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A new estimate of the minimal wave speed for travelling fronts in reaction–diffusion–convection equations

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Abstract. In this paper we prove a new estimate of the threshold wave speed for travelling wavefronts of the reaction–diffusion–convection equations of the type

$$v_\tau + h(v)v_x = [D(v)v_x]_x + f(v)$$

where h is a convective term, D is a positive (potentially degenerate) diffusive term and f stands for a monostable reaction term.

Keywords: reaction–diffusion equations, convective terms, travelling fronts, critical wave speed, singular boundary value problems, heteroclinic solutions.

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1 Introduction

Reaction–diffusion equations have been intensively investigated, since they model various biological and chemical phenomena. The simplest model in this context is the Fisher equation

$$\frac{\partial v}{\partial \tau} = \frac{\partial^2 v}{\partial x^2} + v(1-v).$$

This equations has been subsequently generalized involving a general reaction term $f(v)$, vanishing at $v = 0$ and $v = 1$, a general diffusive term $D(v)$, which can be positive in $[0, 1]$ (non-degenerate case), or positive in $(0, 1]$ with $D(0) = 0$ (degenerate case), and a term $H(v)$ to include possible convective phenomena

$$\frac{\partial v}{\partial \tau} + \frac{\partial H(v)}{\partial x} = \frac{\partial}{\partial x}\left(D(v)\frac{\partial v}{\partial x}\right) + f(v). \tag{1.1}$$

A relevant class of solutions of equation (1.1) is that of travelling wave solutions (t.w.s.), that is solutions of the type $v(\tau, x) := u(x - c\tau)$ for some one-variable function u and real constant c, the wave speed.

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In the monostable case, that is when the reaction term f is positive in $(0,1)$, it is known (see [11]) that there exists a threshold wave speed c^* such that equation (1.1) supports travelling wave solutions having speed c if and only if $c \geq c^*$. The importance of the value c^* is due to the fact that in many cases the t.w.s. having speed c^* is the limit profile, for large times, of the solutions of the equation (1.1) (see, e.g., [3,6,7,9]).

For the value c^* it is known the following estimate (see [11])

$$2\sqrt{(D(u)f(u))'(0)} + h(0) \leq c^* \leq 2\sqrt{\sup_{u\in(0,1]} \frac{D(u)f(u)}{u}} + \max_{u\in[0,1]} h(u) \tag{1.2}$$

where h is the derivative of the one-variable function H. Estimate (1.2) is valid provided that the product function $D \cdot f$ is differentiable at $u = 0$. Notice that, when the convection is constant and the product function $D \cdot f$ is concave, then the previous estimate reduces to an equality: $c^* = 2\sqrt{(D(u)f(u))'(0)} + h(0)$.

In [1] the following upper estimate was achieved for the threshold value c^*, valid just in the case of no convective effects

$$\frac{1}{4}(c^*)^2 \leq \sup_{u\in(0,1]} \frac{2}{u^2}\int_0^u D(s)f(s)\, ds \tag{1.3}$$

which improves the previous upper estimate (1.2) in the particular case $h \equiv 0$. Such a result was achieved by means of a variational approach which seems to be not appropriate for equations involving convective effects.

A different variational principle was proposed in [2], where, under the further assumption that $h \in C^1([0,1])$, it was proved that

$$c^* \leq \inf_{\alpha>0} \sup_{u\in(0,1]} \left(\alpha + \frac{1}{\alpha}\frac{f(u)D(u)}{u} + h(u)\right) \tag{1.4}$$

which improves the upper estimate in (1.2), since the latter can be obtained by (1.4) taking

$$\alpha = \sqrt{\sup_{u\in(0,1]} \frac{f(u)D(u)}{u}}.$$

The value of c^* has been exactly determined for some special type of reaction–diffusion–convection equations. For instance, in the case when $D(u) = u$, $f(u) = u(1-u)$ and $h(u) \equiv 0$, then $c^* = \frac{1}{\sqrt{2}}$ (see [15,16]). Other computations in presence of convective processes have been stated by Gibbs and Murray (see [14]) in the case $D(u) = 1$, $f(u) = u(1-u)$ and $h(u) = ku$, for which it was showed (see also [11]) that

$$c^* = \begin{cases} 2 & \text{for } k \leq 2 \\ \dfrac{2}{k} + \dfrac{k}{2} & \text{for } k > 2. \end{cases}$$

Moreover, in [5] it was considered the equation

$$u_t + \beta u^k u_x = (\alpha u^k u_x)_x + \gamma u(1-u^k),$$

with $\alpha \geq 0$, β, $\gamma \geq 0$ and $k > 0$ given constants, for which the authors proved that the value of c^* is the following

$$c^* = \frac{\beta + \sqrt{\beta^2 + 4\alpha\gamma(k+1)}}{2(k+1)}.$$

Finally, in the case when $f(u) = u^m(1-u)$, $D(u) = h(u) \equiv 0$, the function $c^* = c^*(m)$ has been studied in [4], by means of asymptotical expansions as $m \to +\infty$.

