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Multiple solutions of nonlinear elliptic functional differential equations

Dedicated to Professor László Hatvani on the occasion of his 75th birthday

László Simon ✉

Eötvös Loránd University, Pázmány P. sétány 1/c, Budapest, H–1117, Hungary

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Abstract. We shall consider weak solutions of boundary value problems for elliptic functional differential equations of the form

$$-\sum_{j=1}^{n} D_j[a_j(x,u,Du;u)] + a_0(x,u,Du;u) = F, \qquad x \in \Omega$$

with homogeneous boundary conditions, where $\Omega \subset \mathbb{R}^n$ is a bounded domain and $;u$ denotes nonlocal dependence on u (e.g. integral operators applied to u).

By using the theory of pseudomonotone operators, one can prove existence of solutions.

However, in certain particular cases it is possible to find theorems on the number of solutions. These statements are based on arguments for fixed points of certain real functions and operators, respectively.

Keywords: elliptic functional equations, multiple solutions.

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1 Introduction

It is well known that mathematical models of several applications are functional differential equations of one variable (e.g. delay equations). In the monograph by Jianhong Wu [7] semilinear evolutionary partial functional differential equations and applications are considered, where the book is based on the theory of semigroups and generators. In the monograph by A. L. Skubachevskii [6] linear elliptic functional differential equations (equations with nonlocal terms and nonlocal boundary conditions) and applications are considered. A nonlocal boundary value problem, arising in plasma theory, was considered by A. V. Bitsadze and A. A. Samarskii in [1].

✉ Email: simonl@cs.elte.hu

It turned out that the theory of pseudomonotone operators is useful to study nonlinear (quasilinear) partial functional differential equations (both stationary and evolutionary equations) and to prove existence of weak solutions (see [2,4,5]).

In the present work we shall consider weak solutions of the following elliptic functional differential equations:

$$-\sum_{j=1}^{n} D_j[a_j(x,u,Du;u)] + a_0(x,u,Du;u) = F(x), \qquad x \in \Omega \tag{1.1}$$

(for simplicity) with homogeneous Dirichlet or Neumann boundary condition where $\Omega \subset \mathbb{R}^n$ is a bounded domain and $;u$ denotes nonlocal dependence on u.

By using the theory of pseudomonotone operators one can prove existence theorems on weak solutions. In this paper we shall investigate the number of solutions in certain particular cases and prove existence of multiple solutions, based on fixed points of certain functions and operators, respectively.

2 Number of solutions of equations with real valued functionals of solutions

Denote by $\Omega \subset \mathbb{R}^n$ a bounded domain with sufficiently smooth boundary, $1 < p < \infty$, $W^{1,p}(\Omega)$ the Sobolev space with the norm

$$\|u\| = \left[\int_\Omega \left(\sum_{j=1}^{n} |D_j u|^p + |u|^p\right) dx\right]^{1/p}.$$

Further, let $V \subset W^{1,p}(\Omega)$ be a closed linear subspace of $W^{1,p}(\Omega)$, $V^\star$ the dual space of V, the duality between $V^\star$ and V will be denoted by $\langle \cdot, \cdot \rangle$.

Weak solutions of (1.1) are defined as functions $u \in V$ satisfying

$$\int_\Omega \left[\sum_{j=1}^{n} a_j(x,u,Du;u) D_j v + a_0(x,u,Du;u) v\right] dx = \langle F, v \rangle \quad \text{for all } v \in V$$

where $a_j : \Omega \times \mathbb{R}^{n+1} \times V \to \mathbb{R}$ $(j = 0,1,...,n)$ are given functions. In the case of homogeneous Dirichlet boundary condition, $V = W_0^{1,p}(\Omega)$ (the closure of $C_0^1(\Omega)$ in $W^{1,p}(\Omega)$) and in the case of Neumann boundary condition $V = W^{1,p}(\Omega)$.

By using the theory of monotone type operators one can formulate assumptions on a_j which imply existence of weak solutions (see [2,4,5]). Now we shall consider particular cases when one can prove existence of multiple solutions and statements on the number of solutions.

