity of N^* [8,14,16]. For a nice survey on the subject and further references, see [1].

Only recently have Nicholson systems with patch structure and multiple delays deserved some attention from researchers. Most studies are centred on autonomous systems (or at least with constant coefficients), the main focus under investigation being the existence and global attractivity of a positive equilibrium [5,7,10,12,13].

Here, we investigate the stability and global attractivity of a positive equilibrium N^* for (1.1). Most of the results will be proven for systems (1.1) with $c_i = 1$ for all i, i.e.,

$$N_i'(t) = -d_i N_i(t) + \sum_{j=1, j\neq i}^{n} a_{ij} N_j(t) + \sum_{k=1}^{m} \beta_{ik} N_i(t - \tau_{ik}(t)) e^{-N_i(t-\tau_{ik}(t))}, \; i = 1, \dots, n, \; t \geq 0, \quad (1.2)$$

since, as we shall see, a simple scaling shows that the conclusions obtained for (1.2) hold with small adjustments for the more general family (1.1). We start with a brief overview of some recent and selected results regarding the global asymptotic behaviour of solutions to Nicholson systems. Among them, emphasis is given to the papers of Faria and Röst [7] and Jia et al. [10], which strongly motivated the present work.

In [12], Liu considered an autonomous Nicholson system of the form

$$N_i'(t) = -dN_i(t) + \sum_{j=1}^{n} a_{ij} N_j(t) + \beta N_i(t - r_i) e^{-N_i(t-r_i)}, \quad i = 1, \dots, n, \; t \geq 0, \qquad (1.3)$$

with $d, \beta, r_i > 0, a_{ij} \geq 0$ for $i \neq j$, $1 \leq i, j \leq n$, and $[a_{ij}]$ an irreducible matrix. Moreover, it was assumed in [12] that

$$\sum_{j=1}^{n} a_{ij} = 0, \quad i = 1, \dots, n,$$

so that, when $\beta > d$,

$$N^* = \left(\log \frac{\beta}{d}, \dots, \log \frac{\beta}{d} \right)$$

is the positive equilibrium of (1.3). Under these conditions, the global attractivity of N^* was proven in [12] for the case $\beta/d \in [e, e^2]$. This result was later extended by the same author [13] to more general systems with multiple time-dependent delays of the form (1.2), but again with the same requirements on the coefficients, except that β was replaced by the constant $\beta := \sum_k \beta_{ik}$; note that d, β were still assumed to be independent of i and $\sum_j a_{ij} = 0$. These constraints were relaxed in [5,7], however only the autonomous version,

$$N_i'(t) = -d_i N_i(t) + \sum_{j=1, j\neq i}^{n} a_{ij} N_j(t) + \sum_{k=1}^{m} \beta_{ik} N_i(t - \tau_{ik}) e^{-N_i(t-\tau_{ik})}, \quad i = 1, \dots, n, \; t \geq 0, \quad (1.4)$$

was treated. Here, as well as in [5,7], without loss of generality we assume that $a_{ii} = 0$ for $i = 1, \dots, n$, since each of these coefficients may be incorporated in d_i.

With the terms $d_i - \sum_{j\neq i} a_{ij}$ and $\sum_k \beta_{ik}$ depending naturally on i, a prime concern is to ensure the existence of a positive equilibrium N^*, since it cannot be explicitly computed. This was established in [7] by imposing that the ODE system $x_i' = -d_i x_i + \sum_{j=1, j\neq i}^{n} a_{ij} x_j$ is asymptotically stable (a natural requirement from a biological point of view) and the following condition on the *community matrix*, defined here as $M = \mathrm{diag}\,(\beta_1 - d_1, \dots, \beta_n - d_n) + [a_{ij}]$ where $\beta_i = \sum_k \beta_{ik}$ (see Section 2 for further details): there exists a positive vector v such that $Mv > 0$. In the case of $[a_{ij}]$ an irreducible matrix (a constraint not imposed in [5,7]), this condition turns out to be equivalent to saying that the community matrix M has an eigenvalue with positive real part, and is indeed a necessary and sufficient condition for the existence of a positive equilibrium N^*; otherwise, the equilibrium 0 is a global attractor of all positive solutions of (1.4). Furthermore, under the stronger condition

$$1 < \frac{\beta_i}{d_i - \sum_{j \neq i} a_{ij}} \leq e^2, \quad i = 1, \ldots, n, \tag{1.5}$$

it was also shown in [7] that N^* is globally asymptotically stable.

The first purpose of this paper is to recover and generalize the results on the existence and global attractivity of the positive equilibrium N^* given in [7], so that they apply to the nonautonomous version (1.2), and more generally to (1.1). Note that, although this system has autonomous coefficients, the delays $\tau_{ik}(t)$ are time-dependent, and consequently known results and techniques for *autonomous* Nicholson systems do not apply directly to (1.1); thus, new arguments should be used.

On the other hand, it is well known that the introduction of large delays may induce instability, oscillations, unbounded solutions; contrarily, small delays are expected to be negligible. Delay-dependent criteria for the global attractivity of equilibria are in general more difficult to obtain, even for scalar delay differential equations (DDEs), for which several conjectures on local stability implying global asymptotic stability remain open. For multi-dimensional DDEs, clearly this topic is even harder to address, and only a few results have been produced. See e.g. [17], for 3/2-criteria for the global attractivity of delayed Lotka–Volterra systems.

To the best of our knowledge, for Nicholson systems, criteria for the global attractivity of the positive equilibrium N^* depending on the size of the delays were established for the first time in 2017, in two very recent papers [4,10]. In Jia et al. [10], the quite restrictive assumption

$$\frac{\beta_i}{d_i - \sum_{j \neq i} a_{ij}} = c > 1 \quad \text{for all } 1 \leq i \leq n, \tag{1.6}$$

was still imposed, and consequently the positive equilibrium $N^* = (N_1^*, \ldots, N_n^*)$ exists and has all its components equal to the same constant, $N^* = (\log c, \ldots, \log c)$. On the other hand, El-Morshedy and Ruiz-Herrera [4] gave a result for the global attractivity of the positive equilibrium N^* of the *autonomous* system (1.4) which does not depend on knowing N^* explicitly; nevertheless the authors had to assume that such an equilibrium exists.

This brings us to the second main task of this paper: to generalize the result in [10], by establishing a more general criterion for the global attractivity of N^* depending on the size of the delays $\tau_{ik}(t)$, which in particular not only does not require an a priori explicit knowledge of the equilibrium N^*, much less that the components of N^* are all equal.

The main lines of the work in this paper are as described above, and its organization is as follows. In Section 2, we introduce some notation and recall some preliminary results on persistence and existence of a positive equilibrium N^* for (1.2). In Section 3, sufficient conditions for the absolute global exponential stability of N^* are given. In Section 4, we establish a delay-dependent criterion for the global attractivity of N^*, without assuming condition (1.6), which generalizes the result in [10]. A comparison with the criterion in [4] will also be given. Two illustrative examples will be given at the end.

2 Preliminaries

Consider a Nicholson system with patch structure of the form

$$N_i'(t) = -d_i N_i(t) + \sum_{j=1, j\neq i}^{n} a_{ij} N_j(t) + \sum_{k=1}^{m} \beta_{ik} N_i(t-\tau_{ik}(t)) e^{-N_i(t-\tau_{ik}(t))},\ i=1,\dots,n,\ t\geq 0, \quad (2.1)$$

under the following general assumption on the coefficients and delays:

(H0) $d_i > 0, a_{ij} \geq 0\,(j \neq i), \beta_{ik} \geq 0$ with $\beta_i := \sum_{k=1}^m \beta_{ik} > 0$, $\tau_{ik} : [0,\infty) \to [0,\tau]$ (for some $\tau > 0$) are continuous, for $i, j = 1,\dots,n$, $k = 1,\dots,m$.

