NONLINEAR EIGENVALUE PROBLEMS FOR HIGHER ORDER LIDSTONE BOUNDARY VALUE PROBLEMS

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ABSTRACT. In this paper, we consider the Lidstone boundary value problem $y^{(2m)}(t) = \lambda a(t)f(y(t), \ldots, y^{(2j)}(t), \ldots y^{(2(m-1))}(t))$, $0 < t < 1$, $y^{(2i)}(0) = 0 = y^{(2i)}(1)$, $i = 0, \ldots, m-1$, where $(-1)^m f > 0$ and a is nonnegative. Growth conditions are imposed on f and inequalities involving an associated Green's function are employed which enable us to apply a well-known cone theoretic fixed point theorem. This in turn yields a λ interval on which there exists a nontrivial solution in a cone for each λ in that interval. The methods of the paper are known. The emphasis here is that f depends upon higher order derivatives. Applications are made to problems that exhibit superlinear or sublinear type growth.

1. INTRODUCTION

We consider the nonlinear Lidstone boundary value problem (BVP),

$$y^{(2m)}(t) = \lambda a(t)f(y(t), \ldots, y^{(2j)}(t), \ldots y^{(2(m-1))}(t)), \quad 0 < t < 1, \tag{1.1}$$

$$y^{(2i)}(0) = 0 = y^{(2i)}(1), \quad i = 0, \ldots m-1, \tag{1.2}$$

where $(-1)^m f > 0$ is continuous and a is nonnegative. For more precise conditions on f and a, let $(-1)^j[a,b] = [a,b]$ if j is even and $(-1)^j[a,b] = [-b,-a]$ if j is odd. Let

$$\prod_{j=0}^{m-1}[a_j, b_j] = [a_0, b_0] \times \cdots \times [a_{m-1}, b_{m-1}].$$

We shall require that

(A): $(-1)^m f : \prod_{j=0}^{m-1}(-1)^j[0,\infty) \to [0,\infty)$ is continuous,

(B): $a : [0,1] \to [0,\infty)$ is continuous and does not vanish identically on any subinterval.

This work is primarily motivated by the original work of Erbe and Wang [15] for $m = 1$. To our knowledge, Erbe and Wang [15] are the first to apply the methods employed here to the cases that f is superlinear or sublinear. A flurry of extensions to nth order problems have been obtained in recent years

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(for example, see [1], [2], [22], [23], [11], [12]). Henderson and Wang [17] were the first to introduce the problem as a nonlinear eigenvalue problem.

In all the above cited works, the nonlinear term, f, only depends on position. The primary interest of this paper is that the nonlinear term, f, depends on position, acceleration and other even order derivatives of the unknown function. Recent related works in which dependence on higher order derivatives is allowed can be found in [4] or [9].

The Lidstone boundary value problem (BVP) was first studied by Lidstone [21]; Agarwal and Wong's work [3] has generated renewed interest in the problem. Recently, Davis, Henderson, and Lamar ([6], [7], [10], [19]) have studied the problem intensely. A feature of the Lidstone BVP that is exploited in this paper is that it can be analyzed as a nested family of second order conjugate BVPs. This feature has been employed by Davis, Eloe, Henderson, Islam and Thompson ([14], [8], and [13]). The primary contribution of this paper is that this nested feature is exploited so that the methods employed by Erbe and Wang [15] can be be applyed to the BVP, (1.1), (1.2). Moreover, we indicate that the contribution is of interest by exhibiting applications to problems that exhibit superlinear or sublinear type growth.

We close the introduction with one open question. Can the methods employed here apply to a Lidstone BVP with nonlinear dependence on odd order derivatives of the unknown function? That question is completely open. The problem is that large in norm does not imply large componentwise; by exploiting the nested feature of Lidstone BVPs in this paper, large in norm will, in fact, imply large in the appropriate components.

