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Bifurcation and blow-up results for equations with p-Laplacian and convex-concave nonlinearity

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Abstract. This paper is concerned with the existence of global, blow-up and bifurcation solutions for parametrized families of elliptic and parabolic equations with p-Laplacian and concave-convex nonlinearity. The main results are obtained by means of a generalised Collatz–Wielandt formula.

Keywords: concave–convex nonlinearity, Collatz–Wielandt formula, p-Laplacian, bifurcation, blow up.

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1 Introduction

In this paper we study the following parabolic problem

$$\begin{cases} u_t = \Delta_p u + \lambda f(x)|u|^{\gamma-2}u + q(x)|u|^{\alpha-2}u & \text{in } (0,T)\times\Omega, \\ u = 0 & \text{on } [0,T)\times\partial\Omega, \\ u|_{t=0} = u_0 & \text{in } \Omega, \end{cases} \tag{1.1}$$

and the corresponding stationary problem

$$\begin{cases} -\Delta_p u = \lambda f(x)|u|^{\gamma-2}u + q(x)|u|^{\alpha-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

Here Ω is a bounded domain in $\mathbb{R}^N$ with $C^{1,\beta}$-boundary $\partial\Omega$ for some $\beta \in (0,1)$, $N \geq 1$, $0 < T < \infty$; Δ_p is the p-Laplacian, $1 < \alpha < p < \gamma$, $f := f(x)$ and $q := q(x)$ are measurable functions on Ω. We assume that $u_0 \in W_0^{1,p}(\Omega)$ and by a weak solution of (1.1) we mean a function

$$u \in C(0,T; L^2(\Omega)) \cap L^p(0,T; W_0^{1,p}(\Omega)) \cap L^\infty((0,T)\times\Omega), \qquad u_t \in L^2((0,T)\times\Omega),$$

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satisfying

$$\int_\Omega u(t)\phi(t)dx - \int_\Omega u_0\phi(0)dx = \int_0^t \int_\Omega (u\phi_t - |\nabla u|^{p-2}(\nabla u, \nabla\phi) + \lambda f u^{\gamma-1}\phi + q u^{\alpha-1}\phi)dxdt \quad (1.3)$$

for all $t \in [0, T)$ and for all test functions $\phi \in C^1([0, T) \times \overline{\Omega})$, $\phi = 0$ on $[0, T) \times \partial\Omega$. A weak solution $u \in W_0^{1,p}(\Omega)$ of (1.2) is defined analogously.

Beginning with the well-known results of Ambrosetti, Brezis, Cerami [2], problems with concave-convex nonlinearity of type (1.2) have received a lot of attention (cf., in particular, Ambrosetti, Azorero, Peral [3], De Figueiredo, Gossez, Ubilla [19] and the references therein). In the case $f, q \in C(\overline{\Omega})$, $p \geq 2$, existence of local in time solutions of (1.1) is well understood; see Ladyzhenskaja, Solonnikov, Ural'tseva [30] for $p = 2$ and Zhao [42] for $p \geq 2$. Furthermore, for $p = 2$ and $f(x), q(x) \equiv 1$, Escobedo, Cazenave, Dickstein [18] have proved that there exists a unique positive solution of (1.1) defined on *a maximal time interval* $(0, T_m)$, where the blow up alternative holds: either $T_m = +\infty$, i.e., u_λ is a *global in time solution*, or else $T_m < +\infty$ and u_λ *blows up in finite time* $\|u_\lambda(t)\|_{L^\infty} \to +\infty$ as $t \to T_m$. Furthermore, they found that there exists a thresholds value $\Lambda > 0$ such that (1.1) has a global solution for $0 < \lambda \leq \Lambda$, whereas any positive solution of (1.1) blows up in finite time for $\lambda > \Lambda$. The dividing line Λ coincides with the critical value of Ambrosetti, Brezis, Cerami [2] for the stationary problem (1.2) which separates the interval $(0, \Lambda]$ of the existence of minimal positive solution of (1.2) and the interval $(\Lambda, +\infty)$ where positive solutions of (1.2) are absent. The key tool in [18] relies on the arguments introduced by Brezis, Cazenave, Martel, Ramiandrisoa in [9], which is based on the proving that any global solution $u_\lambda(t)$ of parabolic problem (1.1) converges to a weak solution of the stationary problem (1.2) as $t \to +\infty$. In this way, the blow up behaviour for $\lambda > \Lambda$ is obtained by contradiction.

The purpose of this paper is to investigate the existence of global and blow-up solutions of (1.1) and the existence of bifurcations for branches of positive solutions of (1.2) with respect to the behaviour of the functions f, q and the value of the parameter λ. Our approach is based on the development of the extended functional method [8, 21, 23–26]. The central role in this method is played by the so-called generalised Collatz–Wielandt formula which gives a threshold value λ^* of the existence of positive solutions for nonlinear elliptic boundary value problems [21, 24]. Furthermore, the dual variational problem corresponding to the Collatz–Wielandt formula allows finding a threshold value λ^{**} for the existence of global or blow-up solutions to parabolic problems [23, 25]. Our interest in the development of this approach also emerges from the fact that the Collatz–Wielandt formula gives a simple numerical algorithm for the calculating the threshold value λ^* [26].

