Electronic Journal of Qualitative Theory of Differential Equations
2011, No. **48**, 1-10; **http://www.math.u-szeged.hu/ejqtde/**

LINEAR DIFFERENTIAL EQUATIONS WITH COEFFICIENTS IN FOCK TYPE SPACE

XIANG DONG YANG & JIN TU

ABSTRACT. In this paper we deal with complex differential equations of the form

$$f^{(k)} + a_{k-1}(z)f^{(k-1)} + \cdots + a_1(z)f^{'} + a_0(z)f = 0$$

with the coefficients in Fock type space. The relation between the solutions and coefficients in Fock type space is obtained.

1. INTRODUCTION

Motivated by the work in [6], [7] and [8], we will study complex differential equations of the form

$$f^{(k)} + a_{k-1}(z)f^{(k-1)} + \cdots + a_1(z)f^{'} + a_0(z)f = 0 \tag{1}$$

where the coefficients are entire functions.

In [8], equations of the form (1) with coefficients in weighted Bergman or Hardy spaces are studied. The *direct problem* is proved, that is, if the coefficients $a_j(z)$, $j = 0, ..., k-1$ of (1) belong to the weighted Bergman space, then all solutions are of finite order of growth and belong to weighted Bergman space. The *inverse problem* is also investigated, that is, if all solutions are of finite order of growth, then the coefficient is proved to belong to weighted Bergman space.

The Bargmann-Fock space (see [1], [2]) is the Hilbert space of entire functions equipped with the inner product

$$< f, g >= \frac{1}{\pi}\int_{\mathbb{C}} f(z)\overline{g(z)}e^{-z\cdot\overline{z}}\mathrm{d}x\mathrm{d}y,$$

normed by $\|f\| = \sqrt{< f, f >}$. This space has been studied by many authors and it is rooted from mathematical problems of relativistic physics (see [12]) or from quantum optics (see [10]). In physics the Bargmann-Fock space contains the canonical coherent states, so it is the main tool for studying the bosonic coherent state theory of radiation field (see [11]). The Bargmann-Fock space has also been proved invaluable in the

2000 *Mathematics Subject Classification.* 34M10, 30D35.

Key words and phrases. Linear differential equations, Fock type spaces, Entire functions.

Supported by Natural Science Foundation of Yunnan Province in China (Grant No. 2009ZC013X); Basic Research Foundation of Education Bureau of Yunnan Province in China (Grant No. 09Y0079); Natural Science Foundation of Jiangxi Province in China (Grant No. 2009GQS0013).

theory of the wavelets. In fact, the Bargmann transform is a unitary map from $L^2(\mathbb{R})$ onto the Bargmann-Fock space which transforms the family of evaluation functionals at a point into canonical coherent states which are nothing but the Gabor wavelets.

The Fock-type space F_α (see [3]) is the Hilbert space of entire functions equipped with the inner product

$$< f, g >_\alpha = \int_{\mathbb{C}} f(z)\overline{g(z)} e^{-\alpha(|z|)} \mathrm{d}x\mathrm{d}y,$$

normed by $\|f\|_\alpha = \sqrt{< f, f >_\alpha}$, where $\alpha(r)$ is a nonnegative and non-decreasing function of r. It's obvious that $|z|^2$ is such a function. Thus the Fock-type space F_α is a generalization of the Bargmann-Fock space.

In this paper, we will consider the growth relation between the coefficients and the solutions of (1). We are particularly interested in the Fock-type spaces F_α and F_{e^α} case:

(i)Find the conditions imposed on the coefficients $a_j(z)$, $j = 0, ..., k-1$ of (1) which make all of the solutions belong to the Fock-type space F_{e^α}.

(ii)Suppose that all solutions of (1) belong to the Fock-type space F_{e^α}, find out whether all of the coefficients $a_j(z)$, $j = 0, ..., k-1$ belong to the Fock-type space F_α.

Hereafter, problems (i) and (ii) will be referred to as the *direct problem* and the *inverse problem*, respectively.

Throughout this paper, A will denote positive constants, it may be different at each occurrence.

2. DIRECT PROBLEM

In this section, sufficient conditions for all of the solutions of (1) belong to F_{e^α} will be obtained. We need the following result on growth estimate for solutions of (1) in [6].