However, the exact computation of c^*, or the study of its properties related to some parameters of the equations, can be carried on just for special equations, for which the explicit solution is known. For general equations the value of c^* has to be estimated.

The aim of this paper is to prove the following new upper estimate, valid also for equations involving convective effects,

$$c^* \leq 2\sqrt{\sup_{u\in(0,1]} \frac{1}{u}\int_0^u \frac{D(s)f(s)}{s}\,\mathrm{d}s} + \sup_{u\in(0,1]} \frac{1}{u}\int_0^u h(s)\,\mathrm{d}s \tag{1.5}$$

and we show it improves both (1.2) and (1.3). Moreover, we also show that when h is increasing, then the upper bound in (1.5) can be improved by the following

$$c^* \leq \sup_{u\in(0,1]} \left\{ 2\sqrt{\frac{1}{u}\int_0^u \frac{D(s)f(s)}{s}\,\mathrm{d}s} + \frac{1}{u}\int_0^u h(s)\,\mathrm{d}s \right\} \tag{1.6}$$

We also present some extensions to other reaction–diffusion models, such as equations with bi-stable reaction terms (that is f is negative in $(0,\alpha)$ and positive in $(\alpha,1)$) or reaction–diffusion–aggregation equations, in which the diffusivity D can also assume negative values. For these type of equations, the estimate for the speed c^*, obtained in the monostable reaction–diffusion case, plays a relevant role.

2 Preliminaries

Let us consider equation (1.1). In what follows we will assume that f, D are continuous functions, defined in $[0,1]$, both positive in the open interval $(0,1)$, with $f(0) = f(1) = 0$, and $H \in C^1([0,1])$. In the whole paper h will denote the derivative $\dot{H}$. The diffusive term D can vanish at $u = 0$ or/and $u = 1$ (degenerate and doubly-degenerate case).

Assume that

$$f \cdot D \text{ is differentiable at } u = 0. \tag{2.1}$$

A function $v(\tau, x)$ is said to be a travelling wave solution (t.w.s.) of (1.1) if there exists a real constant c and an one-variable function u such that $v(\tau, x) = u(x - c\tau)$ for every (τ, x) in the domain of existence of v. We are interested in t.w.s. connecting the two stationary states $v = 0$ and $v = 1$, that is satisfying $u(-\infty) = 1$ and $u(+\infty) = 0$. As it is immediate to verify, the profile u of a t.w.s. is a solution of the following second order boundary value problem

$$\begin{cases} (D(u)u')' + (c - h(u))u' + f(u) = 0 \\ u(-\infty) = 1,\ u(+\infty) = 0. \end{cases} \tag{2.2}$$

Moreover, if a t.w.s. with speed c exists, then it is unique (up to shifts) and it is strictly decreasing whenever $0 < u < 1$ (see [11]).

A travelling wave can be a solution in the classical sense (that is a C^1-function such that the product $D(u)u'$ is C^1 as well, or in the weak sense (sharp solutions), that is solutions reaching one or both the equilibria at finite times and with slope possibly non zero. Sharp t.w.s. can appear when the diffusivity vanishes at $u = 0$ (degenerate case) and/or at $u = 1$ (doubly degenerate case). We refer to [10] for a discussion about the characterization of the appearing travelling fronts.

The study of the existence or non-existence of t.w.s. is carried on by investigating the solvability of the following associate singular boundary value problem

$$\begin{cases} \dot{z} = h(u) - c - \dfrac{f(u)D(u)}{z(u)} \\ z(0^+) = z(1^-) = 0 \\ z(u) < 0 \text{ for every } u \in (0,1). \end{cases} \tag{2.3}$$

where the dot stands for derivation with respect to the variable u. Indeed, since the profiles are strictly decreasing whenever $0 < u < 1$, then equation in (2.2) can be handled as a typical autonomous equation, setting $z(u) = D(u)u'(t(u))$, where $t(u)$ is the inverse function of $u(t)$. However, a careful analysis is needed regarding the behaviour at the equilibria, in order to obtain classical or sharp travelling fronts.

The aim of this paper is to improve the estimate of the threshold speed c^*, so from now on we investigate the solvability of the boundary value problem (2.3), referring to [10] for the classification of the related travelling fronts. A key result providing a sufficient condition for the solvability of (2.3) is the following.