Assume that functions a_j have the form

$$a_j(x,\eta,\zeta;u) = \tilde{a}_j(x,\eta,\zeta,M(u)), \qquad j = 0,1,\ldots,n$$

where $M : V \to \mathbb{R}$ is a bounded, continuous (possibly nonlinear) operator and

$$\tilde{a}_j : \Omega \times \mathbb{R}^{n+1} \times \mathbb{R} \to \mathbb{R}$$

satisfy the Carathéodory conditions. (I.e. they are measurable in x and continuous in the other variables.) For arbitrary $\lambda \in \mathbb{R}$ define operator $A_\lambda : V \to V^\star$ by

$$\langle A_\lambda(u), v \rangle = \int_\Omega \left[\sum_{j=1}^{n} \tilde{a}_j(x,u,Du,\lambda) D_j v + \tilde{a}_0(x,u,Du,\lambda) v\right] dx.$$

Theorem 2.1. *Assume that for every $\lambda \in \mathbb{R}$ there exists a unique solution $u_\lambda \in V$ of*

$$A_\lambda(u_\lambda) = F \qquad (F \in V^\star). \tag{2.1}$$

Define the function $g : \mathbb{R} \to \mathbb{R}$ by $g(\lambda) = M(u_\lambda)$. Then a function $u \in V$ is a solution of

$$\int_\Omega \left[\sum_{j=1}^n \tilde{a}_j(x, u, Du, M(u)) D_j v + \tilde{a}_0(x, u, Du, M(u)) v \right] dx = \langle F, v \rangle, \qquad v \in V \tag{2.2}$$

if and only if $\lambda = M(u)$ satisfies $\lambda = g(\lambda)$. Thus the number of solutions of (2.2) equals the number of roots of the equation $\lambda = g(\lambda)$.

Proof. If $u \in V$ satisfies (2.2) then with $\lambda = M(u)$ the function $u_\lambda = u$ satisfies (2.1) and, consequently,

$$g(\lambda) = M(u_\lambda) = M(u) = \lambda.$$

Further, assume that $\lambda \in \mathbb{R}$ satisfies $\lambda = g(\lambda)$. Consider the solution u_λ of (2.1), then, clearly, $u = u_\lambda$ is a solution of (2.2) since $\lambda = g(\lambda) = M(u_\lambda)$. □

Consider the following particular case

$$\tilde{a}_j(x, u, Du, M(u)) = b_j(x, u, Du) h(M(u)),$$

i.e.

$$\tilde{a}_j(x, u, Du, \lambda) = b_j(x, u, Du) h(\lambda),$$

$j = 1, \dots, n$, and

$$\tilde{a}_0(x, u, Du, \lambda) = b_0(x, u, Du) h(\lambda) + \beta(x) l(\lambda),$$

with some continuous functions $h : \mathbb{R} \to \mathbb{R}^+$, $l : \mathbb{R} \to \mathbb{R}$ and $\beta \in L^q(\Omega)$ where $1/p + 1/q = 1$.

Define the operator $B : V \to V^\star$ by

$$\langle B(u), v \rangle = \int_\Omega \left[\sum_{j=1}^n b_j(x, u, Du) D_j v + b_0(x, u, Du) v \right], \qquad u, v \in V. \tag{2.3}$$

Theorem 2.2. *Assume that $B : V \to V^\star$ is a uniformly monotone, bounded, hemicontinuous operator (see, e.g. [8]) then the unique solution of*

$$A_\lambda(u) = F \tag{2.4}$$

is

$$u = u_\lambda = B^{-1} \left(\frac{F - l(\lambda)\beta}{h(\lambda)} \right) \tag{2.5}$$

and thus

$$g(\lambda) = M(u_\lambda) = M \left[B^{-1} \left(\frac{F - l(\lambda)\beta}{h(\lambda)} \right) \right].$$

Proof. In the particular case the equation $\langle A_\lambda(u), v \rangle = \langle F, v \rangle$ has the form

$$\int_\Omega \left[\sum_{j=1}^n b_j(x, u, Du) h(\lambda) D_j v + b_0(x, u, Du) h(\lambda) v + \beta(x) l(\lambda) v \right] dx = \langle F, v \rangle,$$

i.e.

$$\int_\Omega \left[\sum_{j=1}^n b_j(x,u,Du)D_jv + b_0(x,u,Du)v \right] dx = \left\langle \frac{F - l(\lambda)\beta}{h(\lambda)}, v \right\rangle,$$

thus

$$B(u) = \frac{F - l(\lambda)\beta}{h(\lambda)}. \tag{2.6}$$

According to the theory of monotone operators (see, e.g., [8]) the equation (2.6) has a unique solution

$$u = u_\lambda = B^{-1}\left(\frac{F - l(\lambda)\beta}{h(\lambda)}\right) \text{ and}$$
$$g(\lambda) = M(u_\lambda) = M\left[B^{-1}\left(\frac{F - l(\lambda)\beta}{h(\lambda)}\right)\right]. \qquad \square$$

Since $B^{-1} : V^\star \to V$ and $M : V \to \mathbb{R}$, l, h are continuous, $g : \mathbb{R} \to \mathbb{R}$ is a continuous function.