2. The Fixed Point Operator

The method developed by Erbe and Wang [15] employs an application of the cone theoretic fixed point theorem that we credit to Krasnosel'skii [18]. Also see [16]. For simplicity we state the theorem here.

Theorem 2.1. *Let $\mathcal{B}$ be a Banach space, and let $\mathcal{P} \subset \mathcal{B}$ be a cone in $\mathcal{B}$. Assume Ω_1, Ω_2 are open subsets of $\mathcal{B}$ with $0 \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$, and let*

$$\mathcal{T} : \mathcal{P} \cup (\overline{\Omega}_2 \setminus \Omega_1) \to \mathcal{P}$$

be a completely continuous operator such that, either

(i): $||\mathcal{T}u|| \leq ||u||, u \in \mathcal{P} \cup \partial\Omega_1$, *and* $||\mathcal{T}u|| \geq ||u||, u \in \mathcal{P} \cap \partial\Omega_2$, *or*

(ii): $||\mathcal{T}u|| \geq ||u||, u \in \mathcal{P} \cap \partial\Omega_1$, *and* $||\mathcal{T}u|| \leq ||u||, u \in \mathcal{P} \cap \partial\Omega_2$.

Then $\mathcal{T}$ has a fixed point in $\mathcal{P} \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

We now construct the fixed point operator upon which we apply the above fixed point theorem. To do so, we exploit that the Lidstone BVP, (1.1), (1.2), can be constructed as a nested sequence of second order conjugate type BVPs. In particular, we shall construct a second order BVP that is equivalent to (1.1), (1.2). In Section 3 we shall apply the above fixed point theorem to the equivalent second order BVP.

The Green's function for $y''(t) = 0$, $0 < t < 1$, $y(0) = y(1) = 0$, is

$$G(t,s) = \begin{cases} t(s-1), & 0 \le t \le s \le 1, \\ s(t-1), & 0 \le s \le t \le 1. \end{cases}$$

Set $G_1(t,s) := G(t,s)$, and for $j = 2, \dots m$, define recursively

$$G_j(t,s) = \int_0^1 G(t,r)G_{j-1}(r,s)\,dr.$$

As a result, $G_j(t,s)$ is the Green's function for the BVP,

$$y^{(2j)}(t) = 0, \quad 0 < t < 1,$$
$$y^{(2i)}(0) = 0 = y^{(2i)}(1), \quad i = 0, \dots, j-1,$$

for each $j = 1 \dots m$. One can verify this directly ([5], page 192) or see [13] or [19, 20].

For each $j = 1, \dots, m-1$, define $A_j : C[0,1] \to C[0,1]$ by

$$A_j v(t) = \int_0^1 G_j(t,s)v(s)ds.$$

By the construction of A_j it follows that

$$(A_j v)^{(2j)}(t) = v(t), \quad 0 < t < 1,$$
$$(A_j v)^{(2i)}(0) = (A_j v)^{(2i)}(1), \quad i = 0, \dots, j-1.$$

Thus, it follows that the BVP, (1.1), (1.2), has a solution if, and only if, the BVP,

$$v''(t) = f(A_{m-1}v(t), \dots, A_1 v(t), v(t)), \quad 0 < t < 1,$$
$$v(0) = v(1) = 0,$$

has a solution. If y is a solution of the BVP, (1.1), (1.2), then $v = y^{(2(m-1))}$ is a solution of the second order BVP; conversely, if v is a solution of the second order BVP, then $y = A_{m-1}v$ is a solution of the BVP, (1.1), (1.2).

Define $\mathcal{T} : C[0,1] \to C[0,1]$ by

$$\mathcal{T}v(t) = \lambda \int_0^1 a(s)G(t,s)f(A_{m-1}v(s), \dots, A_1 v(s), v(s))ds.$$

The properties of each G_j readily imply that $\mathcal{T} : C[0,1] \to C[0,1]$ is completely continuous. It now follows that there exists a solution of the BVP, (1.1), (1.2), if, and only if, there exists a continuous fixed point of $\mathcal{T}$. Moreover, the relation between solutions, y, of the BVP, (1.1), (1.2), and fixed points, v, of $\mathcal{T}$, is given by $y(t) = A_{m-1}v(t)$, or $y^{(2(m-1))}(t) = v(t)$.