2 Main results

The Collatz–Wielandt formula for the Perron root $r = \max_{x\in(\mathbb{R}^+)^n,\, x\neq 0} L(x)$ of $A_{n\times n} > 0$, where

$$L(x) = \min_{1\leq i\leq n} \left\{ \frac{[Ax]_i}{x_i} : x_i \neq 0 \right\}, \qquad x \in (\mathbb{R}^+)^n, \quad (2.1)$$

was discovered in 1942 by L. Collatz [10] and then developed by H. Wielandt [41] in 1950. Since (2.1) has the following equivalent form (see e.g. [26])

$$L(x) = \min_{z\in(\mathbb{R}^+)^n} \left\{ \frac{\langle Ax, z\rangle}{\langle x, z\rangle} : z \neq 0 \right\}, \qquad x \in (\mathbb{R}^+)^n,$$

it is natural to call

$$\lambda^* = \sup_{u\in\mathcal{C}^+} \inf_{\phi\in\mathcal{C}_0^+} \left\{ L(u,\phi) : \int_\Omega f u^{\gamma-1}\phi \, dx \neq 0 \right\} \tag{2.2}$$

as a *generalized Collatz–Wielandt formula*, where

$$L(u,\phi) := \frac{\int_\Omega (|\nabla u|^{p-2}\nabla u, \nabla\phi)\, dx - \int_\Omega q u^{\alpha-1}\phi \, dx}{\int_\Omega f u^{\gamma-1}\phi \, dx}, \quad \text{for } \int_\Omega f u^{\gamma-1}\phi\, dx \neq 0,$$

$$\mathcal{C}^+ = \{u \in C^1(\overline{\Omega}) |\ u > 0 \text{ in } \Omega,\ u = 0 \text{ on } \partial\Omega\}, \tag{2.3}$$

$$\mathcal{C}_0^+ = \{\phi \in C^1(\overline{\Omega}) |\ \phi(x) \geq 0 \text{ in } \Omega,\ \text{supp}(\phi) \subset \Omega,\ \phi \not\equiv 0\}. \tag{2.4}$$

Remark 2.1. Another type of generalization for the Collatz–Wielandt formula to (1.2) can be obtained directly from (2.1), i.e. as follows

$$\tilde{\lambda}^* = \sup_{u\in C^2(\Omega)} \inf_{x\in\Omega} \left\{ \frac{-\Delta_p u(x) - q(x)u^{\alpha-1}(x)}{f(x)u^{\gamma-1}(x)} : u = 0 \text{ on } \partial\Omega,\ u > 0,\ f(x)u^{\gamma-1}(x) \neq 0 \right\}.$$

For similar approach, the reader is refereed to Barta [4] , Berestycki, Nirenberg, Varadhan [5], Birindelli, Demengel [7], Donsker, Varadhan [17], Berestycki, Coville, Vo [6] and references therein.

Remark 2.2. It is important to emphasise that minimax variational formula (2.2) admits a simple numerical procedure for finding the extremal value λ^* (see [26]).

Along with (2.2), we also need the following equivalent minimax variational formula

$$\lambda^* = \sup_{u\in\mathcal{C}^+} \inf_{\psi\in\mathcal{C}_0^+} \left\{ L(u, \psi^p/u^{p-1}) : \int_\Omega f u^{\gamma-p}\psi^p \, dx \neq 0 \right\}. \tag{2.5}$$

Furthermore, we shall deal with the dual variational formulas for (2.2) and (2.5):

$$\lambda^{**} = \inf_{\phi\in\mathcal{C}_0^+} \sup_{u\in\mathcal{C}^+} \left\{ L(u,\phi) : \int_\Omega f u^{\gamma-1}\phi \, dx \neq 0 \right\}, \tag{2.6}$$

$$\lambda_P^{**} = \inf_{\psi\in\mathcal{C}_0^+} \sup_{u\in\mathcal{C}^+} \left\{ L(u, \psi^p/u^{p-1}) : \int_\Omega f u^{\gamma-p}\psi^p \, dx \neq 0 \right\}, \tag{2.7}$$

respectively. By standard arguments it follows that $\lambda^* \leq \lambda^{**}$ and $\lambda^* \leq \lambda_P^{**}$.

Our main assumptions on f and q are the following.

(F_1) There is an open subset $U \subset \Omega$ such that $\operatorname{ess\,inf}_{x\in U}\{f(x), q(x)\} > 0$.

(F_2) $\operatorname{ess\,sup}_{x\in\Omega}\{f(x), q(x)\} < +\infty$.