Lemma 2.1. *Let f be a solution of* (1) *in the disk $\{z \in \mathbb{C} : |z| < r\}$, where $0 < r \le \infty$, let $n_c \in \{1, ..., k\}$ be the number of nonzero coefficients $a_j(z)$, $j = 0, ..., k-1$, and let $\theta \in [0, 2\pi)$. If $z_\theta = r_0 e^{i\theta} \in \{z \in \mathbb{C} : |z| < r\}$ is such that $a_j(z_\theta) \ne 0$ for some $j = 0, ..., k-1$, then for all $r_0 < r_1 < r$,*

$$|f(r_1 e^{i\theta})| \le A \exp\{n_c \int_{r_0}^{r_1} \max_{j=0,\dots,k-1} |a_j(te^{i\theta})|^{1/(k-j)} \mathrm{d}t\}$$

where A is some positive constant depends on the values of the derivatives of f and the values of $a_j(z_\theta)$ at z_θ.

The main result of this section is as follows.

Theorem 2.1. *Suppose that $\alpha(r)$ is a nonnegative and nondecreasing function of r satisfying*

$$\liminf_{r\to\infty} \frac{\alpha(r)}{\log r} > 1, \tag{2}$$

furthermore, suppose that $a_j(z) \in F_\alpha, j = 0, 1, ..., k-1$, then all solutions of (1) belong to F_{e^α}.

Proof. Since $a_j(z) \in F_\alpha$ for $j = 0, 1, ..., k-1$, we have

$$\int_{\mathbb{C}} |a_j(z)|^2 e^{-\alpha(|z|)} \mathrm{d}x\mathrm{d}y = 2\pi \int_0^\infty |a_j(z)|^2 \mathrm{e}^{-\alpha(\mathrm{r})} \mathrm{r}\mathrm{d}r < \infty.$$

Thus

$$\lim_{r\to\infty} |a_j(z)|^2 e^{-\alpha(r)} r = 0,$$

which yields

$$|a_j(z)|^2 \leq A \frac{e^{\alpha(r)}}{r}$$

for sufficiently large $|z| = r > 0$, or

$$|a_j(z)| \leq A \frac{e^{\alpha(r)/2}}{\sqrt{r}}, r > r_0. \tag{3}$$

If $f(z)$ is a solution of (1), from Lemma 2.1, we have

$$|f(re^{i\theta})| \leq A \exp\left\{n_c \int_{r_0}^r \max_{j=0,\dots,k-1} |a_j(te^{i\theta})|^{1/(k-j)} \mathrm{d}t\right\}$$

where A is some positive constant depends on the values of the derivatives of f and the values of $a_j(z_\theta)$ at z_θ and n_c is defined in Lemma 2.1, combination with (3) yields

$$\begin{aligned} |f(re^{i\theta})| &\leq \exp\left\{A \int_{r_0}^r \frac{e^{\alpha(t)/2}}{\sqrt{t}} \mathrm{d}t\right\} \\ &\leq A \exp\left\{Ar^{1/2} e^{\alpha(r)/2}\right\}. \end{aligned}$$

By (2), we have

$$\alpha(r) > (1+\varepsilon) \log r$$

for any $\frac{1}{2} > \varepsilon > 0$, then

$$2Ar^{1/2} e^{\alpha(r)/2} - e^{\alpha(r)} < e^{\alpha(r)/2}(2Ar^{1/2} - r^{1+\varepsilon}).$$

Thus

$$\begin{aligned} \|f\|_{e^\alpha}^2 &= \int_{\mathbb{C}} |f(z)|^2 e^{-e^{\alpha(|z|)}} \mathrm{d}x\mathrm{d}y \\ &\leq A \int_{r_0}^\infty \exp\left\{2Ar^{1/2} e^{\alpha(r)/2} - e^{\alpha(r)}\right\} r\mathrm{d}r < \infty, \end{aligned}$$

proving that all of solutions of (1) belong to F_{e^α}.