Now consider two particular cases.

1. Assume that B, M are homogeneous in the sense

$$B^{-1}(\mu F) = \mu^{\frac{1}{p-1}} B^{-1}(F) \quad \text{for all } \mu \geq 0 \quad (p > 1),$$
$$M(\mu u) = \mu^\sigma M(u) \qquad \text{for all } \mu \geq 0 \quad (\sigma \geq 0)$$

(M is nonnegative).

Theorem 2.3. *Assume that l, β are arbitrary continuous functions and g is a positive continuous function such that $\lambda = g(\lambda)$ has exactly N roots ($N = 0, 1, \dots, \infty$) then our boundary value problem (with 0 boundary condition) has exactly N solutions with*

$$h(\lambda) = \left[\frac{M\{B^{-1}[F - l(\lambda)\beta]\}}{g(\lambda)}\right]^{\frac{p-1}{\sigma}}.$$

Proof. According to Theorem 2.2 in this particular case

$$g(\lambda) = \frac{M\{B^{-1}[F - l(\lambda)\beta]\}}{h(\lambda)^{\frac{\sigma}{p-1}}},$$

i.e.

$$h(\lambda) = \left[\frac{M\{B^{-1}[F - l(\lambda)\beta]\}}{g(\lambda)}\right]^{\frac{p-1}{\sigma}}.$$

Thus the theorem follows from Theorem 2.1. $\square$

We have this particular case with $\beta = 0$ if e.g. B is defined by the p-Laplacian, i.e.

$$b_j(x,\eta,\zeta) = |\zeta|^{p-2}\zeta, \quad j = 1, \dots, n, \qquad b_0(x,\eta,\zeta) = c|\eta|^{p-2}\eta,$$

$\eta \in \mathbb{R}$, $\zeta \in \mathbb{R}^n$ with some $c > 0$. (If $V = W_0^{1,p}(\Omega)$ then c may be 0, too.) Further,

$$M(u) = \int_\Omega \left[\sum_{j=1}^n a_j(x)|D_ju|^\sigma + a_0(x)|u|^\sigma\right] dx$$

where $a_j \in L^\infty(\Omega)$, $a_j > 0$, $0 < \sigma \leq p$.

2. Assume that B and M are linear

Theorem 2.4. *If g is a positive continuous function such that $\lambda = g(\lambda)$ has N roots ($N = 0, 1, \dots, \infty$) then our boundary value problem has N solutions with*

$$h(\lambda) = \frac{M[B^{-1}(F)] - l(\lambda)M[B^{-1}(\beta)]}{g(\lambda)}$$

and arbitrary continuous function l. Similarly, if $M[B^{-1}(\beta)] \neq 0$ and g is a continuous function such that $\lambda = g(\lambda)$ has N roots then our boundary value problem has N solutions with

$$l(\lambda) = \frac{M[B^{-1}(F)] - g(\lambda)h(\lambda)}{M[B^{-1}(\beta)]}$$

and arbitrary continuous function h.

Proof. According to Theorem 2.2 in this case

$$g(\lambda) = \frac{M[B^{-1}(F)] - l(\lambda)M[B^{-1}(\beta)]}{h(\lambda)},$$

i.e.

$$h(\lambda) = \frac{M[B^{-1}(F)] - l(\lambda)M[B^{-1}(\beta)]}{g(\lambda)}$$

and

$$l(\lambda) = \frac{M[B^{-1}(F)] - g(\lambda)h(\lambda)}{M[B^{-1}(\beta)]}.$$

So Theorem 2.4 follows from Theorem 2.1. □

In this case the operator $M : W^{1,2}(\Omega) \to \mathbb{R}$ may have the form

$$Mu = \int_\Omega \left[\sum_{j=1}^n a_j D_j u + a_0 u \right] + \int_{\partial\Omega} b_0 u d\sigma$$

where $a_j \in L^2(\Omega)$, $b_0 \in L^2(\partial\Omega)$.

3 Number of solutions of equations with nonlocal operators

Now consider equations (1.1) containing nonlinear and nonlocal operators of the form

$$B(u) = F(u) \tag{3.1}$$

where B is given by (2.3) and $F : V \to V^\star$ is a given nonlinear operator. Clearly, $u \in V$ satisfies (3.1) if and only if

$$u = B^{-1}[F(u)] = G(u) \tag{3.2}$$

where $G : V \to V$ is a given operator, i.e. u is a fixed point of G. Then

$$F(u) = B[G(u)]. \tag{3.3}$$

Now we shall consider particular cases for G.