Lemma 2.3. *Let $1 < \alpha < p < \gamma$.*

(a) *Assume (F_1), then $\lambda^{**} < +\infty$, $\lambda^{**}_P < +\infty$ and thus $\lambda^* < +\infty$.*

(b) *Assume $f(x) \geq 0$ in Ω and (F_2), then $\lambda^* > 0$ and thus $\lambda^{**} > 0$, $\lambda^{**}_P > 0$.*

Observe that problem (1.2) has the variational form with the Euler functional $I_\lambda(u)$, defined on $W_0^{1,p}(\Omega) \cap L^\gamma(|f|, \Omega) \cap L^\alpha(|q|, \Omega)$ by

$$I_\lambda(u) = \frac{1}{p}\int_\Omega |\nabla u|^p dx - \frac{\lambda}{\gamma}\int_\Omega f|u|^\gamma dx - \frac{1}{\alpha}\int_\Omega q|u|^\alpha dx. \tag{2.8}$$

Our result on the existence and non-existence of positive solutions and the existence of bifurcation point for stationary problem (1.2) is as follows

Theorem 2.4. *Let $1 < \alpha < p < \gamma$ and Ω be a bounded domain in $\mathbb{R}^N$ with $C^{1,\beta}$-boundary for some $\beta \in (0, 1)$.*

(i) *Assume (F_1), then for any $\lambda > \lambda^*$, (1.2) has no weak solution $u_\lambda \in \mathcal{C}^+$.*

(ii) *Assume (F_1), $f(x) \geq 0$ in Ω and $f, q \in L^\infty(\Omega)$, then for any $\lambda \in (0, \lambda^*)$ there exists a weak solution u_λ of (1.2) such that $u_\lambda \in \mathcal{C}^+$. Moreover, if $\inf_{x\in\Omega} f(x) > 0$, then (1.2) has a weak non-negative solution $u_{\lambda^*} \in L^\gamma(\Omega) \cap W_0^{1,p}(\Omega)$ for $\lambda = \lambda^*$.*

(iii) *Assume $p = 2$, $f, q \in C(\overline{\Omega})$, $\min_{x\in\Omega} f(x) > 0$ and $q(x) \geq 0$ in Ω. Suppose that $u_{\lambda^*} \neq 0$ and $u_{\lambda^*} \in L^\infty(\Omega)$. Then $X_1 := KerD^2_u I_{\lambda^*}(u_{\lambda^*})$ is an one-dimensional subspace of $W_0^{1,p}(\Omega)$ spanned by $\phi^* \in W_0^{1,p}(\Omega)$; i.e., $X_1 = \langle\phi^*\rangle$, $W_0^{1,p}(\Omega) = X_1 \oplus X_2$.*

Furthermore, $(\lambda^*, u_{\lambda^*})$ is a bifurcation point; i.e., there exist an interval $(-a, a) \subset \mathbb{R}$ and C^1 mappings $\lambda : (-a, a) \to \mathbb{R}$ and $u : (-a, a) \to W_0^{1,p}(\Omega)$ such that for each $s \in (-a, a)$ the function $u(s) \in \mathcal{C}^+$ is a weak solution of problem (1.1) for $\lambda = \lambda(s)$, $(u(0), \lambda(0)) = (u_{\lambda^*}, \lambda^*)$, $d\lambda(0)/ds = 0$, $du(0)/ds = \phi^*$ and $\lambda(s) \leq \lambda^*$ for $s \in (-a, a)$. Furthermore, $u(s) = u_{\lambda^*} + s\phi^* + \xi(s)$, where $\xi : (-a, a) \to X_2$, $\xi(0) = 0$, $d\xi(0)/ds = 0$.

Remark 2.5. If one does not take into account that λ^* is expressed in generalized Collatz–Wielandt formula (2.2), then statements **(i)**, **(ii)** of Theorem 2.4 follow from Theorems 2.1, 2.2 in [19].

Remark 2.6. In the case of the subcritical Sobolev exponent $1 < \alpha < p < \gamma < p^*$, where $p^* = pN/(N - p)$ if $N > p$ and $p^* = \infty$ if $N \leq p$, the existence of the weak positive solution u_λ of (1.1) for $\lambda \in (0, \Lambda_\mathcal{N})$, where $\Lambda_\mathcal{N}$ is the so-called extreme value of the Nehari manifold method (see [28]) can be obtained by the Nehari manifold method under weaker assumptions $f \in L^{r_1}(\Omega)$ and $q \in L^{r_2}(\Omega)$ with some $r_1, r_2 \in (1, +\infty]$ (see, e.g., [22]). However, recent investigations Il'yasov, Silva and Silva, Macedo [27, 35] show that, in general, $\Lambda_\mathcal{N}$ does not give the threshold value for the existence of positive solutions of (1.1).

Remark 2.7. Under assumptions **(iii)** of Theorem 2.4, the conditions $u_{\lambda^*} \neq 0$, $u_{\lambda^*} \in L^\infty(\Omega)$ are satisfied, for example, if $1 < q < p < \gamma < p^*$ (see [19]) or $p = 2$, $f(x), q(x) \equiv 1$ and $N \leq 10$ (see Mignot, Puel [32]).

For (1.1) our main result is the following theorem.