These systems are in general used in population dynamics or disease modelling, as they serve as models for the growth of biological populations distributed over n classes or patches, with migration among them: $x_i(t)$ denotes the density of the ith-population, a_{ij} is the rate of the population moving from class j to class i, d_i is the coefficient of instantaneous loss for class i (which integrates both the death rate and the dispersal rates of the population in class i moving to the other classes), and $\beta_{ik} N_i(t-\tau_{ik}(t))e^{-N_i(t-\tau_{ik}(t))}$ are birth functions for class i; as usual, delays are included in the "birth terms", and our model prescribes time-dependent delays. Due to this biological interpretation, it is natural to assume that $a_{ii} = 0$ for all i; however, for different settings, one may still suppose that $a_{ii} = 0$, since, for each i, the term $a_{ii}N_i(t)$ may be incorporated in the term $-d_i N_i(t)$. In biological terms, it is also natural to take $d_i = m_i + \sum_{j\neq i} a_{ji}$, where $m_i > 0$ is the death rate for class i, although here a weaker version of this condition will be required.

Set $\tau = \max_{i,k} \sup_{t\geq 0} \tau_{ik}(t) > 0$. As the phase space for (1.1), take the Banach space $C := C([-\tau, 0]; \mathbb{R}^n)$ endowed with the norm $\|\varphi\| = \max_{t\in[-\tau,0]} |\varphi(t)|$, where the supremum norm $|\cdot|$ in $\mathbb{R}^n$ is fixed, $|v| = |(v_1,\dots,v_n)| = \max_{1\leq i\leq n} |v_i|$. A vector $v \in \mathbb{R}^n$ will be identified with the constant function $\varphi(t) \equiv v$ in C. System (1.1) can be written as an abstract DDE in C, $N'(t) = f(t, N_t)$, where N_t denotes the function in C given by $N_t(\theta) = N(t+\theta)$ for $-\tau \leq \theta \leq 0$. By C^+ we denote the cone in C of nonnegative functions, and write $\varphi \geq 0$ for $\varphi \in C^+$. By a positive vector $v \in \mathbb{R}^n$, we mean a vector whose components are all positive, and write $v > 0$. In a similar way, we denote $\varphi > 0$ for a function in C whose components are positive for all $t \in [-\tau, 0]$.

Bearing in mind the biological interpretation of the family (1.1), the set

$$C_0^+ := \{\varphi \in C^+ : \varphi(0) > 0\}$$

is taken as the set of admissible initial conditions. For simplicity, here initial conditions are given at time $t = 0$,

$$N_0 = \varphi, \quad (2.2)$$

with $\varphi \in C_0^+$, but it is clear that one may give initial conditions of the form $N_{t_0} = \varphi \in C_0^+$ for any $t_0 \geq 0$. As usual, the solution of (2.1) with initial condition $N_{t_0} = \varphi$ is denoted by $N(t, t_0, \varphi)$. In what follows, even if not mentioned, only solutions of (2.1)–(2.2) with $N_0 = \varphi \in C_0^+$ will be considered.

Writing (2.1) in the form $N_i'(t) = -d_i N_i(t) + g_i(t, N_t), 1 \leq i \leq n$, the functions g_i are bounded on bounded sets of $\mathbb{R} \times C^+$ and satisfy $g_i(t,\varphi) \geq 0$ for $t \geq 0, \varphi \in C^+$, thus solutions of (2.1) with initial conditions on C_0^+ are defined and positive on $[0,\infty)$.

Recall that a DDE in $C = C([-\tau, 0]; \mathbb{R}^n)$ given by $x'(t) = f(t, x_t), t \geq 0$, is *dissipative* (in C_0^+) if all its solutions are defined for all $t \geq 0$ and are eventually bounded in norm by a common positive constant; in other words, there exists $M > 0$ such that $\limsup_{t\to\infty} |x(t, 0, \varphi)| \le M$ for all $\varphi \in C_0^+$. The DDE $x'(t) = f(t, x_t), t \ge 0$, is said to be *persistent* (in C_0^+) if all its solutions are defined and bounded below away from zero on $[0, \infty)$, i.e., $\liminf_{t\to\infty} x_i(t, 0, \varphi) > 0$ for all $1 \le i \le n, \varphi \in C_0^+$; and (2.1) is *uniformly persistent* if all positive solutions are defined on $[0, \infty)$ and there is a uniform lower bound $m > 0$, i.e., $\liminf_{t\to\infty} x_i(t, 0, \varphi) \ge m$ for all $1 \le i \le n, \varphi \in C_0^+$.

To simplify the notation, define the $n \times n$ matrices

$$A = [a_{ij}], \quad B = \text{diag}\,(\beta_1, \dots, \beta_n), \quad D = \text{diag}\,(d_1, \dots, d_n), \tag{2.3}$$

where $a_{ii} := 0$ $(1 \le i \le n)$, and the so-called *community matrix*

$$M = B - D + A. \tag{2.4}$$

The properties of the matrices $D - A$ and M play an important role in the global asymptotic behaviour of solutions of (2.1). The following algebraic concept is timely.

Definition 2.1. A square matrix $N = [n_{ij}]$ with nonpositive off-diagonal entries (i.e., $n_{ij} \le 0$ for $i \ne j$) is said to be a **non-singular M-matrix** if all its eigenvalues have positive real parts.

If $n_{ij} \le 0$ for $i \ne j$, it is well known that $N = [n_{ij}]$ is a non-singular M-matrix if and only if there exists a positive vector v such that $Nv > 0$ [3]. Hence, $D - A$ is a non-singular M-matrix if and only if $(D - A)v > 0$ for some vector $v = (v_1, \dots, v_n) > 0$, i.e.,

$$d_i v_i - \sum_{j=1, j\ne i}^{n} a_{ij} v_j > 0, \quad i = 1, \dots, n.$$

Clearly, this is equivalent to saying that the ODE $x' = -(D - A)x$ is asymptotically stable.

Throughout this paper, we shall assume a stronger hypothesis:

(H1) there exists a vector $v = (v_1, \dots, v_n) > 0$ such that

$$\gamma_i(v) := \frac{\beta_i v_i}{d_i v_i - \sum_{j\ne i} a_{ij} v_j} > 1, \quad i = 1, \dots, n. \tag{2.5}$$

Since $\beta_i > 0$ for all i, note that **(H1)** is equivalent to saying that there is a positive vector v that satisfies both $(D - A)v > 0$ and $Mv > 0$.

Remark 2.2. Since M is a *cooperative matrix* (also called a *Metzler matrix*), i.e., all its off-diagonal entries are nonnegative, by using the theory of Perron–Frobenius one can show that, if $Mc > 0$ for some vector $c > 0$, then the *spectral bound* $s(M) = \max\{\text{Re}\,\lambda : \lambda \in \sigma(M)\}$ of M is positive; in fact, the converse is also true when M is irreducible. Moreover, when $D - A$ is a non-singular M-matrix, **(H1)** is satisfied if and only if there exists a positive vector c such that $Mc > 0$ (see [7]). This means that, if there are positive vectors u, w such that $(D - A)u > 0$ and $Mw > 0$, then there is a positive vector v for which both conditions $(D - A)v > 0$ and $Mv > 0$ are satisfied.

Remark 2.3. From a biological viewpoint, it is quite natural to assume that $D - A$ is a non-singular M-matrix. In fact, as mentioned above, for models from population dynamics we take $d_i - \sum_{j\ne i} a_{ji} = m_i > 0\,(1 \le i \le n)$, where m_i is the death rate for the population in patch i. Thus $D - A^T$ is diagonally dominant, i.e., $[D - A^T]\mathbf{1} > 0$ where $\mathbf{1} := (1, \dots, 1)$. In particular, the matrix $D - A^T$ is a non-singular M-matrix, which implies that $D - A$ is a non-singular M-matrix as well.

The main purpose of this paper is to investigate the stability and global attractivity of a positive equilibrium N^*, when it exists. Some standard definitions are given below.

Definition 2.4. A positive equilibrium N^* of (2.1) is said to be **globally attractive** (in C_0^+) if $N(t,0,\varphi) \to N^*$ as $t \to \infty$, for all solutions of (2.1) with initial conditions $N_0 = \varphi \in C_0^+$; N^* is **globally asymptotically stable** if it is stable and globally attractive. If there are $K > 0, \alpha > 0$ such that $|N(t,0,\varphi) - N^*| \le Ke^{-\alpha t}\|\varphi - N^*\|$, for all $t \ge 0, \varphi \in C_0^+$, then N^* is said to be **globally exponentially stable**.