1.

$$[G(u)](x) = [K(u)](x) = \int_\Omega \mathcal{K}(x, y)u(y)dy \tag{3.4}$$

where $\mathcal{K} \in L^2(\Omega \times \Omega)$, $u \in V \subset W^{1,2}(\Omega)$ and B is a linear strongly elliptic differential operator.

Theorem 3.1. *If $\mathcal{K}$ is sufficiently smooth and good then the solution of (3.2) belongs to V and by (3.3)*

$$[F(u)](x) = \int_\Omega B_x[\mathcal{K}(x,y)]u(y)dy$$

and (3.1) has the form

$$[B(u)](x) = \int_\Omega B_x[\mathcal{K}(x,y)]u(y)dy. \tag{3.5}$$

Further, if 1 is an eigenvalue of G with multiplicity k then (3.5) has k solutions.

Proof. Equation (3.5) is equivalent with

$$u(x) = \int_\Omega \mathcal{K}(x,y)u(y)dy$$

which implies Theorem 3.1. □

2.

$$G(u) = Ku + h(P(u))g \tag{3.6}$$

where K is given by (3.4), $P : V \to \mathbb{R}$ is a linear continuous functional, $h : \mathbb{R} \to \mathbb{R}$ is a continuous function and $g \in V$. Assume that $\mathcal{K}$ is the function before, 1 is not an eigenvalue of the operator K and B is a linear strongly elliptic differential operator.

Theorem 3.2. *In this case equation (3.1) has the form*

$$B(u) = \int_\Omega B_x[\mathcal{K}(x,y)]u(y)dy + h(P(u))Bg. \tag{3.7}$$

Further, u is a solution of (3.7) if and only if $u = h(\lambda)[I-K]^{-1}(g)$ where λ is a root of the equation

$$\lambda = h(\lambda)P([I-K]^{-1}(g)). \tag{3.8}$$

Thus the number of solutions of (3.7) equals the number of solutions of equation (3.8).

Proof. Equation (3.7) is fulfilled if and only if

$$u = \int_\Omega \mathcal{K}(x,y)]u(y)dy + h(P(u))g$$

i.e.

$$(I-K)u = h(P(u))g,$$
$$u = h(P(u)(I-K)^{-1}g. \tag{3.9}$$

Let $u_\lambda = h(\lambda)(I-K)^{-1}g$, then

$$P(u_\lambda) = h(\lambda)P[(I-K)^{-1}g].$$

Consequently, (3.9) (and so (3.7)) is satisfied if and only if $\lambda = P(u)$ satisfies (3.8). □

3.

$$[G(u)](x) = P(u)h(u(x)) \tag{3.10}$$

where $h : \mathbb{R} \to \mathbb{R}$ is a given C^1 function such that $u \in V$ implies $h(u) \in V$ ($V \subset W^{1,p}(\Omega)$) and $P : V \to \mathbb{R}$ is defined by $P(u) = \int_\Omega g(y)u(y)dy$ where $g \in L^q(\Omega)$ $(1/p + 1/q = 1)$, $g \geq 0$. Further, let B be homogeneous in the sense $B(\mu F) = \mu^{p-1}B(F)$ for $\mu \geq 0$ $(p > 1)$.

Theorem 3.3. *In this case* (3.1) *has the form*

$$B(u) = [P(u)]^{p-1} B[h(u)]. \tag{3.11}$$

Assume that there exists an interval (z_1, z_2) *such that*

$$h(z)/z = c_0 \text{ (some constant)} \quad \text{for } z \in (z_1, z_2) \tag{3.12}$$

and

$$z_1 \int_\Omega g(y)dy < 1/c_0 < z_2 \int_\Omega g(y)dy \tag{3.13}$$

then there is an infinite number of functions satisfying (3.11).

If there is no interval (z_1, z_2) *satisfying* (3.12) *then solutions of* (3.1) *may be only constant functions*

$$u(x) = c, \quad x \in \Omega \tag{3.14}$$

where c *satisfies*

$$h(c) \int_\Omega g(y)dy = 1. \tag{3.15}$$

Thus, in this case the number of solutions of (3.1) *equals the number of roots* c *of* (3.15) *which may be any finite or infinite number depending on the function* h.

(In the last case $V = W^{1,p}(\Omega)$, i.e. we have solutions of the Neumann problem with homogeneous boundary condition.)