Theorem 2.8. *Let $1 < \alpha < p < \gamma$ and Ω be a bounded domain in $\mathbb{R}^N$ with $C^{1,\beta}$-boundary for some $\beta \in (0, 1)$.*

(i) *Assume* (F_1) *is satisfied and* $\inf_{x\in\Omega} f(x) > 0$*. Let* u_λ *be a weak non-negative solution of* (1.1) *defined on a maximal time interval* $(0, T_m)$*.*

- *Suppose* $p = 2$ *and* $\lambda > \lambda^{**}$*. Then* $T_m < +\infty$ *and* u_λ *blows up in finite time, i.e.,* $\|u_\lambda(t)\|_{L^\infty} \to +\infty$ *as* $t \to T_m$*.*
- *Suppose* $1 < p < 2$*,* $\gamma > 2$*,* $\lambda > \lambda_p^{**}$ *and* $u_\lambda \in C^1([0,T_m) \times \overline{\Omega})$*,* $u_\lambda > 0$ *in* $[0,T_m) \times \Omega$*. Then* $T_m < +\infty$ *and* u_λ *blows up in finite time, i.e.,* $\|u_\lambda(t)\|_{L^\infty} \to +\infty$ *as* $t \to T_m$*.*

(ii) *Assume* (F_1)*,* $f(x) \ge 0$ *in* Ω *and* $f, q \in L^\infty(\Omega)$*. Then* (1.1) *possesses global in time weak positive solution* u_λ *for any* $\lambda \in (0, \lambda^*)$*.*

As it was mention above, from [2, 18] it follows that if $f(x), q(x) \equiv 1$ and $p = 2$, then there exists $\Lambda > 0$ such that for $\lambda \in (0, \Lambda)$ parabolic problem (1.1) possesses a global in time solution whereas for $\lambda > \Lambda$ any positive solution u_λ blows up in finite time. Hence, Theorem 2.8 yields the following result on the saddle-point property for (2.2) and (2.6).

Corollary 2.9. *Assume that* $f(x), q(x) \equiv 1$*,* $p = 2$ *and* $1 < \alpha < 2 < \gamma$*,* Ω *is a bounded domain in* $\mathbb{R}^N$ *with* C^1*-boundary. Then variational formulas* (2.2) *and* (2.6) *satisfy the saddle-point property:* $\lambda^* = \lambda^{**} = \Lambda$*.*

3 Proof of Lemma 2.3

(a) Let us prove that $\lambda_p^{**} < +\infty$. The proof of $\lambda^{**} < +\infty$ is similar. Assume (F_1). Take a ball $B \subset U$. Consider the first eigenpair (λ_1, ϕ_1) of the operator $-\Delta_p$ on B with the zero Dirichlet boundary condition. It is well known that the eigenvalue λ_1 is positive, simple and isolated, and the corresponding eigenfunction ϕ_1 is positive and $\phi_1 \in C^1(\overline{B})$. Evidently $\phi_1^p / u^{p-1} \in C^1(\overline{\Omega})$ for any $u \in \mathcal{C}^+$. Hence by Allegretto, Xi [1] there holds

$$\left(|\nabla u|^{p-2} \nabla u, \nabla \frac{\phi_1^p}{u^{p-1}} \right) \le |\nabla \phi_1|^p \quad \text{in } \Omega, \ \forall u \in \mathcal{C}^+.$$

In view of (F_1), there is $\delta > 0$ such that $f(x) > \delta$, $q(x) > \delta$ a.e. on $\overline{B}$. This implies that there exists a sufficiently large $\Lambda > 0$ such that

$$\lambda_1 < \Lambda \delta s^{\gamma - p} + \delta s^{\alpha - p} \le \Lambda f(x) s^{\gamma - p} + q(x) s^{\alpha - p} \quad \text{a.e. in } B, \ \forall s > 0.$$

Hence

$$L(u, \phi_1^p / u^{p-1}) \le \frac{\int_B (\lambda_1 - q(x) u^{\alpha - p}) \phi_1^p \, dx}{\int_B f(x) u^{\gamma - p} \phi_1^p \, dx} < \Lambda, \qquad \forall u \in \mathcal{C}^+,$$

which implies that $\lambda_p^{**} < +\infty$.

(b) Since (F_2), there exists $K > 0$ such that $f(x) < K$, $q(x) < K$ a.e. in Ω. Following [2], let us consider

$$-\Delta_p e = 1 \quad \text{in } \Omega,$$
$$e|_{\partial\Omega} = 0.$$

By the maximum principle (see Tolksdorf [38], Trudinger [39], Vázquez [40]) and the regularity arguments (see DiBenedetto [14], Lieberman [31], Tolksdorf [37]) one has $e \in \mathcal{C}^+$. Furthermore, it is easily seen that for any sufficiently small $\lambda > 0$, there is $M = M(\lambda) > 0$

such that $M^{p-1} - KM^{\alpha-1}\|e\|_\infty^{\alpha-1} > \lambda KM^{\gamma-1}\|e\|_\infty^{\alpha-1}$. Hence and in view of that $f(x) \geq 0$ in Ω, by (2.2) we have

$$\lambda^* \geq \inf_{\phi \in \mathcal{C}_0^+} L(Me, \phi) \geq \inf_{\phi \in \mathcal{C}_0^+} \frac{\int_B (M^{p-1} - KM^{\alpha-1}\|e\|_\infty^{\alpha-1})\phi\, dx}{M^{\gamma-1}\int_B f(x)e^{\gamma-1}\phi\, dx} > \lambda > 0.$$

4 Proof of Theorem 2.4

(i) By (F_1), Lemma 2.3 implies that $\lambda^* < +\infty$. Let $\lambda > \lambda^*$ and suppose, contrary to our claim, that there exists a weak solution u_λ of (1.2) such that $u_\lambda \in \mathcal{C}^+$. By (2.2), there is $\phi_\lambda \in \mathcal{C}_0^+$ such that $L(u_\lambda, \phi_\lambda) < \lambda$ and $\int_\Omega f u_\lambda^{\gamma-1}\phi_\lambda\, dx \neq 0$. Assume, for instance, that $\int_\Omega f u_\lambda^{\gamma-1}\phi_\lambda\, dx > 0$. Then

$$\int_\Omega (|\nabla u_\lambda|^{p-2}\nabla u_\lambda, \nabla\phi_\lambda)\, dx - \int_\Omega q u_\lambda^{\alpha-1}\phi_\lambda\, dx - \lambda\int_\Omega f u_\lambda^{\gamma-1}\phi_\lambda\, dx < 0$$

which is a contradiction.