Theorem 2.1 ([13, Lemma 2.1])**.** *If there exists a function $\zeta \in C^1(0,1)$ such that*

$$\dot{\zeta}(u) \geq h(u) - c - \frac{f(u)D(u)}{\zeta(u)} \quad \text{for every } u \in (0,1)$$

with $\zeta(0^+) = 0$ and $\zeta(u) < 0$ for every $u \in (0,1)$, then problem (2.3) admits a (unique) solution z satisfying

$$\zeta(u) \leq z(u) < 0 \quad \text{for every } u \in (0,1).$$

So, in order to prove the solvability of (2.3), it suffices to show the existence of a negative upper-solution for the equation in (2.3), approaching the origin.

3 The new estimates

We now state our main results.

Theorem 3.1. *Assume (2.1). Let*

$$c > 2\sqrt{\sup_{u \in (0,1]} \frac{1}{u} \int_0^u \frac{D(s)f(s)}{s}\, \mathrm{d}s} + \sup_{u \in (0,1]} \frac{1}{u} \int_0^u h(s)\, \mathrm{d}s. \tag{3.1}$$

Then problem (2.3) admits a solution.

Proof. Put

$$\gamma(u) := \begin{cases} \dfrac{f(u)D(u)}{u} & \text{for } 0 < u \leq 1, \\ (f \cdot D)'(0) & \text{for } u = 0. \end{cases}$$

Since $f(u)D(u)$ is differentiable at $u = 0$, we get that γ is continuous in $[0,1]$.

Put $T := \{(x,u) \in \mathbb{R}^2 : \ 0 \le x \le u \le 1\}$ and let $G : T \to \mathbb{R}$ be the function defined by

$$G(x,u) := \begin{cases} \dfrac{1}{u-x}\displaystyle\int_x^u \gamma(s)\, ds & \text{for } 0 \le x < u \le 1, \\ \gamma(u) & \text{for } 0 \le x = u \le 1. \end{cases}$$

The function G is continuous in the compact triangle T. Indeed, the continuity is obvious at every point $(x_0, u_0) \in T$ with $x_0 < u_0$, whereas as for the continuity at the points of bisector $u = x$ observe that for every pair $(x,u) \in T$ there exists a value $\sigma_{x,u} \in [x,u]$ such that $G(x,u) = \gamma(\sigma_{x,u})$. So, if $(x,u) \to (u_0,u_0)$ for some $u_0 \in [0,1]$, we get $\sigma_{x,u} \to u_0$ and then also $G(x,u) = \gamma(\sigma_{x,u}) \to \gamma(u_0) = G(u_0,u_0)$.

Let $H : T \to \mathbb{R}$ be the function defined by

$$H(x,u) := \begin{cases} \dfrac{1}{u-x}\displaystyle\int_x^u h(s)\, ds & \text{for } 0 \le x < u \le 1, \\ h(u) & \text{for } 0 \le x = u \le 1. \end{cases}$$

Similarly to what we have done above, it is easy to show that H is continuous on the triangle T.

Put $M := \max_{u\in[0,1]} G(0,u)$ and $N := \max_{u\in[0,1]} H(0,u)$. We have

$$M = \max_{u\in[0,1]} G(0,u) = \sup_{u\in(0,1]} G(0,u) = \sup_{u\in(0,1]} \frac{1}{u}\int_0^u \gamma(s)\, ds; \tag{3.2}$$

$$N = \max_{u\in[0,1]} H(0,u) = \sup_{u\in(0,1]} H(0,u) = \sup_{u\in(0,1]} \frac{1}{u}\int_0^u h(s)\, ds. \tag{3.3}$$

By (3.1) we have $\frac{1}{4}(c-N)^2 > M$ and $c > N$, so for some $\varepsilon > 0$ we have $\frac{1}{4}(c-(N+\varepsilon))^2 > M + \varepsilon$ and $c > N + \varepsilon$. By the uniform continuity of the functions G and H in the triangle T, there exists a real $\delta = \delta_\epsilon > 0$ such that

$$G(x,u) \le G(0,u) + \varepsilon \le M + \epsilon \quad \text{for every } x \in [0,\delta] \text{ and } u \in [x,1]; \tag{3.4}$$

$$H(x,u) \le H(0,u) + \varepsilon \le N + \epsilon \quad \text{for every } x \in [0,\delta] \text{ and } u \in [x,1]. \tag{3.5}$$

Let us choose a costant L such that $M + \varepsilon < L < \frac{1}{4}(c-(N+\varepsilon))^2$ and put

$$K := \frac{1}{2}\left(c - N - \epsilon + \sqrt{(c-N-\epsilon)^2 - 4L}\right).$$

Notice that $K^2 - (c-N-\epsilon)K = -L$ and $K > 0$ since $c > N + \varepsilon$. Moreover, by (3.4) we deduce that

$$G\left(\frac{1}{n}, u\right) = \frac{1}{u-\frac{1}{n}}\int_{\frac{1}{n}}^u \gamma(s)\, ds < L \quad \text{for every } n > \frac{1}{\delta} \text{ and } u \in \left(\tfrac{1}{n}, 1\right].$$