In what concerns the existence and uniqueness of an equilibrium $N^* > 0$, observe that the equilibria of (2.1) coincide with the equilibria of the autonomous ODE

$$N_i'(t) = -d_iN_i(t) + \sum_{j=1, j\neq i}^{n} a_{ij}N_j(t) + \beta_iN_i(t)e^{-N_i(t)}, \quad i = 1, \dots, n. \tag{2.6}$$

The nonlinearity $h(x) = xe^{-x}$ is bounded on $[0, \infty)$, hence a simple use of the variation of constants formula shows that, if the linear ODE $x' = -[D - A]x$ is exponentially stable, then (2.6) is dissipative. Since $\mathbb{R}^n_+$ is positively invariant for (2.6) and the system is dissipative, by [9] there is at least a saturated equilibrium of (2.6) in $\mathbb{R}^n_+$. Under the assumption **(H1)**, by exploiting the properties of the cooperative matrix M, it was shown in [7] that such an equilibrium is forcefully positive and unique.

Some preliminary results on the global asymptotic behaviour of (2.1) are collected in the theorem below. In spite of the situation with time-dependent delays, the statements are easily deduced by repeating the arguments in [7], so the proofs are omitted (see also [5, 10]).

Theorem 2.5. *For system* (2.1), *assume* **(H0)** *and that $D - A$ is a non-singular M-matrix. Then:*

(i) (2.1) *is dissipative;*

(ii) if $s(M) \le 0$, the equilibrium 0 is globally asymptotically stable;

(iii) if **(H1)** *holds,* (2.1) *is uniformly persistent and there is a unique positive equilibrium N^*;*

(iv) if (1.5) *holds, the positive equilibrium N^* is globally asymptotically stable.*

Remark 2.6. One can check that the statements in Theorem 2.5 (i)–(iii) are valid with $h(x) = xe^{-x}$ replaced in each equation by smooth functions $h_i(x)$ with $h_i(x) > 0$ for $x > 0$, $h_i(0) = 0, h_i'(0) = 1, h_i(\infty) = 0$ and $h_i(x)/x$ decreasing on $(0, \infty)$. Nevertheless, good criteria for the attractivity of the equilibrium $N^* > 0$ would depend heavily on the shape of the nonlinearities $h_i(x)$.

3 Absolute exponential stability of the positive equilibrium

If the coefficients $\gamma_i(v)$ defined in (2.5) satisfy suitable upper bounds, the positive equilibrium N^* has its components in the interval $(0, 2)$, where the nonlinearity $h(x) = xe^{-x}$ has very specific properties. This allows us to derive the *absolute* global exponential stability of the positive equilibrium N^* of (2.1), where as usual the term 'absolute' refers to the fact that such a stability holds regardless of the size of the delay functions $\tau_{ik}(t)$, provided that they remain bounded.

Before the main theorem of this section, we state two auxiliary results. The first lemma is a simplified version of [6, Lemma 3.2], while the second refers to properties of the nonlinearity xe^{-x}.

Lemma 3.1 ([6]). *Let $S \subset C$ be the set of initial conditions for a DDE $x'(t) = f(t, x_t)$ $(t \geq t_0)$ in C, where $f : [t_0, \infty) \times S \to \mathbb{R}^n$ is continuous. For $|\cdot|$ the maximum norm in $\mathbb{R}^n$, suppose that $f = (f_1, \dots, f_n)$ satisfies*

(H) for all $t \geq t_0$ and $\varphi \in S$, whenever $|\varphi(\theta)| < |\varphi(0)|$ for $\theta \in [-\tau, 0)$, then $\varphi_i(0) f_i(t, \varphi) < 0$ for some i such that $|\varphi(0)| = |\varphi_i(0)|$.

Then the solutions $x(t)$ of $x'(t) = f(t, x_t)$ with initial conditions $x_{t_0} = \varphi \in S$ are defined and bounded for $t \geq t_0$ and, if $x_t \in S$ for all $t \geq t_0$, the solution satisfies $|x(t)| \leq \|x_{t_1}\|$ for all $t \geq t_1 \geq t_0$.

Lemma 3.2 ([5]). *Fix $x \in (0, 2]$. Then $|ye^{-y} - xe^{-x}| < e^{-x}|y - x|$ for any $y > 0, y \neq x$.*

Theorem 3.3. *Consider system (2.1) under the general condition* **(H0)**. *Further assume that*

(H2) *there exists a vector $v = (v_1, \dots, v_n) > 0$ such that*

$$1 < \gamma_i(v) \leq e^{2 \min\limits_{k,j}(v_k/v_j)}, \quad i = 1, \dots, n, \tag{3.1}$$

where $\gamma_i(v)$ are defined as in (2.5).

Then, the positive equilibrium N^* of (2.1) is uniformly stable. Moreover, if

(H2*) *there exists a vector $v = (v_1, \dots, v_n) > 0$ such that*

$$1 < \gamma_i(v) < e^{2 \min\limits_{k,j}(v_k/v_j)} \quad i = 1, \dots, n, \tag{3.2}$$

the positive equilibrium N^* of (2.1) is globally exponentially stable. In particular, if

$$1 < \frac{\beta_i}{d_i - \sum_{j \neq i} a_{ij}} < e^2, \quad i = 1, \dots, n, \tag{3.3}$$

N^* is globally exponentially stable.

Proof. Assume **(H2)**. The existence and uniqueness of a positive equilibrium N^* is guaranteed by Theorem 2.5. The equilibrium $N^* = (N_1^*, \dots, N_n^*)$ is determined by the system

$$\beta_i N_i^* e^{-N_i^*} = d_i N_i^* - \sum_{j \neq i} a_{ij} N_j^*, \quad 1 \leq i \leq n.$$

Fix $i \in \{1, \dots, n\}$ such that $N_i^*/v_i = \max_j(N_j^*/v_j)$. Since $\beta_i e^{-N_i^*} \geq d_i - \frac{1}{v_i} \sum_{j \neq i} a_{ij} v_j$, it follows that

$$e^{N_i^*} \leq \gamma_i(v).$$

Hence from (3.1) we have $N_i^* \leq 2m_0$ where $m_0 = \min\limits_{k,j}(v_k/v_j)$, implying that

$$N_j^* \leq v_j \frac{N_i^*}{v_i} \leq 2m_0 \frac{v_j}{v_i} \leq 2, \quad 1 \leq j \leq n. \tag{3.4}$$

On the other hand, as all coordinates of N^* lie in $(0, 2]$, from Lemma 3.2 we have

$$|h(N_i^*(1 + x)) - h(N_i^*)| < e^{-N_i^*} N_i^* |x|, \quad 1 \leq i \leq n, \tag{3.5}$$

for all $x > -1, x \neq 0$, where as before $h(x) = xe^{-x}$.

Returning to (2.1), through the change of variables

$$x_i(t) = \frac{N_i(t)}{N_i^*} - 1, \quad 1 \le i \le n, \tag{3.6}$$

system (2.1) becomes

$$\begin{aligned} x_i'(t) &= -d_i x_i(t) + \sum_{j=1, j\neq i}^{n} \hat{a}_{ij} x_j(t) + \sum_{k=1}^{m} \beta_{ik} \frac{1}{N_i^*} \left[h\Big(N_i^*(1 + x_i(t - \tau_{ik}(t)))\Big) - h(N_i^*)\right] \\ &=: f_i(t, x_t), \quad 1 \le i \le n, \end{aligned} \tag{3.7}$$

where $\hat{a}_{ij} = \frac{N_j^*}{N_i^*} a_{ij}$, $i \neq j$. Note that $S = \{\varphi \in C : \varphi(\theta) \ge -1 \text{ for } \theta \in [-\tau, 0) \text{ and } \varphi(0) > -1\}$ is the set of admissible initial conditions for the transformed system (3.7).