(ii) Since (F_2) and $f(x) \geq 0$ in Ω, Lemma 2.3 implies that $\lambda^* > 0$. Let $0 < \lambda < \lambda^*$. By (2.2), one can find $\hat{u}_\lambda \in \mathcal{C}^+$ such that $L(\hat{u}_\lambda, \phi) > \lambda$ for all $\phi \in \mathcal{C}_0^+$. Hence and since $f(x) \geq 0$, $\hat{u}_\lambda$ is a super-solution of (1.2). Take $\check{u} = 0$ for a sub-solution. Consider

$$\hat{I}_\lambda = \min\{I_\lambda(u) \mid u \in M_\lambda\}, \tag{4.1}$$

where $M_\lambda = \{u \in W_0^{1,p}(\Omega) |\ 0 \leq u \leq \hat{u}_\lambda\}$. In view of that $f, q \in L^\infty(\Omega)$, we may apply Proposition 3.1 from [19] (see also for semilinear case Theorem 2.4 in Struwe [36]). Thus for any $\lambda \in (0, \lambda^*)$ there exists a minimizer $u_\lambda \in M_\lambda$ of (4.1) which weakly satisfies (1.2).

Using (F_1) it is not hard to show that there exists $u_0 \in M_\lambda$ such that

$$\int q(x)|u_0|^\alpha dx > 0 \quad \text{and} \quad \int f(x)|u_0|^\gamma dx > 0.$$

This and the assumption $1 < \alpha < p < \gamma$ imply that there is a sufficiently small $t > 0$ such that $tu_0 \in M_\lambda$ and $I_\lambda(tu_0) < 0$. Thus $\hat{I}_\lambda = I_\lambda(u_\lambda) < 0$ and therefore $u_\lambda \neq 0$.

Since $u_\lambda \leq \hat{u}_\lambda$ in Ω, one has $u_\lambda \in L^\infty(\Omega)$. Furthermore, by the assumptions $\partial\Omega$ is $C^{1,\beta}$-manifold for some $\beta \in (0,1)$. Hence, by $C^{1,\alpha}$-regularity results [14,31,37] we have $u_\lambda \in C^{1,\alpha}(\overline{\Omega})$ for some $\alpha \in (0, \beta)$. Finally, the maximum principle [38–40] implies that $u_\lambda > 0$ in Ω for all $\lambda \in (0, \lambda^*)$.

Let us show that there exists a limit solution u_{λ^*}. Since $I_\lambda(u_\lambda) < 0$ and $D_u I_\lambda(u_\lambda)(u_\lambda) = 0$, we have

$$\frac{(\gamma - p)}{p}\|u_\lambda\|_1^p - \frac{(\gamma - \alpha)}{\alpha}\int q(x)|u_\lambda|^\alpha dx < 0, \tag{4.2}$$

$$\lambda\frac{(\gamma - p)}{\gamma}\int f(x)|u_\lambda|^\gamma dx - \frac{(p - \alpha)}{\alpha}\int q(x)|u_\lambda|^\alpha dx < 0, \tag{4.3}$$

$\forall \lambda \in (0, \lambda^*)$. Here and what follows we denote by $\|\cdot\|_1$ the norm in the space $W_0^{1,p}(\Omega)$. In view of that $q(x) < +\infty$ in Ω, inequality (4.2) implies that $\|u_\lambda\|_1 < C_1 < +\infty$ and $\int q(x)|u_\lambda|^\alpha dx < C_2 < +\infty$, where C_1, C_2 do not depend on $\lambda \in (0, \lambda^*)$. Hence by (4.3), $\int f(x)|u_\lambda|^\gamma dx < C_3 < +\infty$. Consequently using $\inf_{x\in\Omega} f(x) > 0$ we derive that $\|u_\lambda\|_{L^\gamma} < C_4 < +\infty$, where C_3, C_4 do not depend on $\lambda \in (0, \lambda^*)$. Now the Banach–Alaoglu and Sobolev theorems imply that there exists a sequence λ_n such that $\lambda_n \uparrow \lambda^*$ and $u_{\lambda_n} \to u_{\lambda^*}$ weakly in $W_0^{1,p}$, strongly in $L^\alpha(\Omega)$ and $u_{\lambda_n} \to u_{\lambda^*} \geq 0$ a.e. in Ω as $n \to \infty$. Furthermore, since

$u_{\lambda_n} \to u_{\lambda^*}$ a.e. in Ω and $||u_\lambda||_{L^\gamma} < C_4$, we have $u_{\lambda_n} \to u_{\lambda^*} \in L^\gamma(\Omega)$ weakly in $L^\gamma(\Omega)$ (see, e.g, Theorem 13.44 in Hewitt, Stromberg [20]). By the same arguments $u_{\lambda_n}^{\gamma-1} \to u_{\lambda^*}^{\gamma-1}$ weakly in $L^{\gamma/(\gamma-1)}(\Omega)$. Hence in virtue of that $f, q \in L^\infty(\Omega)$, we may pass to the limit in (1.2) as $n \to \infty$. Thus u_{λ^*} weakly satisfies (1.2) for $\lambda = \lambda^*$. This completes the proof of **(ii)**.