Therefore, for every $n > \frac{1}{\delta}$ and every $u \in \left(\frac{1}{n}, 1\right]$ we have

$$K^2 - (c-(N+\varepsilon))K = -L < -\frac{1}{u-\frac{1}{n}}\int_{\frac{1}{n}}^u \gamma(s)\, ds$$

implying

$$K\left(u - \frac{1}{n}\right) - c\left(u - \frac{1}{n}\right) + (N+\varepsilon)\left(u - \frac{1}{n}\right) + \int_{\frac{1}{n}}^u \frac{\gamma(s)}{K}\, ds < 0. \tag{3.6}$$

Moreover, by (3.5) we get

$$H\left(\frac{1}{n},u\right)=\frac{1}{u-\frac{1}{n}}\int_{\frac{1}{n}}^{u}h(s)\,\mathrm{d}s<N+\varepsilon.$$

So, by (3.6) we deduce

$$K\left(u-\frac{1}{n}\right)-c\left(u-\frac{1}{n}\right)+\int_{\frac{1}{n}}^{u}h(s)\,\mathrm{d}s+\int_{\frac{1}{n}}^{u}\frac{\gamma(s)}{K}\,\mathrm{d}s<0,$$

hence

$$-Ku>-K\frac{1}{n}+\int_{\frac{1}{n}}^{u}\left(-c+h(s)-\frac{\gamma(s)}{-K}\right)\mathrm{d}s=-K\frac{1}{n}+\int_{\frac{1}{n}}^{u}\left(-c+h(s)-\frac{f(s)D(s)}{-Ks}\right)\mathrm{d}s$$

that is, put $\phi(u):=-Ku$, for every $n>\frac{1}{\delta}$ and every $u\in\left(\frac{1}{n},1\right]$, we have

$$\phi(u)>\phi\Big(\frac{1}{n}\Big)+\int_{\frac{1}{n}}^{u}\left(-c+h(s)-\frac{f(s)D(s)}{\phi(s)}\right)\mathrm{d}s. \tag{3.7}$$

Therefore, put $\zeta_n(u):=\phi(\frac{1}{n})+\int_{\frac{1}{n}}^{u}\big(-c+h(s)-\frac{f(s)D(s)}{\phi(s)}\big)\mathrm{d}s$, we have

$$\dot{\zeta}_n(u)=-c+h(u)-\frac{f(u)D(u)}{\phi(u)}>-c+h(u)-\frac{f(u)D(u)}{\zeta_n(u)}$$

and $\zeta_n(\frac{1}{n})=\phi(\frac{1}{n})$. Hence, ζ_n is an upper-solution for the problem

$$\begin{cases}\dot{z}=-c+h(u)-\dfrac{f(u)D(u)}{z} & \text{for } u\in[\frac{1}{n},1]\\ z(\frac{1}{n})=\phi(\frac{1}{n})\end{cases} \tag{3.8}$$

implying that $\zeta_n(u)>z_n(u)$ for every $u\in\left(\frac{1}{n},1\right]$, where z_n is the (unique) solution of problem (3.8), in the interval $\left[\frac{1}{n},1\right]$. So, we conclude that

$$\phi(u)>\zeta_n(u)>z_n(u)\quad \text{for every } u\in\left(\tfrac{1}{n},1\right].$$

Let us now continue each function z_n in the whole interval $[0,1]$ by setting $z_n(u):=\phi(u)=-Ku$ for every $u\in[0,\frac{1}{n})$, denoting them again z_n.

Let us now prove that $z_{n+1}(u)\le z_n(u)$ for every $u\in[0,1]$. By the construction of z_n, this is obvious for $0\le u\le\frac{1}{n}$, while, for $u>\frac{1}{n}$, since $z_{n+1}\big(\frac{1}{n}\big)<z_n\big(\frac{1}{n}\big)$, if $z_{n+1}(\bar{u})=z_n(\bar{u})$ for some $\bar{u}\in\big(\frac{1}{n},1\big)$, this contradicts the uniqueness of the solution of the Cauchy problem

$$\begin{cases}\dot{z}=-c+h(u)-\dfrac{f(u)D(u)}{z} & \text{for } u\in(\frac{1}{n},1],\\ z(\bar{u})=z_n(\bar{u}).\end{cases}$$

So, $z_{n+1}(u)\neq z_n(u)$ for every $u\in\big(\frac{1}{n+1},1\big)$, implying $z_{n+1}(u)<z_n(u)$ for every $u\in\big(\frac{1}{n},1\big)$.