We now apply Lemma 3.1 to show that N^* is uniformly stable. For any $t \ge 0$ and $\varphi \in S$ with $|\varphi(\theta)| < |\varphi(0)|$ for $\theta \in [-\tau, 0)$, we need to verify that $\varphi_i(0) f_i(t, \varphi) < 0$ for some i such that $|\varphi(0)| = |\varphi_i(0)|$.

Let φ be as above and fix i such that $|\varphi(0)| = |\varphi_i(0)|$. We only consider the case $\varphi_i(0) > 0$, since the case $\varphi_i(0) < 0$ is treated in a similar way. For $t \ge 0$, the estimates in (3.5) yield

$$\left| h\Big(N_i^*(1 + \varphi_i(-\tau_{ik}(t)))\Big) - h(N_i^*)\right| < N_i^* e^{-N_i^*} \varphi_i(0),$$

and consequently

$$\begin{aligned} f_i(t, \varphi) &= -d_i \varphi_i(0) + \sum_{j \neq i} \hat{a}_{ij} \varphi_j(0) + \sum_{k} \beta_{ik} \frac{1}{N_i^*} \left[h\Big(N_i^*(1 + \varphi_i(-\tau_{ik}(t)))\Big) - h(N_i^*)\right] \\ &< \varphi_i(0) \left(-d_i + \sum_{j \neq i} \hat{a}_{ij} + \beta_i e^{-N_i^*} \right) = 0. \end{aligned} \tag{3.8}$$

From Lemma 3.1, it follows that, for any solution $x(t)$ of (3.7) with initial condition in S, the function $t \mapsto \|x_t\|$ is non-increasing. This shows that the equilibrium N^* of (2.1) is uniformly stable.

Next, we assume the strict inequalities in (3.2). Proceeding as in (3.4), one obtains that all the components N_i^* of N^* are in the interval $(0, 2)$. From the boundedness and persistence of solutions to (2.1), one may fix $m, L > 0$ such that the components of the solution $x(t)$ of (3.7) satisfy $-1 + m \le x_i(t) \le L$ for t sufficiently large. On the other hand, since $|h'(N_i^*)| < e^{-N_i^*}$ for $0 < N_i^* < 2$, the estimates (3.5) lead to

$$\max_{x \in [-1+m, L]} |g_i(x)| < e^{-N_i^*}, \quad 1 \le i \le n,$$

where $g_i(x)$ is the continuous function given by $g_i(x) = \frac{h(N_i^*(1+x)) - h(N_i^*)}{N_i^* x}$ if $x \neq 0$, $g_i(0) = h'(N_i^*)$. Hence, one can choose a small $\delta > 0$ such that

$$|h(N_i^*(1 + x)) - h(N_i^*)| \le e^{-\delta r_i}(e^{-N_i^*} - \sqrt{\delta}) N_i^* |x|, \quad 1 \le i \le n,$$

for all $x \in [-1 + m, L]$, where $r_i = \max_{1 \le k \le m} \sup_{t \ge 0} \tau_{ik}(t)$.

Now, effect a second change of variables by setting $\bar{x}_i(t) = e^{\delta t} x_i(t)$. System (3.7) becomes

$$\begin{aligned}\bar{x}_i'(t) &= -(d_i-\delta)\bar{x}_i(t) + \sum_{j=1, j\neq i}^{n} \hat{a}_{ij}\bar{x}_j(t) \\ &\quad + \sum_{k=1}^{m} \beta_{ik}\frac{e^{\delta t}}{N_i^*}\left[h\Big(N_i^*(1+e^{-\delta(t-\tau_{ik}(t))}\bar{x}_i(t-\tau_{ik}(t)))\Big) - h(N_i^*)\right] \\ &=: \bar{f}_i(t,\bar{x}_t), \quad 1\le i\le n,\end{aligned} \tag{3.9}$$

with $\bar{S} = \{\varphi \in C : \varphi(\theta)e^{\delta\theta} \geq -1 \text{ for } \theta \in [-\tau, 0) \text{ and } \varphi(0) > -1\}$ as the set of initial conditions. Arguing as in (3.8), for $t \geq 0$ and $\varphi \in \bar{S}$ with $|\varphi(\theta)| < |\varphi(0)| = \varphi_i(0)$ for $\theta \in [-\tau, 0)$ and some $i \in \{1, \ldots, n\}$, we obtain

$$\begin{aligned}\bar{f}_i(t,\varphi) &= -(d_i-\delta)\varphi_i(0) + \sum_{j\neq i}\hat{a}_{ij}\varphi_j(0) + \sum_k \beta_{ik}\frac{e^{\delta t}}{N_i^*}\left[h\big(N_i^*(1+e^{-\delta t}e^{\delta\tau_{ik}(t)}\varphi_i(-\tau_{ik}(t)))\big) - h(N_i^*)\right] \\ &\le \varphi_i(0)\left(-(d_i-\delta) + \sum_{j\neq i}\hat{a}_{ij} + \beta_i(e^{-N_i^*} - \sqrt{\delta})\right) = \varphi_i(0)\sqrt{\delta}(\sqrt{\delta}-\beta_i),\end{aligned}$$

hence $\bar{f}_i(t, \varphi) < 0$ if δ is sufficiently small. From Lemma 3.1, $t \mapsto \|\bar{x}_t\|$ is non-increasing. This implies that the solutions $N(t)$ of (2.1) satisfy

$$|N(t) - N^*| \leq e^{-\delta t}\max_i(N_i^*)\|N_0\|,$$

thus N^* is globally exponentially stable. □

Analysis of the above proof shows that, under the existence of the positive equilibrium N^*, hypotheses **(H2)** and **(H2*)** were used only to derive that all its components N_i^* are in the interval $(0, 2]$, respectively $(0, 2)$. Therefore, a weaker version of Theorem 3.3 is obtained as follows.

Theorem 3.4. *For system (2.1), suppose* **(H0), (H1)**. *If the positive equilibrium $N^* = (N_1^*, \ldots, N_n^*)$ (whose existence is given in Theorem 2.5) satisfies $\max_{1\le i\le n} N_i^* \leq 2$, respectively $\max_{1\le i\le n} N_i^* < 2$, then N^* of (2.1) is uniformly stable, respectively globally exponentially stable.*

For the more general Nicholson system (1.1), we obtain the following generalization of Theorem 3.3. Clearly, Theorem 3.4 can be generalized in a similar way.

Theorem 3.5. *Consider system (1.1), where $c_i > 0\,(1 \le i \le n)$ and the other coefficients and delay functions satisfy* **(H0)**. *For any positive vector $v = (v_1, \ldots, v_n)$, let $\gamma_i(v)$, $i = 1, \ldots, n$, be as in (2.5).*

(i) If there exists $v > 0$ such that

$$1 < \gamma_i(v) \leq e^{2\min\limits_{k,j}(\frac{v_k c_j}{v_j c_k})}, \quad i = 1, \ldots, n, \tag{3.10}$$

the positive equilibrium N^* of (1.1) is uniformly stable.

(ii) If

$$1 < \gamma_i(v) < e^{2\min\limits_{k,j}(\frac{v_k c_j}{v_j c_k})}, \quad i = 1, \ldots, n, \tag{3.11}$$

the positive equilibrium N^* of (1.1) is globally exponentially stable.