(iii) Let $p = 2$. Since $u_{\lambda^*} \neq 0$ and $u_{\lambda^*} \in L^\infty(\Omega)$, the standard theory of regularity solutions and maximum principle for elliptic equations yield $u_{\lambda^*} \in C^1(\overline{\Omega}) \cap C^2(\Omega)$, $u_{\lambda^*} > 0$. Furthermore, since $f(x) > 0$ and $q(x) \geq 0$ in Ω, Hoph's lemma implies (see Protter, Weinberger [34]) that $\partial u_{\lambda^*}/\partial \nu < 0$ on $\partial\Omega$, where $\nu := \nu(x)$ denotes the exterior unit normal to $\partial\Omega$ at $x \in \partial\Omega$.

Consider the eigenvalue problem

$$\begin{cases} -\Delta\psi - [\lambda^*(\gamma-1)fu_{\lambda^*}^{\gamma-2} + (\alpha-1)qu_{\lambda^*}^{\alpha-2}]\psi = \mu\psi & \text{in } \Omega, \\ \psi = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.4}$$

Then there exists a first eigenpair (μ_1, ϕ^*) of (4.4) such that $\phi^* > 0$, $\phi^* \in C^2(\Omega) \cap C^1(\overline{\Omega})$ and

$$\mu_1 = \inf_{\psi \in W_0^{1,2}(\Omega)\setminus 0} \left\{ \frac{\int |\nabla\psi|^2 dx - \int [\lambda^*(\gamma-1)fu_{\lambda^*}^{\gamma-2} + (\alpha-1)qu_{\lambda^*}^{\alpha-2}]\psi^2 dx}{\int \psi^2 dx} \right\}. \tag{4.5}$$

Indeed, this can be shown by arguments introduced Díaz, Hernández [13], Díaz, Hernández, Il'yasov [12]. Let us give a sketch of its proof. Since $\partial u_{\lambda^*}/\partial\nu < 0$ on $\partial\Omega$, one has $c\,d(x) \leq u_{\lambda^*}(x) \leq C\,d(x)$ for $x \in \Omega$ with some constants $0 < c, C < +\infty$, where $d(x) := \mathrm{dist}(x, \partial\Omega)$. Hence by the monotonicity properties of eigenvalues it is sufficient to show that the first eigenvalue of the problem

$$\begin{cases} -\Delta\psi - \left(\lambda^*(\gamma-1)fu_{\lambda^*}^{\gamma-2} + q\dfrac{(\alpha-1)}{d(x)^{2-\alpha}}\right)\psi = \mu\psi & \text{in } \Omega, \\ \psi = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.6}$$

is well-defined and has the usual properties. Assume first that $\mu > 0$. Then (4.6) is equivalent to the existence of μ such that $r(\mu) = 1$, where $r(\mu)$ is the first eigenvalue for the associated problem

$$\begin{cases} -\Delta\psi = r(\mu)\left(\lambda^*(\gamma-1)fu_{\lambda^*}^{\gamma-2} + q\dfrac{(\alpha-1)}{d(x)^{2-\alpha}} + \mu\right)\psi & \text{in } \Omega, \\ \psi = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.7}$$

That $r(\mu) > 0$ is well-defined follows by showing that (4.7) is equivalently formulated as $Tw = rw$, with $T = i \circ P \circ F$, where $F : L^2(\Omega, d^{2-\alpha}) \to W^{-1,2}(\Omega)$ defined by

$$F(\psi) = \left(\lambda^*(\gamma-1)fu_{\lambda^*}^{\gamma-2} + q\frac{(\alpha-1)}{d(x)^{2-\alpha}} + \mu\right)\psi,$$

$P : W^{-1,2}(\Omega) \to W_0^{1,2}(\Omega)$ is the solution operator for the linear problem

$$\begin{cases} -\Delta z = h(x) & \text{in } \Omega, \\ z = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.8}$$

for $h \in W^{-1,2}(\Omega)$, and where $i : W_0^{1,2}(\Omega) \to L^2(\Omega, d^{2-\alpha})$ is the standard embedding. Then F and P are continuous and the map i is compact (see Kufner [29]). Hence, it is possible to apply the Krein–Rutman theorem in the formulation by Daners, Koch-Medina [16]. Thus we have the variational formulation

$$r(\mu) = \inf_{w\in W_0^{1,2}(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla w|^2 dx}{\int \left([\lambda^*(\gamma-1)fu_{\lambda^*}^{\gamma-2} + (\alpha-1)qu_{\lambda^*}^{\alpha-2}]w^2 + \mu w^2\right)\, dx}. \tag{4.9}$$