Summarizing, we have

$$-cu+\int_0^u h(s)\,\mathrm{d}s\le z_{n+1}(u)\le z_n(u)\le -Ku\quad \text{for every } u\in[0,1].$$

Let $Z(u) := \lim_{n\to+\infty} z_n(u)$. Notice that the sequence $(z_n)_n$ is equibounded and equicontinuous in each compact subinterval $[\tilde{u}, 1]$, with $\tilde{u} > 0$. Indeed, for n sufficiently large and every $u \in [\tilde{u}, 1]$ we have

$$-c + h(u) \le \dot{z}_n(u) = -c + h(u) - \frac{f(u)D(u)}{z_n(u)} \le -c + h(u) + \frac{f(u)D(u)}{Ku},$$

hence the convergence of the sequence $(z_n)_n$ towards the function Z is uniform in every compact subinterval $[\tilde{u}, 1]$, implying that

$$\dot{Z}(u) = -c + h(u) - \frac{f(u)D(u)}{Z(u)} \quad \text{for every } u \in (0,1).$$

Finally, being $-cu + \int_0^u h(s)\,\mathrm{d}s \le Z(u) \le -Ku$, we have $Z(0) = 0$ and Z continuous at 0. Therefore, by Theorem 2.1, problem (2.3) admits a solution. □

When h is increasing, then the upper bound given by (1.5) can be improved by (1.6), as stated in the following result.

Theorem 3.2. *Assume* (2.1). *If h is increasing and*

$$c > \sup_{u\in(0,1]} \left\{ 2\sqrt{\frac{1}{u}\int_0^u \frac{D(s)f(s)}{s}\,\mathrm{d}s} + \frac{1}{u}\int_0^u h(s)\,\mathrm{d}s \right\} \tag{3.9}$$

then problem (2.3) *admits a solution.*

Proof. Let us fix $u \in (0,1]$ and consider the following trinomial in the variable λ:

$$\lambda^2 + \left(cu - \int_0^u h(s)\,\mathrm{d}s \right)\lambda + u\int_0^u \frac{f(s)D(s)}{s}\,\mathrm{d}s. \tag{3.10}$$

By (3.9) we can deduce that the discriminant of the trinomial (3.10) is positive, so, if $\lambda_1(u)$, $\lambda_2(u)$ are the two roots, put

$$\phi(u) := \frac{1}{2}(\lambda_1(u) + \lambda_2(u)) = \frac{1}{2}\left(\int_0^u h(s)\,\mathrm{d}s - cu \right)$$

we get

$$\phi^2(u) + \left(cu - \int_0^u h(s)\,\mathrm{d}s \right)\phi(u) + u\int_0^u \frac{f(s)D(s)}{s}\,\mathrm{d}s < 0 \quad \text{for every } u \in (0,1]$$

and since $\phi(u) < 0$ for every $u \in (0,1]$ we infer

$$\phi(u) > \int_0^u (h(s) - c)\,\mathrm{d}s - \frac{u}{\phi(u)}\int_0^u \frac{f(s)D(s)}{s}\,\mathrm{d}s \quad \text{for every } u \in (0,1] \tag{3.11}$$

Let us now consider the function $\psi(u) := \frac{u}{|\phi(u)|} = \frac{2}{c - \frac{1}{u}\int_0^u h(s)\,\mathrm{d}s}$, $u \in (0,1]$. Since h is increasing, also the function $u \mapsto \frac{1}{u}\int_0^u h(s)\,\mathrm{d}s$ is increasing, and so also ψ does. Therefore,

$$\begin{aligned} -\frac{u}{\phi(u)}\int_0^u \frac{f(s)D(s)}{s}\,\mathrm{d}s &= \psi(u)\int_0^u \frac{f(s)D(s)}{s}\,\mathrm{d}s \\ &\ge \int_0^u \psi(s)\frac{f(s)D(s)}{s}\,\mathrm{d}s \\ &= -\int_0^u \frac{f(s)D(s)}{\phi(s)}\,\mathrm{d}s \end{aligned}$$

and taking (3.11) into account we get

$$\phi(u) > \int_0^u \left(h(s) - c - \frac{f(s)D(s)}{\phi(s)} \right) \mathrm{d}s \quad \text{for every } u \in (0,1]. \tag{3.12}$$

Hence, put $\xi(u) := \int_0^u \left(h(s) - c - \frac{f(s)D(s)}{\phi(s)}\right)\mathrm{d}s$, we have

$$\dot{\xi}(u) = h(u) - c - \frac{f(u)D(u)}{\phi(u)} > h(u) - c - \frac{f(u)D(u)}{\xi(u)} \quad \text{for every } u \in (0,1]$$

with $\xi(0^+) = 0$ and $\xi(u) < \phi(u) < 0$ for every $u \in (0,1]$. So, by virtue of Theorem 2.1, problem (2.3) admits a solution. □

Remark 3.3. Notice that estimates (1.5) and (1.6) are not comparable, since when h is increasing the upper bound given by (1.6) can be strictly less than the one given by (1.5), as Example 4.4 in the next section shows. Instead, the validity of (1.6) for a generic convective term h, not necessarily increasing, remains an open problem.