Proof. Effect the scalings $\tilde{N}_i(t) = c_i N_i(t), i = 1, \dots, n$. System (1.1) is transformed into a system of the form (2.1), with the coefficients a_{ij} replaced by

$$\tilde{a}_{ij} := \frac{c_i}{c_j} a_{ij}, \quad i, j = 1, \dots, n, \ j \neq i.$$

For $v = (v_1, \dots, n) > 0$, note that

$$\tilde{\gamma}_i(v) := \frac{\beta_i v_i}{d_i v_i - \sum_{j \neq i} \tilde{a}_{ij} v_j} = \gamma_i(\tilde{v}), \quad i = 1, \dots, n, \tag{3.12}$$

where $\tilde{v} = (\frac{v_1}{c_1}, \dots, \frac{v_n}{c_n})$. The result now follows from Theorem 3.3. □

Remark 3.6. For the case of (2.1) with autonomous discrete delays $\tau_{ik}(t) \equiv \tau_{ik}$, the global asymptotic stability (but not the exponential stability) of the positive equilibrium was proven in [7] under **(H2)** with $v = \mathbf{1} := (1, \dots, 1)$:

$$1 < \gamma_i(\mathbf{1}) := \frac{\beta_i}{d_i - \sum_{j \neq i} a_{ij}} \leq e^2, \quad i = 1, \dots, n.$$

However, the arguments in [7] do not carry out for the present situation, since properties of ω-limit sets for *autonomous* DDEs were employed to derive the result. Although Theorem 3.3 and Theorem 3.5 address the more general situation of Nicholson systems with time-varying discrete delays, with $\gamma_i(\mathbf{1})$ replaced by $\gamma_i(\mathbf{v})$ for some positive vector v, the global attractivity of N^* cannot be derived when $\gamma_i(\mathbf{1}) = e^2$ for some i. On the other hand, combining the techniques above with the ones in [7], for the autonomous case of (1.1) with $\tau_{ik}(t) \equiv \tau_{ik}$ it follows that (3.10) is a sufficient condition for the global asymptotic stability of N^*.

4 Global attractivity under small delays

In this section, the goal is to prove that a condition on the size of the delays, together with **(H1)**, implies that the positive equilibrium is a global attractor. Here, some ideas in So and Yu [16] (for the scalar case) and Jia et al. [10] (for the n-dimensional case) are followed. However, significant adjustments to the arguments in [10] have to be performed, in order to eliminate the quite restrictive assumption (1.6). We emphasize that, without imposing (1.6), not only are the components of N^* in general different from each other, but also N^* cannot be computed explicitly.

To simplify some arguments, we write (2.1) as

$$N_i'(t) = -d_i N_i(t) + \sum_{j \neq i} a_{ij} N_j(t) + \sum_{k=1}^{m_i} \beta_{ik} N_i(t - \tau_{ik}(t)) e^{-N_i(t - \tau_{ik}(t))}, \quad i = 1, \dots, n, \tag{4.1}$$

where $m_1, \dots, m_n \in \mathbb{N}$, all the coefficients and delays are as in (2.1) and moreover we now demand that $\beta_{ik} > 0$ for all $i = 1, \dots, n, k = 1, \dots, m_i$. In what follows, as before we always assume $a_{ii} = 0$, and denote

$$\beta_i = \sum_{k=1}^{m_i} \beta_{ik}, \quad r_i = \max_{1 \leq k \leq m_i} \sup_{t \geq 0} \tau_{ik}(t), \quad \tau = \max_{1 \leq i \leq n} r_i \quad \text{for } i = 1, \dots, n.$$

Assume **(H0), (H1)**, and effect again the change of variables (3.6), which transforms system (2.1) into (3.7), also written as

$$x_i'(t) = -d_i(x_i(t)+1) + \sum_{j\neq i} \hat{a}_{ij}(x_j(t)+1) + \sum_{k=1}^{m_i} \beta_{ik}(x_i(t-\tau_{ik}(t))+1)e^{-N_i^*(x_i(t-\tau_{ik}(t)+1)}, \quad 1\le i\le n, \tag{4.2}$$

with $\hat{a}_{ij} = \frac{N_j^*}{N_i^*}a_{ij}$, $i\neq j$. In this way, the global attractivity of the equilibrium N^* for (2.1) is equivalent to the global attractivity of the trivial solution for (4.2).

We start with a useful lemma, whose elementary proof can be checked by the reader or found in [11, p. 122] or [16].

Lemma 4.1. *Let $u\ge 0, v\ge 0$ be such that $u\le e^v-1$ and $v\le 1-e^{-u}$. Then $u=v=0$.*

To prove that solutions $x(t)$ of (4.2) satisfy $x(t)\to 0$ as $t\to\infty$, we will ask for the following condition on the size of the delays:

(H3) $$(e^{d_ir_i}-1)\frac{\beta_i}{d_i}N_i^*e^{-N_i^*}\le 1, \quad \text{for all } i=1,\dots,n.$$

Theorem 4.2. *Assume* **(H0), (H1)** *and* **(H3)**. *Then, the positive equilibrium N^* of the Nicholson system* (4.1) *is a global attractor of all positive solutions; i.e., all solutions of* (2.1) *with initial conditions $N_0\in C_0^+$ satisfy*

$$\lim_{t\to\infty} N(t) = N^*.$$

Proof. For solutions $x(t)=(x_1(t),\dots,x_n(t))$ of (4.2), define

$$\underline{\lambda} = \min_{1\le i\le n}\liminf_{t\to\infty} x_i(t), \quad \overline{\mu} = \max_{1\le i\le n}\limsup_{t\to\infty} x_i(t).$$

Observe that, from Theorem 2.5, $-1<\underline{\lambda}\le\overline{\mu}<\infty$. Our aim is to show that $\max(\overline{\mu},-\underline{\lambda})=0$, since this implies that $\overline{\mu}=\underline{\lambda}=0$.

For the sake of contradiction, assume that $\max(\overline{\mu},-\underline{\lambda})>0$ holds. Suppose that $\overline{\mu}=\max(\overline{\mu},-\underline{\lambda})>0$ (the case $-\underline{\lambda}=\max(\overline{\mu},-\underline{\lambda})>0$ is analogous).

Fix i_1 such that $\overline{\mu}=\limsup_{t\to\infty}x_{i_1}(t)$. To simplify the notation, denote $\mu=\overline{\mu}$ and $\lambda=\liminf_{t\to\infty}x_{i_1}(t)$.

By the fluctuation lemma [15], there exists an increasing sequence (t_q) such that $t_q\to\infty$, $x_{i_1}(t_q)>0$ with $x_{i_1}(t_q)\to\mu, x_{i_1}'(t_q)\to 0$. We divide the rest of the proof into several steps.

Step 1. We first prove that there exists $q_0\in\mathbb{N}$ such that, whenever $q\ge q_0$, there is $l_q\in[t_q-r_{i_1},t_q)$ such that $x_{i_1}(l_q)=0$ and $x_{i_1}(t)>0$, for $t\in(l_q,t_q)$.

Suppose the assertion is false. Then there is a subsequence of t_q, which we also denote by t_q, such that $x_{i_1}(t)>0$, for all $t\in[t_q-r_{i_1},t_q)$. We have

$$\begin{aligned} x_{i_1}'(t_q) = & -d_{i_1}(x_{i_1}(t_q)+1) + \sum_{j=1}^n \hat{a}_{i_1j}(x_j(t_q)+1) \\ & + \sum_{k=1}^{m_{i_1}} \beta_{i_1k}(x_{i_1}(t_q-\tau_{i_1k}(t_q))+1)e^{-N_{i_1}\left(x_{i_1}(t_q-\tau_{i_1k}(t_q))+1\right)} \\ < & -d_{i_1}(x_{i_1}(t_q)+1) + \sum_{j=1}^n \hat{a}_{i_1j}(x_j(t_q)+1) + e^{-N_{i_1}^*}\sum_{k=1}^{m_{i_1}} \beta_{i_1k}(x_{i_1}(t_q-\tau_{i_1k}(t_q))+1), \end{aligned} \tag{4.3}$$

as by assumption $x_{i_1}(t_q - \tau_{i_1k}(t_q)) > 0$. Now we claim that, for all k, $\lim x_{i_1}(t_q - \tau_{i_1k}(t_q)) = \mu$. On the one hand, $\limsup x_{i_1}(t_q - \tau_{i_1k}(t_q)) \le \mu$. On the other hand, taking $\liminf$ on (4.3), observing again that $x_{i_1}(t_q - \tau_{i_1k}(t_q)) > 0$ and with $\alpha_k := \liminf x_{i_1}(t_q - \tau_{i_1k}(t_q))$, we get