Note that $G_1 < 0$ on $(0,1) \times (0,1)$, and $(-1)^j G_j > 0$ on $(0,1) \times (0,1)$. Thus, y is a positive solution of the BVP, (1.1), (1.2), if, and only if,

$(-1)^{m-1}y^{(2(m-1))} = (-1)^{m-1}v$ is positive, where v is the corresponding continuous fixed point of $\mathcal{T}$. In the next section then, we restrict our analysis to finding λ intervals upon which $\mathcal{T}$ generates a nontrivial fixed point, v, such that $(-1)^{m-1}v(t) > 0$, $0 < t < 1$.

3. Existence of Positive Solutions

We remind the reader of two fundamental bounds involving the Green's function, G_1.

$$0 < -(G_1(t,s)) \leq s(1-s), \quad 0 < t, s < 1,$$

$$-(G_1(t,s)) \geq (1/4)s(1-s), \quad (1/4) \leq t \leq (3/4), \quad 0 \leq s \leq 1.$$

From here, four estimates which we shall employ are readily obtained.

$$\max_{0 \leq t \leq 1} \int_0^1 |G_1(t,s)| ds \leq (1/8), \tag{3.1}$$

$$\min_{1/4 \leq t \leq 3/4} \int_{1/4}^{3/4} |G_1(t,s)| ds \geq (1/16). \tag{3.2}$$

If $v \in C[0,1]$, then

$$||A_j v|| \leq ||v||/8^j, \quad j = 1, \ldots, m-1. \tag{3.3}$$

($||\cdot||$ denotes the usual supremum norm on $[0,1]$.) If $(-1)^{(m-1)}v(t) > 0, 0 < t < 1$, and if $(-1)^{(m-1)}v(t) > ||v||/4$ for $1/4 \leq t \leq 3/4$, then

$$\min_{1/4 \leq t \leq 3/4} |A_j v(t)| \geq ||v||/(4(16)^j), \quad j = 1, \ldots, m-1. \tag{3.4}$$

(3.3) and (3.4) are readily obtained from (3.1) and (3.2). (3.3) and (3.4) motivate conditions (C1) and (C2) given below.

(C1): There exist $k_{0j} \geq (1/8)^{j-1}$, $j = m, \ldots, 2$, such that

$$\lim_{(-1)^{m-1}v \to 0^+} (-1)^m f((-1)^{m-1}k_{0m}v, \ldots, -k_{02}v, v)/(-1)^{m-1}v = f_0,$$

and there exist $0 < k_{\infty j} \leq (1/16)^{j-1}$, $j = m, \ldots, 2$, such that

$$\lim_{(-1)^{m-1}v \to \infty} (-1)^m f((-1)^{m-1}k_{\infty m}v, \ldots, -k_{\infty 2}v, v)/(-1)^{m-1}v = f_\infty.$$

(C2): There exist $0 < k_{0j} \leq (1/16)^{j-1}$, $j = m, \ldots, 2$, such that

$$\lim_{(-1)^{m-1}v \to 0^+} (-1)^m f((-1)^{m-1}k_{0m}v, \ldots, -k_{02}v, v)/(-1)^{m-1}v = f_0,$$

and there exist $k_{\infty j} \geq (1/8)^{j-1}$, $j = m, \ldots, 2$, such that

$$\lim_{(-1)^{m-1}v \to \infty} (-1)^m f((-1)^{m-1}k_{\infty m}v, \ldots, -k_{\infty 2}v, v)/(-1)^{m-1}v = f_\infty.$$

(D): $(-1)^m f(u_0, u_1, \ldots, u_{m-1})$ is increasing in each u_{2j} and decreasing in each u_{2j+1} for $(u_0, u_1, \ldots, u_{m-1}) \in \prod_{j=0}^{m-1}(-1)^j[0,\infty)$.