Hence a positive eigenvalue of (4.7) exits if and only if there is a $\mu > 0$ such that $r(\mu) = 1$. Analogous argument gives the formulation for $\mu < 0$

$$r_1(\mu) = \inf_{w\in W_0^{1,2}(\Omega)\setminus\{0\}} \frac{\int_\Omega \left(|\nabla w|^2 - \mu w^2\right)\, dx}{\int [\lambda^*(\gamma-1)fu_{\lambda^*}^{\gamma-2} + (\alpha-1)qu_{\lambda^*}^{\alpha-2}]w^2\, dx}. \tag{4.10}$$

It is not hard to show that $r(\mu)$ ($r_1(\mu)$) is decreasing (increasing) in μ and $r(\mu) \to 0$ ($r_1(\mu) \to +\infty$) as $\mu \to +\infty$ ($\mu \to -\infty$). Observe

$$r(0) = r_1(0) = \inf_{w\in W_0^{1,2}(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla w|^2\, dx}{\int [\lambda^*(\gamma-1)fu_{\lambda^*}^{\gamma-2} + (\alpha-1)qu_{\lambda^*}^{\alpha-2}]w^2\, dx}.$$

Thus, there exists a positive eigenvalue of (4.7) if $r(0) > 1$ and a negative one if $r(0) < 1$. Hence $-\infty < \mu_1 < +\infty$ and there exists a minimizer ϕ^* of (4.5) such that $\phi^* \in C^1(\overline{\Omega}) \cap C^2(\Omega)$, $\phi^* > 0$, $\partial\phi^*/\partial\nu < 0$ on $\partial\Omega$ and

$$\begin{cases} -\Delta\phi^* - (\lambda^*(\gamma-1)fu_{\lambda^*}^{\gamma-2} + (\alpha-1)qu_{\lambda^*}^{\alpha-2})\phi^* = \mu_1\phi^* & \text{in } \Omega, \\ \phi^* = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.11}$$

Let us show that $\mu_1 = 0$. Assume the converse $\mu_1 \neq 0$ and suppose, for instance, that $\mu_1 > 0$. Consider $u_\varepsilon = u_{\lambda^*} + \varepsilon\phi^*$. It is readily seen that $u_\varepsilon \in \mathcal{C}^+$ for sufficient small ε. The equations (1.2) and (4.11) imply the following equality

$$\int (\nabla u_\varepsilon, \nabla\psi)dx - \int qu_\varepsilon^{\alpha-1}\psi dx = \lambda^* \int fu_\varepsilon^{\gamma-1}\psi dx + \varepsilon\mu_1 \int \phi^*\psi dx + \bar{o}(\varepsilon)$$

which holds uniformly with respect to $\psi \in B^1 := \{\psi \in \mathcal{C}_0^+ : \|\psi\|_{W^{1,2}} \leq 1\}$ so that $\bar{o}(\varepsilon) = r(\varepsilon,\psi)$, where $|r(\varepsilon,\psi)| < C\varepsilon^2$ and $C < +\infty$ does not depend on $\psi \in B^1$. Hence there exists $\varepsilon_0 > 0$ such that

$$\inf_{\psi\in\mathcal{C}_0^+} \frac{\int(\nabla u_{\varepsilon_0}, \nabla\psi)dx - \int qu_{\varepsilon_0}^{\alpha-1}\psi dx}{\int fu_{\varepsilon_0}^{\gamma-1}\psi dx} > \lambda^*, \tag{4.12}$$

which contradicts (2.2). The maximum principle for elliptic boundary value problems (see e.g. [34]) implies that the minimal eigenvalue μ_1 is simple. Consequently, the kernel $X_1 :=$ Ker $D_u^2 I_{\lambda^*}(u_{\lambda^*})$ is the one-dimensional subspace in $W_0^{1,p}(\Omega)$ spanned by ϕ^*.

The proof of the second part of assertion (iii) follows from the bifurcation theorem of Crandall and Rabinowitz [11].

5 Blow up and global solutions

(i) Let $p = 2$. Since (F_1), Lemma 2.3 implies that $\lambda^{**} < +\infty$. Let $\lambda > \lambda^{**}$. Take $\varepsilon > 0$ such that $\lambda - \varepsilon > \lambda^{**}$. Then by (2.6), there exists $\phi_\lambda \in \mathcal{C}_0^+$ such that

$$\sup_{u \in \mathcal{C}^+} \frac{\int_\Omega (\nabla u, \nabla \phi_\lambda)\, dx - \int_\Omega q u^{\alpha-1} \phi_\lambda \, dx}{\int_\Omega f u^{\gamma-1} \phi_\lambda \, dx} < \lambda - \varepsilon,$$

that is

$$\int_\Omega (\nabla u, \nabla \phi_\lambda)\, dx - \lambda \int_\Omega f u^{\gamma-1} \phi_\lambda \, dx - \int_\Omega q u^{\alpha-1} \phi_\lambda \, dx < -\varepsilon \int_\Omega f u^{\gamma-1} \phi_\lambda \, dx. \tag{5.1}$$

By the assumptions there is $a_0 > 0$ such that $f(x) \geq a_0$ a.e. in Ω. Hence, Jensen's inequality yields

$$\left(\int_\Omega u \phi_\lambda \, dx \right)^{\gamma-1} \leq c_0 \int_\Omega f u^{\gamma-1} \phi_\lambda \, dx, \tag{5.2}$$

where $0 < c_0 < \infty$ does not depend on $u \in \mathcal{C}^+$. Thus, one has the inequality

$$\int_\Omega (\nabla u, \nabla \phi_\lambda)\, dx - \lambda \int_\Omega f u^{\gamma-1} \phi_\lambda \, dx - \int_\Omega q u^{\alpha-1} \phi_\lambda \, dx < -\varepsilon c_0 \left(\int_\Omega u \phi_\lambda \, dx \right)^{\gamma-1},$$

which holds by continuity for any $u \in W_0^{1,p}(\Omega)$, $u \geq 0$ in Ω.