4 Comparisons with the previous known estimates

In order to compare the various available estimates for c^*, it is convenient to discuss separately the case of equations without convection and those with convection.

4.1 Comparison for reaction–diffusion equation (no convective effects)

Observe that in this case (1.5) and (1.6) provide the same upper bound:

$$c^* \leq 2\sqrt{\sup_{u\in(0,1]} \frac{1}{u} \int_0^u \frac{f(s)D(s)}{s}\, \mathrm{d}s} \tag{4.1}$$

which obviously improves the one given by (1.2), by the mean value theorem. Moreover, in this case the upper bound in (1.4) becomes

$$\inf_{\alpha>0} \sup_{u\in(0,1]} \left(\alpha + \frac{1}{\alpha} \frac{f(u)D(u)}{u} \right) = \inf_{\alpha>0} \left(\alpha + \frac{1}{\alpha} \sup_{u\in(0,1]} \frac{f(u)D(u)}{u} \right) = 2\sqrt{\sup_{u\in(0,1]} \frac{f(u)D(u)}{u}}$$

that is (1.4) furnishes the same upper bound as (1.2).

Finally, the next result states that (4.1) improves also the upper bound given by (1.3) (which has not an analogous version for reaction–diffusion–convection equations).

Theorem 4.1. *Let f, D be satisfying assumption (2.1). Then,*

$$\sup_{u\in(0,1]} \frac{1}{u} \int_0^u \frac{f(s)D(s)}{s}\, \mathrm{d}s \;\leq\; \sup_{u\in(0,1]} \frac{2}{u^2} \int_0^u f(s)D(s)\, \mathrm{d}s. \tag{4.2}$$

Moreover, if relation (4.2) holds as an equality then necessarily

$$\sup_{u\in(0,1]} \frac{1}{u} \int_0^u \frac{f(s)D(s)}{s}\, \mathrm{d}s \;=\; \sup_{u\in(0,1]} \frac{2}{u^2} \int_0^u f(s)D(s)\, \mathrm{d}s = (Df)'(0). \tag{4.3}$$

Proof. For each $u \in (0,1]$ let us consider the functions

$$G(u) := \frac{1}{u}\int_0^u \frac{f(s)D(s)}{s}\,\mathrm{d}s, \qquad H(u) := \frac{1}{u^2}\int_0^u f(s)D(s)\,\mathrm{d}s$$

and

$$\begin{aligned} T(u) &:= u(G(u)-H(u)) - \int_0^u H(s)\,\mathrm{d}s \\ &= \int_0^u \frac{f(s)D(s)}{s}\,\mathrm{d}s - \frac{1}{u}\int_0^u f(s)D(s)\,\mathrm{d}s - \int_0^u H(s)\,\mathrm{d}s. \end{aligned}$$

Of course, T is differentiable in $(0,1]$, with

$$T'(u) = \frac{f(u)D(u)}{u} + \frac{1}{u^2}\int_0^u f(s)D(s)\,\mathrm{d}s - \frac{f(u)D(u)}{u} - H(u) = 0, \quad \text{for all } u \in (0,1],$$

so T is constant in $(0,1]$. Moreover, by the differentiability of the product function $f(u)D(u)$ at $u=0$, we have $T(0^+)=0$. Therefore, $T(u)=0$ for all $u \in (0,1]$, implying that

$$G(u) = H(u) + \frac{1}{u}\int_0^u H(s)\,\mathrm{d}s, \quad \text{for all } u \in (0,1].$$

Hence,

$$\sup_{u\in(0,1]} G(u) \le \sup_{u\in(0,1]} H(u) + \sup_{u\in(0,1]} \frac{1}{u}\int_0^u H(s)\,\mathrm{d}s \le 2\sup_{u\in(0,1]} H(u), \tag{4.4}$$

which proves inequality (4.2).

Moreover, if relation (4.2) holds as equality then by (4.4) we get that

$$\sup_{u\in(0,1]} H(u) = \sup_{u\in(0,1]} \frac{1}{u}\int_0^u H(s)\,\mathrm{d}s. \tag{4.5}$$

Hence, the following two situations can occur:

$$\sup_{u\in(0,1]} H(u) = \sup_{u\in(0,1]} \frac{1}{u}\int_0^u H(s)\,\mathrm{d}s = H(0^+) = \frac{1}{2}(fD)'(0); \tag{4.6}$$

or

$$H(u) \le \frac{1}{\delta}\int_0^\delta H(s)\,\mathrm{d}s = \sup_{v\in(0,1]} \frac{1}{v}\int_0^v H(s)\,\mathrm{d}s \quad \text{for every } u \in (0,\delta]. \tag{4.7}$$