We shall now state and prove a typical result as an application of Theorem 2.1. To apply Theorem 2.1 let $\mathcal{B}$ denote the Banach space $C[0,1]$ with the supremum norm

$$||v|| = \max_{0\le t\le 1} |v(t)|$$

and define the cone $\mathcal{P} \subset \mathcal{B}$ by $\mathcal{P} :=$

$$\{v \in \mathcal{B} : (-1)^{m-1}v(t) \ge 0, 0 \le t \le 1, \min_{1/4\le t\le 3/4} (-1)^{m-1}v(t) \ge (1/4)||v||\}.$$

Theorem 3.1. *Assume that conditions (A), (B), (C1), and (D) are satisfied. Then, for each λ satisfying*

$$4/(\int_{1/4}^{3/4} -G(1/2,s)a(s)dsf_\infty) < \lambda < 1/(\int_0^1 s(1-s)a(s)dsf_0), \tag{3.5}$$

there is at least one nontrivial solution, y, of the BVP, (1.1), (1.2), *such that $v = y^{(2(m-1))}$ belongs to $\mathcal{P}$.*

Proof. Let λ satisfy (3.5) and let $\varepsilon > 0$ be such that

$$4/(\int_{1/4}^{3/4} -G(1/2,s)a(s)ds(f_\infty - \varepsilon)) \le \lambda \le 1/(\int_0^1 s(1-s)a(s)ds(f_0+\varepsilon)).$$

Define $\mathcal{T} : \mathcal{P} \to \mathcal{B}$ by

$$\mathcal{T}v(t) = \lambda \int_0^1 a(s)G(t,s)f(A_{m-1}v(s),\ldots,A_1v(s),v(s))ds.$$

To see that $\mathcal{T} : \mathcal{P} \to \mathcal{P}$ apply Conditions (A) and (B) and recall that G is negative on $(0,1)\times(0,1)$. Moreover, $(-1)^m(\mathcal{T}v)''(t) = a(t)(-1)^m f \ge 0$, $0 < t < 1$, by Conditions (A) and (B), and $\mathcal{T}v(0) = \mathcal{T}v(1) = 0$; in particular, due to concavity,

$$\min_{1/4\le t\le 3/4} (-1)^{m-1}\mathcal{T}v(t) \ge ||\mathcal{T}v||/4.$$

We now construct the domains, Ω_1 and Ω_2 in order to apply Theorem 2.1 (i). Apply Condition (C1) and find $H_1 > 0$ such that

$$(-1)^m f((-1)^{m-1}k_{0m}v,\ldots,-k_{02}v,v) \le (f_0+\varepsilon)(-1)^{m-1}v,$$

for all $0 < (-1)^{m-1}v \le H_1$. Let $v \in \mathcal{P}$ with $||v|| = H_1$. Apply (3.3) and Condition (D) to see that

$$\begin{aligned}(-1)^m f(A_{m-1}v,\ldots,v) &\le (-1)^m f(||v||/8^{m-1},\ldots,(-1)^{m-1}||v||)\\ &\le (-1)^m f(k_{0,m}||v||,\ldots,(-1)^{m-1}||v||).\end{aligned}$$

Thus,

$$\begin{aligned}|\mathcal{T}v(t)| &\le \lambda \int_0^1 s(1-s)a(s)(-1)^m f(A_{m-1}v(s),\ldots,v(s))ds\\ &\le \lambda \int_0^1 s(1-s)a(s)ds(f_0+\varepsilon)||v|| \le ||v||.\end{aligned}$$