$$\begin{aligned}0 &\le (\mu+1)\left(-d_{i_1} + \sum_{j\neq i_1} \hat{a}_{i_1j}\right) + e^{-N^*_{i_1}} \sum_{k=1}^{m_{i_1}} \beta_{i_1k}(\alpha_k+1)\\ &= -(\mu+1)e^{-N^*_{i_1}}\beta_{i_1} + e^{-N^*_{i_1}} \sum_{k=1}^{m_{i_1}} \beta_{i_1k}(\alpha_k+1)\\ &= -\mu e^{-N^*_{i_1}}\beta_{i_1} + e^{-N^*_{i_1}} \sum_{k=1}^{m_{i_1}} \beta_{i_1k}\alpha_k,\end{aligned}$$

which implies $\mu \le \frac{1}{\beta_{i_1}} \sum_{k=1}^{m_i} \beta_{i_1k}\alpha_k$. However, since $\alpha_k \le \mu$, one also has $\frac{1}{\beta_{i_1}} \sum_{k=1}^{m_i} \beta_{i_1k}\alpha_k \le \mu$, so that

$$\mu \le \frac{1}{\beta_{i_1}} \sum_{k=1}^{m_i} \beta_{i_1k}\alpha_k \le \mu.$$

This is only possible with $\alpha_k = \mu$, for all k. Therefore, for every $k = 1, ..., m_{i_1}$,

$$\mu = \liminf x_{i_1}(t_q - \tau_{i_1k}(t_q)) \le \limsup x_{i_1}(t_q - \tau_{i_1k}(t_q)) \le \mu,$$

and consequently $\lim x_{i_1}(t_q - \tau_{i_1k}(t_q))$ exists and is equal to μ, for all k. Taking limits in (4.3) yields

$$\begin{aligned}0 &\le (\mu+1)\Big(-d_{i_1} + \sum_{j\neq i_1} \hat{a}_{i_1j}\Big) + (\mu+1)\beta_{i_1} e^{-N^*_{i_1}(\mu+1)}\\ &= (\mu+1)\Big(-d_{i_1} + \sum_{j\neq i_1} \hat{a}_{i_1j}\Big)\left(1 - e^{-N^*_{i_1}\mu}\right) < 0,\end{aligned}$$

which is not possible. This finishes *Step 1*.

Step 2. Next, we show that λ, μ satisfy

$$\begin{cases} N^*_{i_1}\lambda \ge e^{-N^*_{i_1}\mu} - 1, \\ N^*_{i_1}\mu \le e^{-N^*_{i_1}\lambda} - 1. \end{cases}$$

Let $\varepsilon > 0$ be given so that $\lambda - \varepsilon \ge \underline{\lambda} - \varepsilon > -1$. By the definition of λ and μ, there exists $q_1 > q_0$ such that

$$\lambda - \varepsilon < x_{i_1}(t), \quad x_j(t) < \mu + \varepsilon, \quad 1 \le j \le n$$

whenever $t > \min\{t_{q_1}, s_{q_1}\} - 2\tau$. Considering separately the cases $x_{i_1}(t - \tau_{i_1k}(t)) \le 0$ and $x_{i_1}(t - \tau_{i_1k}(t)) > 0$, it is clear that

$$x_{i_1}(t - \tau_{i_1k}(t))e^{-N^*_{i_1}x_{i_1}(t-\tau_{i_1k}(t))} < \mu + \varepsilon, \tag{4.4}$$

for all i, k and t sufficiently large.

For l_q as in *Step 1*, multiplying the i_1-equation of (4.2) by $e^{d_{i_1}t}$ and integrating over the interval $[l_q, t_q]$ gives

$$(1 + x_{i_1}(t_q))e^{d_{i_1}t_q} - e^{d_{i_1}l_q} = A_q + B_q, \tag{4.5}$$

where

$$A_q = \sum_{j\neq i_1} \hat{a}_{i_1 j} \int_{l_q}^{t_q} (x_j(t)+1)e^{d_{i_1}t}dt \leq (\mu+\varepsilon+1)\left(\sum_{j\neq i_1} \hat{a}_{i_1 j}\right) \frac{e^{d_{i_1}t_q}-e^{d_{i_1}l_q}}{d_{i_1}},$$

and

$$B_q = \sum_{k=1}^{m_{i_1}} \beta_{i_1 k} \int_{l_q}^{t_q} (x_{i_1}(t-\tau_{i_1 k}(t))+1)e^{-N_{i_1}^*(x_{i_1}(t-\tau_{i_1 k}(t))+1)}e^{d_{i_1}t}dt\,.$$

From (4.4), we obtain

$$\begin{aligned}
B_q &\leq \sum_{k=1}^{m_{i_1}} \beta_{i_1 k} \int_{l_q}^{t_q} \left((\mu+\varepsilon)e^{-N_{i_1}^*} + e^{-N_{i_1}^*(x_{i_1}(t-\tau_{i_1 k}(t))+1)}\right) e^{d_{i_1}t}dt \\
&\leq \sum_{k=1}^{m_{i_1}} \beta_{i_1 k} \left((\mu+\varepsilon)e^{-N_{i_1}^*} + e^{-N_{i_1}^*(1+\lambda-\varepsilon)}\right) \int_{l_q}^{t_q} e^{d_{i_1}t}dt \\
&= \beta_{i_1}e^{-N_{i_1}^*}\left(\mu+\varepsilon+e^{-N_{i_1}^*(\lambda-\varepsilon)}\right) \frac{e^{d_{i_1}t_q}-e^{d_{i_1}l_q}}{d_{i_1}}.
\end{aligned}$$

Inserting these upper bounds for A_q and B_q in (4.5), we derive

$$\begin{aligned}
&x_{i_1}(t_q)e^{d_{i_1}t_q} \\
&\quad\leq -(e^{d_{i_1}t_q}-e^{d_{i_1}l_q}) + A_q + B_q \\
&\quad\leq \left[-d_{i_1} + (\mu+\varepsilon+1)\left(\sum_{j\neq i_1}\hat{a}_{i_1 j}\right) + \beta_{i_1}e^{-N_{i_1}^*}\left(\mu+\varepsilon+e^{-N_{i_1}^*(\lambda-\varepsilon)}\right)\right] \frac{e^{d_{i_1}t_q}-e^{d_{i_1}l_q}}{d_{i_1}} \\
&\quad= \left[-d_{i_1} + (\mu+\varepsilon+1)\left(-\beta_{i_1}e^{-N_{i_1}^*}+d_{i_1}\right) + \beta_{i_1}e^{-N_{i_1}^*}\left(\mu+\varepsilon+e^{-N_{i_1}^*(\lambda-\varepsilon)}\right)\right] \frac{e^{d_{i_1}t_q}-e^{d_{i_1}l_q}}{d_{i_1}} \\
&\quad= \left[(\mu+\varepsilon)d_{i_1} + \beta_{i_1}e^{-N_{i_1}^*}(e^{-N_{i_1}^*(\lambda-\varepsilon)}-1)\right] \frac{e^{d_{i_1}t_q}-e^{d_{i_1}l_q}}{d_{i_1}},
\end{aligned}$$

so that

$$x_{i_1}(t_q) \leq \left[(\mu+\varepsilon)d_{i_1} + \beta_{i_1}e^{-N_{i_1}^*}(e^{-N_{i_1}^*(\lambda-\varepsilon)}-1)\right] \frac{1-e^{-d_{i_1}r_{i_1}}}{d_{i_1}}.$$

Letting $q\to\infty$ and $\varepsilon\to 0^+$ yields

$$\mu \leq \mu(1-e^{-d_{i_1}r_{i_1}}) + \beta_{i_1}e^{-N_{i_1}^*}(e^{-N_{i_1}^*\lambda}-1)\frac{1-e^{-d_{i_1}r_{i_1}}}{d_{i_1}},$$

hence

$$0 < \mu \leq (e^{d_{i_1}r_{i_1}}-1)\frac{\beta_{i_1}}{d_{i_1}}e^{-N_{i_1}^*}(e^{-N_{i_1}^*\lambda}-1)\,. \tag{4.6}$$

In particular, this implies that $\lambda < 0$.