When (4.6) holds, we have $\sup_{u\in(0,1]} G(u) = (fD)'(0)$, hence (4.3) holds. Instead, when (4.7) holds, then H is constant in $[0,\delta]$, so

$$\sup_{u\in(0,1]} \frac{1}{u}\int_0^u H(s)\,\mathrm{d}s = \frac{1}{\delta}\int_0^\delta H(s)\,\mathrm{d}s = \frac{1}{2}H(0^+) = \frac{1}{2}(fD)'(0)$$

and (4.3) holds. □

Remark 4.2. Notice that when (4.3) holds, all the estimates (1.2), (1.3) and (1.5) reduce to the equality $c^* = \frac{1}{2}\sqrt{(f\cdot D)'(0)}$. This situation occurs, for instance, when the product function $f \cdot D$ is concave in $[0,1]$, or is concave in $[0,\bar{u}]$ and convex in $[\bar{u},1]$, for some $\bar{u} \in (0,1)$. Theorem 4.1 states that when the right-hand side of the estimates is greater than the left-hand one, that is when c^* is just unknown, then estimate (1.5) is properly sharper then the other ones.

Example 4.3. Just to furnish a numerical example, note that for the very simple case $D(u) = u$ and $f(u) = u(1-u)$, estimate (1.2) provides $c^* \le 1$, estimate (1.3) provides $c^* \le \frac{2}{3}\sqrt{2} \sim 0.94$, finally estimate (4.1) provides $c^* \le \frac{1}{2}\sqrt{3} \sim 0.87$. However, for this special case the exact value of c^* is known: $c^* = \frac{1}{2}\sqrt{2} \sim 0.71$ (see [16]).

4.2 Comparison for reaction–diffusion–convection equations

Obviously, estimate (1.5) improves (1.2), by the mean value theorem, and (1.6) improves (1.5) in the case when h is increasing. In this case, (1.6) provides a better upper bound than (1.5), as the following example shows. Moreover, the relation between (1.6) and (1.4) is unclear, but in the following example, (1.6) provides a better upper bound also with respect to (1.4).

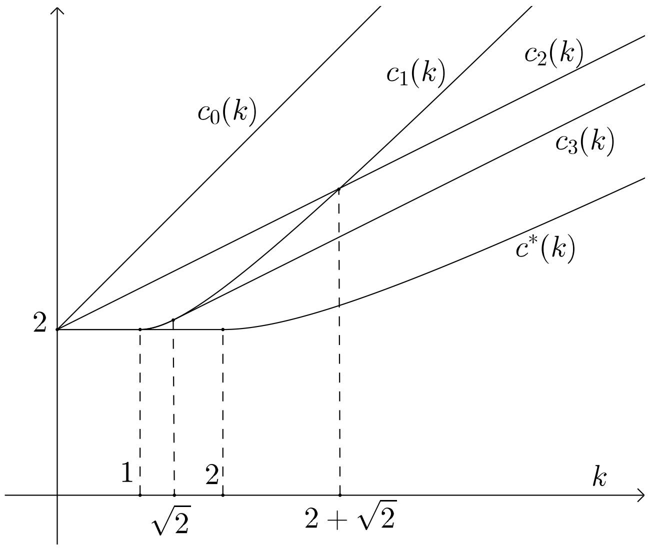

Figure 4.1: Graphics of the exact minimal speed $c^*(k)$ and the upper bounds $c_0(k)$ (by (1.2)), $c_1(k)$ (by (1.4)), $c_2(k)$ (by (1.5)) and $c_3(k)$ (by (1.6)).

Example 4.4. Let us consider the model studied by Gibbs and Murray: $D(u) \equiv 1$, $f(u) = u(1-u)$ and $h(u) = ku$ $(k > 0)$, for which the exact value of c^* is known (see Introduction). Let $c_0(k)$, $c_1(k)$, $c_2(k)$ and $c_3(k)$ respectively denote the upper bounds derived from estimates (1.2), (1.4), (1.5) and (1.6). As it is easy to check, $c_0(k) = 2+k$ and $c_2(k) = 2+\frac{1}{2}k$, and clearly $c_2(k) < c_0(k)$. As for $c_1(k)$, notice that for every $\alpha > 0$ we have

$$\sup_{u\in(0,1]}\left(\alpha + \frac{1}{\alpha}\frac{f(u)D(u)}{u} + h(u)\right) = \alpha + \frac{1}{\alpha} + \sup_{u\in(0,1]}\left(k - \frac{1}{\alpha}\right)u = \begin{cases}\alpha + k & \text{if } \alpha \geq \frac{1}{k},\\ \alpha + \frac{1}{\alpha} & \text{if } \alpha \leq \frac{1}{k}.\end{cases}$$