□

Remark 2.1. *Although we are unable to show the sharpness of the constant 1 in* (4)*, we remark it is necessary. Actually, if* (4) *does not hold, we may suppose*

$$\limsup_{r\to\infty}\frac{\alpha(r)}{\log r}<1.$$

Taking $\alpha(r)\sim\log r^{\eta_0}$ *where* $0<\eta_0<\frac{1}{2}$ *, for example, here the symbol* $\sim$ *denotes that* $\alpha(r)$ *and* $\log r^{\eta_0}$ *have the same growth as* r *tends to infinity, then*

$$2Ar^{1/2}e^{\alpha(r)/2}-e^{\alpha(r)}=e^{\alpha(r)/2}(2Ar^{1/2}-e^{\alpha(r)/2})\sim r^{\frac{\eta_0+1}{2}},$$

and

$$\int_{r_0}^{\infty}\exp\{2Ar^{1/2}e^{\alpha(r)/2}-e^{\alpha(r)}\}r\mathrm{d}r=\infty.$$

By the proof of Theorem 2.1, we know that $f\notin F_{e^\alpha}$ *holds in this case.*

3. INVERSE PROBLEM

To study the inverse problem, we need some background knowledge and some lemmas.

We present the following elementary result on inequality in [4] for later use.

Lemma 3.1. *Let* $a_k\geq 0$ *for* $k=1,...,n$*. Then*

$$(\sum_{k=1}^{n}a_k)^p\leq n^{p-1}(\sum_{k=1}^{n}a_k^p)$$

for $1\leq p<\infty$*.*

We also need the growth estimates of meromorphic functions in [5].

Lemma 3.2. *Let f(z) be a transcendental meromorphic function, furthermore, let* $\beta>1$ *be a positive constant. Then there exist a set* $E\subset[0,2\pi)$ *that has linear measure zero, a constant* $A>0$ *that depends only on* β*, and a constant* $r_0=r_0(\theta)>1$ *such that*

$$\left|\frac{f^{(m)}(re^{i\theta})}{f(re^{i\theta})}\right|\leq A\left[T(\beta r,f)\frac{\log^\beta r}{r}\log T(\beta r,f)\right]^m,\ m\in\mathbb{N},$$

where $r>r_0$ *and* $\theta\in[0,2\pi)\setminus E$ *.*

Recall that the order reduction procedure is as follows(see [8]): if $\{f_1, ..., f_k\}$ is a solution base of (1) in $|z| < r$, then the first order reduction of (1) results in

$$\nu_1^{(k-1)} + a_{1,k-2}(z)\nu_1^{(k-2)} + \cdots + a_{1,0}(z)\nu_1 = 0, \tag{4}$$

where

$$a_{1,j}(z) = a_{j+1}(z) + \sum_{m=1}^{k-j-1} \binom{j+1+m}{m} a_{j+1+m}(z) \frac{f_1^{(m)}(z)}{f_1(z)} \tag{5}$$

for $j = 0, ..., k-2$ and the meromorphic functions

$$\nu_{1,j}(z) = \frac{d}{dz}\left(\frac{f_{j+1}(z)}{f_1(z)}\right) \tag{6}$$

for $j = 0, ..., k-1$ are linearly independent solutions of (5) in $|z| < r$.

We have the following relations between the solutions and its reductions.

Lemma 3.3. *Suppose $\alpha(r)$ is a nonnegative and nondecreasing continuous function of r satisfying*

$$\int_{r_0}^{\infty} e^{(\tau_0\alpha(\beta_0 r) - \alpha(\beta r))} r\mathrm{d}r < \infty, \tag{7}$$

for $r > r_0$, where τ_0, β_0 and β are some positive constants satisfying $\tau_0 > 4k$, $\beta_0 > 1$, and $\beta > \beta_0$. Let $\{f_1, ..., f_k\}$ be a solution base of of equation (1). If $f_j \in F_{e^{\alpha(r)}}, j = 1, 2, ...,$ then for any nonnegative integer $m \leq 2$ and $l = 1, 2, ..., k$,

$$\int_{\mathbb{C}} \left|\frac{f_j^{(l)}(z)}{f_j(z)}\right|^{2m} e^{-\alpha(\beta|z|)} \mathrm{d}x\mathrm{d}y < \infty, \tag{8}$$

$$\int_{\mathbb{C}} \left|\frac{v_{1,j}^{(l)}(z)}{v_{1,j}(z)}\right|^{2m} e^{-\alpha(\beta|z|)} \mathrm{d}x\mathrm{d}y < \infty, \tag{9}$$

and

$$\int_{\mathbb{C}} \left|\frac{v_{k,j}^{(l)}(z)}{v_{k,j}(z)}\right|^{2m} e^{-\alpha(\beta|z|)} \mathrm{d}x\mathrm{d}y < \infty. \tag{10}$$