Define

$$\Omega_1 = \{x \in \mathcal{B} : ||x|| < H_1\},$$

and we have shown that

$$||\mathcal{T}v|| \leq ||v||, \quad v \in \mathcal{P} \cap \partial\Omega_1.$$

To find Ω_2, find $H > 0$ such that

$$(-1)^m f((-1)^{m-1} k_{\infty m} v, \ldots, -k_{\infty 2} v, v) \geq (f_\infty - \varepsilon)(-1)^{m-1} v,$$

for all $0 < (-1)^{m-1} v \geq H$. Let $H_2 = \max\{2H_1, 4H\}$ and set

$$\Omega_2 = \{x \in \mathcal{B} : ||x|| < H_2\}.$$

Let $v \in \mathcal{P}, ||v|| = H_2$. Then

$$\min_{1/4 \leq t \leq 3/4} (-1)^{m-1} v(t) \geq ||v||/4 \geq H.$$

Apply (3.4) and Condition (D), for $s \in [1/4, 3/4]$, to obtain

$$\begin{aligned}(-1)^m f(A_{m-1} v(s), \ldots, v(s)) &\geq (-1)^m f(||v||/4(16)^{m-1}, \ldots, (-1)^{m-1}||v||/4) \\ &\geq (-1)^m f(k_{\infty,m}||v||/4, \ldots, (-1)^{m-1}||v||/4).\end{aligned}$$

Thus,

$$\begin{aligned}|\mathcal{T}v(1/2)| &\geq \lambda \int_{1/4}^{3/4} -G(1/2,s)a(s)(-1)^m f(A_{m-1}v(s), \ldots, v(s))ds \\ &\geq \lambda \int_{1/4}^{3/4} -G(1/2,s)a(s)ds(f_\infty - \varepsilon)(||v||/4) \geq ||v||.\end{aligned}$$

In particular, define

$$\Omega_2 = \{x \in \mathcal{B} : ||x|| < H_2\},$$

and

$$||\mathcal{T}v|| \geq ||v||, \quad v \in \mathcal{P} \cap \partial\Omega_2.$$

This completes the proof of Theorem 3.1.

We remark that if f is superlinear (i.e., $f_0 = 0, f_\infty = \infty$) then the proof of Theorem 3.1 is readily adapted to show that the BVP, (1.1), (1.2), has a nontrivial solution, y, such that $v = y^{(2(m-1))}$ belongs to $\mathcal{P}$, for each $0 < \lambda < \infty$. To illustrate that this observation is of interest, set $m = 2$ and consider the fourth order Lidstone BVP that relates to the cantilever beam problem. Note that each of

$$f_1(u,v) = u^2 + v^2, \quad f_2(u,v) = -uv$$

satisfy conditions (A), (C1) and (D).

We will state without proof a second application of Theorem 2.1. The proof, when f depends only on position is standard (see [15]) and the extension to the problem addressed here is completely analogous to the extension illustrated in the proof of Theorem 3.1.

Theorem 3.2. *Assume that conditions (A), (B), (C2), and (D) are satisfied. Then, for each λ satisfying*

$$4/(\int_{1/4}^{3/4} -G(1/2,s)a(s)ds f_0) < \lambda < 1/(\int_0^1 s(1-s)a(s)ds f_\infty),$$

there is at least one nontrivial solution, y, of the BVP, (1.1), (1.2), *such that $v = y^{(2(m-1))}$ belongs to $\mathcal{P}$.*

In the case that f is sublinear (i.e., $f_0 = \infty, f_\infty = 0$) the proof of Theorem 3.2 is readily adapted to show that the BVP, (1.1), (1.2), has a nontrivial solution, y, such that $v = y^{(2(m-1))}$ belongs to $\mathcal{P}$, for each $0 < \lambda < \infty$. To illustrate that this observation is of interest, again set $m = 2$ and note that each of

$$f_3(u,v) = u^{2/3} + v^{2/3}, \quad f_4(u,v) = -(uv)^{1/3}$$

satisfy conditions (A), (C2) and (D).

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