So,

$$c_1(k) = \inf_{\alpha>0}\sup_{u\in(0,1]}\left(\alpha + \frac{1}{\alpha}\frac{f(u)D(u)}{u} + h(u)\right) = \begin{cases}2 & \text{if } k \leq 1,\\ k+\frac{1}{k} & \text{if } k \geq 1.\end{cases}$$

Hence, $c_1(k) < c_0(k)$ for every $k > 0$, but $c_1(k) > c_2(k)$ whenever $k > 2+\sqrt{2}$. Finally, by simple computations one has

$$c_3(k) = \begin{cases}2 & \text{if } 0 \leq k \leq 1,\\ k + \dfrac{1}{k} & \text{if } 1 \leq k \leq \sqrt{2},\\ \sqrt{2} + \dfrac{1}{2}k & \text{if } k \geq \sqrt{2},\end{cases}$$

and $c_1(k) > c_3(k)$ for every $k > 0$. Hence, for every $k > 0$ we have $c_3(k) < c_2(k)$ and $c_3(k) < c_1(k)$.

5 Extension to other reaction–diffusion models

The estimate for the threshold wave c^* for the reaction–diffusion–convection equations plays a relevant role also for other type of models. So, we now present the natural extension of known results which can be achieved taking estimate (1.5) into account. Of course, analogous results can be derived using (1.6) when h is increasing.

5.1 Reaction–diffusion–convection with bi-stable reaction terms

Another well-known model for reaction–diffusion equations concerns the case of bi-stable reaction terms, that is when the continuous function f is assumed to be negative in $(0, \alpha)$ and positive in $(\alpha, 1)$ for some $\alpha \in (0, 1)$. In this case, it was shown that there exists a unique value $\tilde{c}$ such that equation (1.1) supports t.w.s. having speed $c = \tilde{c}$. As for the estimate of $\tilde{c}$, in [12] it was proved the following estimate

$$-2\sqrt{\sup_{u\in[0,\alpha)}\left|\frac{f(u)D(u)}{u-\alpha}\right|} - \max_{u\in[0,\alpha]} h(u) \le \tilde{c} \le 2\sqrt{\sup_{u\in(\alpha,1]}\left|\frac{f(u)D(u)}{u-\alpha}\right|} - \min_{u\in[\alpha,1]} h(u).$$

Such a relation has been obtained starting from estimate (1.2) for the mono-stable case. Using the same proof as in [12], taking the present estimate (1.5) into account, the estimate of $\tilde{c}$ can be improved by the following

$$\begin{aligned} -2\sqrt{\sup_{u\in[0,\alpha)}\left|\frac{1}{\alpha-u}\int_u^\alpha \frac{f(s)D(s)}{s-\alpha}\,\mathrm{d}s\right|} - \sup_{u\in[0,\alpha]}\frac{1}{\alpha-u}\int_u^\alpha h(s)\,\mathrm{d}s \le \tilde{c} \\ \le 2\sqrt{\sup_{u\in(\alpha,1]}\left|\frac{1}{u-\alpha}\int_\alpha^u \frac{f(s)D(s)}{s-\alpha}\,\mathrm{d}s\right|} - \inf_{u\in[\alpha,1]}\frac{1}{u-\alpha}\int_\alpha^u h(s)\,\mathrm{d}s. \qquad (5.1)\end{aligned}$$

5.2 Extension to diffusion–aggregation models

In [8] it was proposed a model of diffusion–aggregation–reaction process (without convective effects), in which the diffusive term can assume negative values too, in order to describe the behaviour of a population tending to cluster into groups for some value of the density u. Assuming the existence of a value $\beta \in (0, 1)$ such that $D(u)(\beta - u) > 0$ for every $u \in (0, 1)$, $u \ne \beta$, it was proved that there exists a threshold value c^* such that equation

$$\frac{\partial v}{\partial \tau} = \frac{\partial}{\partial x}\left(D(v)\frac{\partial v}{\partial x}\right) + f(v)$$

supports t.w.s. (classical or sharp) if and only if $c \ge c^*$, with c^* satisfying the following upper estimate

$$\frac{1}{4}(c^*)^2 \le \max\left\{\sup_{0<u\le\beta}\frac{f(u)D(u)}{u}, \sup_{\beta\le u<1}\frac{f(u)D(u)}{u-1}\right\} \qquad (5.2)$$

obtained from the classical estimate (1.2) for the merely diffusive model. So, taking account of the present estimate (1.5), inequality (5.2) can be improved by

$$\frac{1}{4}(c^*)^2 \le \max\left\{\sup_{0<u\le\beta}\frac{1}{u}\int_0^u \frac{f(s)D(s)}{s}\,\mathrm{d}s, \sup_{\beta\le u<1}\frac{1}{1-u}\int_u^1 \frac{f(s)D(s)}{s-1}\,\mathrm{d}s\right\}.$$

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