Proof. Since $f_j(z) \in F_{e^{\alpha}}$ for $j = 1, ..., k$, we have

$$\int_{\mathbb{C}} |f_j(z)|^2 e^{-e^{\alpha(|z|)}} \mathrm{d}x\mathrm{d}y = 2\pi \int_0^{\infty} |f_j(z)|^2 \mathrm{e}^{-e^{\alpha(r)}} \mathrm{r}\mathrm{d}r < \infty.$$

Thus

$$\lim_{r\to\infty} |f_j(z)|^2 e^{-e^{\alpha(r)}} r = 0,$$

which yields

$$|f_j(z)|^2 \le A\frac{e^{e^{\alpha(r)}}}{r}$$

for sufficiently large $|z| = r > 0$, or

$$\log M(r, f_j) \le \frac{1 + o(1)}{2} e^{\alpha(r)}.$$

Thus,

$$(T(\beta_0 r, f_j))^m \le A e^{m\alpha(\beta_0 r)} \tag{11}$$

for any nonnegative integer $m \le 2$ and $\beta_0 > 1$. By Lemma 3.2, for given positive constant $\beta_0 > 1$, there exist some set $E \subset [0, 2\pi)$ of measure zero and some constant $r_1 = \sup_{\theta_0 \in [0,2\pi) \setminus E} r_0(\theta_0) > 0$ such that

$$\left| \frac{f_j^{(m)}(z)}{f_j(z)} \right| \le A|T(\beta_0 r, f_j) \frac{\log^\beta r}{r} \log T(\beta_0 r, f_j)|^m \tag{12}$$

for $\arg z \in \{\theta_0 : \theta_0 \in [0, 2\pi) \setminus E\}$ and $|z| \ge r_1$. Combine (7) with (11) and (12), we have

$$\int_{\mathbb{C}} \left| \frac{f_j^{(l)}(z)}{f_j(z)} \right|^{2m} e^{-\alpha(\beta|z|)} \mathrm{d}x\mathrm{d}y \le \mathrm{A} \int_{\mathbb{C}} \mathrm{e}^{(\tau_0 \alpha(\beta_0 \mathrm{r}) - \alpha(\beta \mathrm{r}))} \mathrm{r}\mathrm{d}r < \infty,$$

for any nonnegative integer $m \le 2$, $\beta > \beta_0$ and $l = 1, 2, ..., k$, which is (8). The same reasoning yields (9) and (10).

We also need the following result on reduction.

Lemma 3.4. *Suppose $\alpha(r)$ is a nonnegative and nondecreasing continuous function of r satisfying (7) for some $\tau_0 > 4k$ and $\beta > \beta_0 > 1$ furthermore, assume that for any nonnegative integer $m \le 2$ and $\beta > \beta_0$*

$$\int_{\mathbb{C}} |a_{1,j}(z)|^{2m} e^{-\alpha(\beta|z|)} \mathrm{d}x\mathrm{d}y < \infty, \tag{13}$$

where $a_{1,j}(z)$ is defined in (5), then

$$\int_{\mathbb{C}} |a_j(z)|^{2m} e^{-\alpha(\beta|z|)} \mathrm{d}x\mathrm{d}y < \infty, \tag{14}$$

where $a_j(z)(j = 0, ..., k - 1)$ are coefficients of (1).