Reasoning as above, we take an increasing sequence (s_q) such that $s_q\to\infty$ and $x_{i_1}(s_q)\to\lambda$, $x_{i_1}'(s_q)\to 0$. Next, as in Step 1, we can find $q_2 > q_1$ such that, if $q\geq q_2$, there is $p_q\in[s_q-r_{i_1}, s_q)$ such that $x_{i_1}(p_q)=0$ and $x_{i_1}(t)<0$, for $t\in(p_q,s_q)$. As above in this step, similar arguments now show that

$$x_{i_1}(s_q) \geq \left[(\lambda-\varepsilon)d_{i_1} + \beta_{i_1}e^{-N_{i_1}^*}\left(e^{-(\mu+\varepsilon)N_{i_1}^*}-1\right)\right]\frac{1-e^{-d_{i_1}r_{i_1}}}{d_{i_1}}$$

and consequently, taking limits as $q \to \infty$ and $\varepsilon \to 0^+$,

$$\lambda \geq (e^{d_{i_1} r_{i_1}} - 1) \frac{\beta_{i_1}}{d_{i_1}} e^{-N^*_{i_1}} \left(e^{-N^*_{i_1} \mu} - 1 \right). \tag{4.7}$$

We finally apply hypothesis **(H3)**. With **(H3)**, the estimates (4.6) and (4.7) lead to

$$N^*_{i_1} \mu \leq e^{-N^*_{i_1} \lambda} - 1 \quad \text{and} \quad N^*_{i_1} \lambda \geq e^{-N^*_{i_1} \mu} - 1, \tag{4.8}$$

which proves the claim.

Step 3. Define $u = N^*_{i_1} \mu$, $v = N^*_{i_1} \lambda$, and note that $u, v > 0$. From (4.8), we obtain

$$\begin{cases} v \leq 1 - e^{-u}, \\ u \leq e^v - 1. \end{cases}$$

From Lemma 4.1, we have $u = v = 0$. Consequently $\mu = \lambda = 0$, which contradicts the assumption of $\mu > 0$. This finishes the proof. □

As in Section 3, the above theorem is easily extended to more general Nicholson systems (1.1) by effecting the scalings $N_i(t) \mapsto c_i N_i(t)$, $i = 1, \ldots, n$. For such systems, Theorem 4.2 reads as follows.

Theorem 4.3. *For system* (1.1), *assume* **(H0), (H1)**, $c_i > 0$ *and*

$$(e^{d_i r_i} - 1) \frac{\beta_i}{d_i} c_i N^*_i e^{-c_i N^*_i} \leq 1, \quad \textit{for all } i = 1, \ldots, n.$$

Then, the positive equilibrium N^* of (1.1) *is a global attractor (in C_0^+).*

Condition **(H3)** is a condition on the size of the delays, as $\lim_{r_i \to \infty} (e^{d_i r_i} - 1) = \infty$, $\lim_{r_i \to 0^+} (e^{d_i r_i} - 1) = 0$, thus **(H3)** fails to be true if the delays are too large. On the other hand, **(H3)** involves the a priori knowledge of N^*, therefore it is relevant to obtain criteria that do not depend on it.

Corollary 4.4. *For system* (1.1), *assume* **(H0), (H1)**, $c_i > 0$ *and*

$$(e^{d_i r_i} - 1) \frac{\beta_i}{d_i} \leq e, \quad \textit{for all } i = 1, \ldots, n. \tag{4.9}$$

Then, the positive equilibrium N^* of (1.1) *is a global attractor (in C_0^+).*

Proof. Since $x e^{-x} \leq e^{-1}$ for $x \geq 0$, condition (4.9) implies **(H3)**. □

Corollary 4.5. *Assume that* **(H1)** *holds, for some $v > 0$ for which*

(H3*) $$(e^{d_i r_i} - 1) \max_{1 \leq j,k \leq n} \left(\frac{v_j}{v_k} \log \gamma_k(v) \right) \leq 1, \quad \textit{for all } i = 1, \ldots, n,$$

where $\gamma_i(v)$ are as in (2.5). *Then there is a positive equilibrium N^* of* (2.1), *which is a global attractor (in C_0^+).*

Proof. Let $\frac{N_k^*}{v_k} = \max_i \frac{N_i^*}{v_i}$. Proceeding as in the proof of Theorem 3.3, we obtain $e^{N_k^*} \le \gamma_k(v)$, hence

$$N_k^* \le \log \gamma_k(v).$$

Moreover, $\frac{\beta_i}{d_i} e^{-N_i^*} = \frac{d_i - \sum \hat{a}_{ij}}{d_i} \le 1$, where $\hat{a}_{ij} = a_{ij} N_j^*/N_i^*$ for $j \ne i$. Finally note that, for any given j,

$$N_j^* = \frac{N_j^*}{v_j} v_j \le \frac{N_k^*}{v_k} v_j \le \frac{v_j}{v_k} \log \gamma_k(v),$$

and thus for any $i = 1, \dots, n$ we have

$$(e^{d_i r_i} - 1) \frac{\beta_i}{d_i} \bar{N}_i^* e^{-N_i^*} \le (e^{d_i r_i} - 1) \max_{1 \le j,k \le n} \left(\frac{v_j}{v_k} \log \gamma_k(v) \right) \le 1.$$

The statement is now a consequence of Theorem 4.2. □

When **(H1), (H3*)** hold with $v = (1, \dots, 1)$, we obtain a simpler version of the above statement.

Corollary 4.6. *Assume* **(H0)***,*

$$\gamma_i := \frac{\beta_i}{d_i - \sum_{j \ne i} a_{ij}} > 1 \quad \text{and} \quad (e^{d_i r_i} - 1) \log \left(\max_{1 \le j \le n} \gamma_j \right) \le 1, \quad \textit{for all } i = 1, \dots, n.$$

Then there is a positive equilibrium N^* of (2.1), which is a global attractor (in C_0^+).

We now adapt e.g. Corollary 4.6 to more general systems (1.1).

Corollary 4.7. *Assume that* **(H0)** *holds, $c_i > 0$ and*

$$(e^{d_i r_i} - 1) \log \left(\max_{1 \le j \le n} \gamma_j(c^{-1}) \right) \le 1, \quad \textit{for all } i = 1, \dots, n, \tag{4.10}$$

where $\gamma_i(c^{-1}) := \frac{\beta_i c_i^{-1}}{d_i c_i^{-1} - \sum_{j \ne i} a_{ij}} c_j^{-1} > 1$ $(1 \le i \le n)$. Then there is a positive equilibrium N^* of (1.1), which is a global attractor (in C_0^+).

Proof. As before, effect the change of variables $\tilde{N}_i(t) = c_i N_i(t)$, $i = 1, \dots, n$, and recall that (1.1) is transformed into a system of the form (2.1), with the coefficients a_{ij} replaced by $\tilde{a}_{ij} := \frac{c_i}{c_j} a_{ij}$. With the notation in (3.12), we have

$$\tilde{\gamma}_i(\mathbf{1}) = \gamma_i(c^{-1}), \quad i = 1, \dots, n.$$

Hence, the result follows from the above corollary. □

Remark 4.8. In [10], the authors considered (2.1) but, instead of assuming $a_{ii} = 0$ and incorporating this coefficient in d_i, for each i, they assumed that $\sum_{j=1}^n a_{ij} = 0$. Changing accordingly to the notation followed here, it was assumed in [10] that $\gamma_i = \gamma_i(\mathbf{1})$ is the same constant $c > 1$ for all i, i.e., condition (1.6) is satisfied, so that $N^* = (\log c, \dots, \log c)$ is the positive equilbrium. With this notation, Jia et al. [10] proved the global attractivity of N^* under the additional condition

$$(e^{d_i \tau} - 1) \frac{d_i - \sum_{j \ne i} a_{ij}}{d_i} \log c \le 1, \quad i = 1, \dots, n, \tag{4.11}$$

where $\tau = \max_{1\le i\le n} r_i$. Note however that, in this situation, (4.11) is equivalent to

$$(e^{d_i\tau}-1)\frac{\beta_i}{d_i}c^{-1}\log c \le 1, \quad i=1,\dots,n,$$

and $c^{-1}\log c = e^{-N_i^*}N_i^*$ for $i=1,\dots,n$. This shows that the criterion in [10] is just a particular case of our Theorem 4.2.