Proof. Equation (3.11) is equivalent with

$$u(x) = P(u)h(u(x)),$$

i.e.

$$u(x) = h(u(x)) \int_\Omega g(y)u(y)dy, \qquad x \in \Omega. \tag{3.16}$$

If (3.12) holds then (3.16) (and (3.11)) means that

$$u(x) = c_0 u(x) \int_\Omega g(y)u(y)dy, \qquad x \in \Omega,$$

i.e. (except of the trivial case $u(x) = 0$)

$$\int_\Omega g(y)u(y)dy = 1/c_0.$$

If the condition (3.13) is fulfilled then, clearly, there is an infinite number of functions u such that

$$z_1 < u(y) < z_2 \quad \text{and} \quad \int_\Omega g(y)u(y)dy = 1/c_0.$$

If there is no interval (z_1, z_2) satisfying (3.12) then, clearly, u may satisfy (3.11) only if $u(x) = c$ where

$$c = ch(c) \int_\Omega g(y)dy, \qquad 1 = h(c) \int_\Omega g(y)dy.$$ □

4.

$$[G(u)](x) = P(u)u(\psi(x))$$

where $\psi : \overline{\Omega} \to \overline{\Omega}$ is a C^1 function with C^1 inverse, P is a (possibly nonlinear) functional over V and B is as before in **3.**

Theorem 3.4. *In this case equation* (3.1) *has the form*

$$B(u) = [P(u)]^{p-1} B[u(\psi)] \tag{3.17}$$

and u is a solution of (3.17) *if and only if*

$$u(x) = P(u)u(\psi(x)), \quad x \in \Omega. \tag{3.18}$$

Clearly, $u = 0$ is a trivial solution of (3.18)*, i.e. of* (3.17)*.*

A continuous function u is a nontrivial solution of (3.17) *if and only if*

$$P(u) = 1 \quad \text{and} \quad u(x) = u(\psi(x)) \quad \text{for all } x \in \Omega \tag{3.19}$$

or

$$P(u) = -1 \quad \text{and} \quad u(x) = -u(\psi(x)) \quad \text{for all } x \in \Omega. \tag{3.20}$$

$u = c$ is a constant solution of (3.18) *if $P(c) = 1$. If Ω is symmetric with respect to 0 and $\psi(x) = -x$ ($x \in \overline{\Omega}$) then u satisfies* (3.19) *with the properties $u(-x) = u(x)$, $x \in \Omega$ and $P(u) = 1$. Further, u satisfies* (3.20) *with the properties $u(-x) = -u(x)$, $x \in \Omega$ and $P(u) = -1$.*

Proof. Assume that a (nonidentically 0) continuous u satisfies (3.18), then we have

$$|u(x)| = |P(u)||u(\psi(x))|, \qquad x \in \Omega. \tag{3.21}$$

Then there is $x^{(0)} \in \overline{\Omega}$ such that

$$|u(x^{(0)})| = \sup_{x \in \overline{\Omega}} |u(x)| > 0. \tag{3.22}$$

By (3.21)

$$|u(x^{(0)})| = |P(u)||u(\psi(x^{(0)}))|. \tag{3.23}$$

Assume that $|P(u)| < 1$. Then by (3.23) $|u(\psi(x^{(0)})| > |u(x^{(0)})|$ which is impossible by (3.22).

On the other hand,

$$|u(\psi^{-1}(x^{(0)}))| = |P(u))|u(x^{(0)})|,$$

thus $|P(u)| > 1$ would imply $|u(\psi^{-1}(x^{(0)}))| > |u(x^{(0)})|$ which is impossible by (3.22). Consequently, $|P(u)| = 1$, i.e. either $P(u) = 1$ or $P(u) = -1$. □

If P is a linear functional over V and Ω is symmetric with respect to 0 then multiplying a function u with the property $u(-x) = u(x)$ (resp. $u(-x) = -u(x)$) by a suitable constant, we have $P(u) = 1$ (resp. $P(u) = -1$). Thus, in this case (3.19) (resp. (3.20) and so (3.17)) has infinitely many solutions.

Another particular case for the (nonlinear) functional P:

$$P(u) = 1 + \left[\int_\Omega (a_1 - u)^2 dx\right] \dots \left[\int_\Omega (a_m - u)^2 dx\right]$$

with different real numbers $a_1, \dots, a_m$. Then all the solutions of (3.19) are constant functions

$$u_j(x) = a_j, \quad x \in \Omega, \quad j = 1, \dots, m.$$

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