Assume that there exists a non-negative weak solution u of (1.1) defined on a maximal time interval $(0, T_m)$. Suppose, contrary to our claim, that $T_m = +\infty$.

Consider $\eta(t) = \int_\Omega u(t) \phi_\lambda \, dx$. Then by (1.3) we have

$$\frac{d}{dt} \eta(t) = \int_\Omega (-(\nabla u, \nabla \phi_\lambda) + (\lambda f u^{\gamma-1} + q u^{\alpha-1}) \phi_\lambda)\, dx > \varepsilon c_0 (\eta(t))^{\gamma-1} \quad \text{a.e. in } (0, +\infty).$$

However, then

$$\eta(t) > C_1 \left(\frac{1}{1 - C_2 t} \right)^{1/(\gamma-2)}$$

with some constants $0 < C_1, C_2 < +\infty$. Hence and since $\gamma > 2$, we have

$$\eta(t) \equiv \int_\Omega u(t) \phi_\lambda \, dx \to +\infty \quad \text{as } t \to 1/C_2.$$

But this is possible only if $\|u(t)\|_{L^\infty} \to +\infty$ as $t \to T^*$.

Consider the case $1 < p < 2$. By Lemma 2.3, $\lambda_P^{**} < +\infty$. Take $\lambda > \lambda_P^{**}$. Then there is $\varepsilon > 0$ such that $\lambda - \varepsilon > \lambda_P^{**}$. By (2.7), there exists $\phi_\lambda \in \mathcal{C}_0^+$ such that

$$\sup_{u \in \mathcal{C}^+} \frac{\int_\Omega (|\nabla u|^{p-2} \nabla u, \nabla (\phi_\lambda^p / u^{p-1}))\, dx - \int_\Omega q u^{\alpha-p} \phi_\lambda^p \, dx}{\int_\Omega f u^{\gamma-p} \phi_\lambda^p \, dx} < \lambda - \varepsilon.$$

As above, we may assume that $f(x) \geq a_0$ a.e. in Ω for some $a_0 > 0$. In view of that $1 < p < 2$ and $\gamma > 2$, Jensen's inequality yields

$$\left(\int_\Omega u^{2-p} \phi_\lambda^p \, dx \right)^{\frac{\gamma-p}{2-p}} \leq c_0 \int_\Omega f u^{\gamma-p} \phi_\lambda^p \, dx, \tag{5.3}$$

where $0 < c_0 < \infty$ does not depend on $u \in \mathcal{C}^+$. Thus, one has the inequality

$$\int_\Omega (|\nabla u|^{p-2}\nabla u, \nabla(\phi_\lambda^p/u^{p-1})\,dx - \lambda \int_\Omega f u^{\gamma-p}\phi_\lambda^p\,dx - \int_\Omega q u^{\alpha-p}\phi_\lambda^p\,dx$$
$$< -C_0\left(\int_\Omega u^{2-p}\phi_\lambda^p\,dx\right)^{\frac{\gamma-p}{2-p}},$$

for any $u \in \mathcal{C}^+$ with $C_0 = \varepsilon c_0 > 0$.

Assume that there exists a weak positive solution $u \in C^1([0,T_m) \times \overline{\Omega})$ of (1.1). Suppose, contrary to our claim, that $T_m = +\infty$.

Consider $\zeta(t) = \int_\Omega u(t)^{2-p}\phi_\lambda^p\,dx$. Then by (1.3) we have

$$\frac{d}{dt}\zeta(t) =(2-p)\int_\Omega(-(|\nabla u|^{p-2}\nabla u, \nabla(\phi_\lambda^p/u^{p-1})\,) + (\lambda f u^{\gamma-p} + q u^{\alpha-p})\phi_\lambda)\,dx > C_0'(\zeta(t))^{\frac{\gamma-p}{2-p}}$$

a.e. in $(0,+\infty)$. Hence,

$$\frac{d}{dt}\zeta(t) > C_0'(\zeta(t))^{\frac{\gamma-p}{2-p}} \quad \text{a.e. in } (0,+\infty), \tag{5.4}$$

which implies that

$$\zeta(t) \equiv \int_\Omega u(t)^{2-p}\phi_\lambda^p\,dx \to +\infty \quad \text{as } t \to T^*$$

for some $T^* > 0$.

(ii) By Theorem 2.4 **(ii)**, for $\lambda \in (0,\lambda^*)$ there exists a positive weak solution u_λ of (1.2) which is a positive stationary solution of (1.1) defined globally in the time interval $[0,+\infty)$. This completes the proof of **(ii)**, Theorem 2.8.

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