Remark 4.9. In the recent paper [4], El-Morshedy and Ruiz-Herrera considered an abstract setting, in which they developed a geometric method to prove the global attractivity of non-trivial equilibria for systems of *autonomous* DDEs. They also gave an application to an **autonomous** Nicholson system of the form (1.4), where $\tau_{ik}(t)\equiv\tau_{ik}\ge 0$ for all i,k. However, two major constraints are imposed in [4]: first, the authors assume the **existence** of a positive equilibrium N^* of (1.4); secondly, the matrix $D-A$ is assumed to be not only a non-singular M-matrix, but **diagonally dominant**, i.e., $d_i-\sum_{j\ne i}a_{ij}>0$ for $i=1,\dots,n$. For γ_i as above, and denoting $\rho=\max_{1\le i,k\le m}(\tau_{ik}d_i)$, it was shown in [4] that N^* is a global attractor for (1.4) if

$$\log(\gamma_i)\le 1+\frac{e^\rho}{e^\rho-1}, \quad i=1,\dots,n. \tag{4.12}$$

This criterion is not always comparable with the ones presented in Theorem 4.2 and its corollaries, in the sense that for different concrete values of the coefficients and delays our results might provide better criteria than the one in [4], and vice-versa. On the other hand, condition (4.12) provides both a delay-independent and a delay-dependent criterion for global attractivity. In fact, observe that the function $g(\rho)=1+\frac{e^\rho}{e^\rho-1}$ is decreasing on $(0,\infty)$ with $g(0^+)=\infty, g(\infty)=2$, thus (4.12) is always satisfied if $1<\gamma_i\le e^2$ for $i=1,\dots,n$. This means that the nice result in [4] in particular recovers the requirement in [7] for the **absolute** global attractivity of N^* (see also Remark 3.6); at the same time, (4.12) is clearly satisfied if $\tau=\max_{1\le i,k\le m}\tau_{ik}$ is small.

5 Examples

Example 5.1. In (1.1), let $n=2, m=1, a_{12}=a_{21}=1, d_1=3, d_2=2, c_1=c_2=1$:

$$\begin{aligned} N_1'(t) &= -3N_1(t)+N_2(t)+\beta_1 N_1(t-\tau_1(t))e^{-N_1(t-\tau_1(t))}\\ N_2'(t) &= -2N_2(t)+N_1(t)+\beta_2 N_2(t-\tau_2(t))e^{-N_2(t-\tau_2(t))}, \end{aligned} \tag{5.1}$$

where $\beta_i>0, \tau_i(t)$ are continuous, nonnegative and bounded, $i=1,2$. Below, we shall use the notation $\gamma_i(v)$ as in (2.5) for $v\in\mathbb{R}^2_+$, and $\gamma_i=\gamma_i(\mathbf{1})$, $i=1,2$.

Now, choose $\beta_1=1, \beta_2=3$. For $\mathbf{1}=(1,1)$, we get $\gamma_1(\mathbf{1})=\frac{1}{2}$, hence **(H1)** is not satisfied with $v=\mathbf{1}$. With the notation in (2.3)–(2.4),

$$D-A=\begin{bmatrix} 3 & -1\\ -1 & 2\end{bmatrix}, \quad M=\begin{bmatrix} -2 & 1\\ 1 & 1\end{bmatrix}.$$

One easily verifies that the eigenvalues of $D-A$ are real and positive and that $s(M)>0$. Hence, **(H1)** is satisfied (see Remark 2.2). We look for some $v>0$ for which (2.5) holds true.

For a vector of the form $v=(1,v_2)>0$, we have $\gamma_1(v)=1/(3-v_2), \gamma_2(v)=3v_2/(2v_2-1)$, hence v satisfies (2.5) if and only if $v=(1,2+\varepsilon)$ with $\varepsilon\in(0,1)$. In order to have **(H2)** satisfied as well, we must find $\varepsilon\in(0,1)$ such that

$$\frac{1}{1-\varepsilon}<e^{\frac{2}{2+\varepsilon}}, \quad \frac{6+3\varepsilon}{3+2\varepsilon}<e^{\frac{2}{2+\varepsilon}},$$

which is clearly possible if $\varepsilon > 0$ is sufficiently small. From Theorem 3.3, (5.1) has a globally exponentially stable positive equilibrium N^*.

Next, we consider the same system (5.1) with $\beta_1 = 1$, but choose $\beta_2 = 5$. We still have that **(H1)** is satisfied with v of the form $v = (1, 2+\varepsilon) =: v_\varepsilon$ with $\varepsilon \in (0,1)$. Note however that

$$\Gamma_2(\varepsilon) := \gamma_2(v_\varepsilon) = \frac{10+5\varepsilon}{3+2\varepsilon} > \Gamma_2(1^-) = 3 > e^{\frac{2}{2+\varepsilon}}, \quad \text{for } \varepsilon \in (0,1),$$

and therefore it is not possible to find $v > 0$ such that **(H2)** holds. In this way, we cannot guarantee that the equilibrium $N^* > 0$ is globally attractive for large delays $\tau_1(t), \tau_2(t)$. We now try to find conditions on the size of $r_i = \sup_{t\geq 0} \tau_i(t), i = 1,2$, so that **(H3*)** is satisfied. For e.g. $\varepsilon = 0.001$, we obtain $\max\{\gamma_1(v_\varepsilon), \gamma_2(v_\varepsilon)\} = \frac{10.005}{3.002}$ and $2.001 \times \log(\frac{10.005}{3.002}) \approx 2.4088$. For this choice,

$$\ell := \log\left(1 + \left[2.001 \times \log\left(\frac{10.005}{3.002}\right)\right]^{-1}\right) \approx 0.34723$$

and condition **(H3*)** reads as $3r_1 \leq \ell, 2r_2 \leq \ell$. In this case, Corollary 4.5 allows us to conclude that the equilibrium N^* is globally attractive.

Alternatively, the use of Corollary 4.4 gives less restrictive conditions on the size of the delays: $3r_1 \leq \ell_1$ and $2r_2 \leq \ell_2$, where $\ell_1 = \log(1+3e) \approx 2.21428$, $\ell_1 = \log(1+2e/5) \approx 0.73588$.

Example 5.2. Consider

$$\begin{aligned} N_1'(t) &= -2N_1(t) + N_2(t) + 3N_1(t-\tau_1(t))e^{-N_1(t-\tau_1(t))} \\ N_2'(t) &= -2N_2(t) + N_1(t) + 15N_2(t-\tau_2(t))e^{-N_2(t-\tau_2(t))}, \end{aligned} \tag{5.2}$$

with $\tau_i(t)$ continuous, nonnegative and bounded, $i = 1,2$.

For a vector $v = (1, v_2) > 0$, assumption **(H1)** is satisfied if and only if $1/2 < v_2 < 2$. In particular, there exists a positive equilibrium N^* for (5.2). Write $\gamma_1(v) = \frac{3}{2-v_2}, \gamma_2(v) = \frac{15v_2}{2v_2-1}$. Define $m_0 = \max(1/v_2, v_2)$. Separating the cases $1/2 < v_2 \leq 1$ and $1 \leq v_2 < 2$, one can verify that

$$e^{\frac{2}{m_0}} \leq e^2.$$

On the other hand, $\gamma_2(v) > \gamma_2(v)|_{v_2=2} = 10 > e^2$, which implies that **(H2)** is never satisfied.

Observe however that $\gamma_1 = 3, \gamma_2 = 15$. With $\tau = \sup_{t\geq 0} \max\{\tau_1(t), \tau_2(t)\}$, from Corollary 4.6 the assumption

$$2\tau \leq \log\left(1 + (\log 15)^{-1}\right) \approx 0.31428, \tag{5.3}$$

implies the global attractivity of N^*. For instance, with $\tau_1(t) = \frac{1}{10}\arctan t, \tau_2(t) = \frac{\cos^2 t}{8}$, the positive equilibrium N^* is always a global attractor.

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