Proof. Note that $a_k(z) \equiv 1$, for $j = 1, 2, ..., k$

$$a_{1,k-2}(z) = a_{k-1}(z) + k\frac{f_1'(z)}{f_1(z)}. \tag{15}$$

Let $f_1(z)$ denote any transcendental entire solution of (1). For any nonnegative integer $m \le 2$, by Lemma 3.2, there exist some set $E \subset$

$[0, 2\pi)$ of measure zero and some constant $r_1 = \sup_{\theta_0 \in [0,2\pi)\setminus E} r_0(\theta_0) > 0$ such that

$$\left|\frac{f_1^{(m)}(z)}{f_1(z)}\right| \le A|T(\beta_0 r, f_1)\frac{\log^\beta r}{r}\log T(\beta_0 r, f_1)|^m \tag{16}$$

for $\arg z \in \{\theta_0 : \theta_0 \in [0, 2\pi) \setminus E\}$ and $|z| \ge r_1, \beta_0 > 1$. Combination of (15) and (16) yields

$$\big|a_{k-1}(z)\big| \le |a_{1,k-2}(z)| + |T(\beta_0 r, f_1)\frac{\log^\beta r}{r}\log T(\beta_0 r, f_1)|^m \tag{17}$$

for $|z| \ge r_1, \arg z = \theta \in [0, 2\pi) \setminus E$.

From the proof of Lemma 3.3, we know that for $f_1 \in F_{e^\alpha}$,

$$(T(\beta_0 r, f_j))^m \le Ae^{m\alpha(\beta_0 r)}. \tag{18}$$

Square both sides, multiply both sides of (17) by $e^{-\alpha(\beta|z|)}$, then integrate over the annulus, combine with Lemma 3.1, (18) and the fact (7), we have

$$\begin{aligned}
&\int_{\mathbb{C}} |a_{k-1}(z)|^{2m} e^{-\alpha(\beta|z|)} \mathrm{d}x\mathrm{d}y\\
&= \int_0^{2\pi}\mathrm{d}\theta \int_0^\infty |a_{k-1}(re^{i\theta})|^{2m} e^{-\alpha(\beta r)} r\mathrm{d}r\\
&\le \int_0^{2\pi}\mathrm{d}\theta \int_0^{r_0} |a_{k-1}(re^{i\theta})|^{2m} e^{-\alpha(\beta r)} r\mathrm{d}r\\
&+A\int_0^{2\pi}\mathrm{d}\theta \int_{r_0}^\infty |a_{1,k-2}(re^{i\theta})|^{2m} e^{-\alpha(\beta r)} r\mathrm{d}r\\
&+A\int_{r_0}^\infty |T(\beta_0 r, f_1)\frac{\log^\beta r}{r}\log T(\beta_0 r, f_1)|^{2m} e^{-\alpha(\beta r)} r\mathrm{d}r\\
&\le A\int_{r_0}^\infty |a_{1,k-2}(re^{i\theta})|^{2m} e^{-\alpha(\beta r)} \mathrm{d}x\\
&+A\int_{r_0}^\infty e^{(\tau_0\alpha(\beta_0 r)-\alpha(\beta r))} r\mathrm{d}r\\
&< \infty,
\end{aligned}$$

where A is some positive constant .

Suppose that the assertion is proved for $j = k-1, ..., k-l, l \in \{1, ..., k-2\}$. From

$$a_{1,k-(l+1)}(z) = a_{k-(l+2)}(z) + \sum_{m=1}^{l+1}\binom{k-l-1+m}{m} a_{k-(l+1)+m}(z)\frac{f_1^{(m)}(z)}{f_1(z)},$$

we have

$$|a_{k-(l+1)}(z)| \leq |a_{1,k-(l+2)}(z)| + A\left|\frac{f_1^{(l+1)}(z)}{f_1(z)}\right|$$
$$+ A\sum_{m=1}^{l} |a_{k-(l+1)+m}(z)|\left|\frac{f_1^{(m)}(z)}{f_1(z)}\right|.$$

Thus, by Lemma 3.1, there exists a positive constant A such that

$$|a_{k-(l+1)}(z)|^2 \leq A(|a_{1,k-(l+2)}(z)|^2 + \left|\frac{f_1^{(l+1)}(z)}{f_1(z)}\right|^2$$
$$+ \sum_{m=1}^{l} \left(\left|a_{k-(l+1)+m}(z)\frac{f_1^{(m)}(z)}{f_1(z)}\right|^2\right).$$

Multiply both sides of (17) by $e^{-\alpha(\beta|z|)}$, the Cauchy-Schwartz inequality yields,

$$\int_{\mathbb{C}} |a_{k-(l+1)}(z)|^{2m} e^{-\alpha(\beta|z|)} \mathrm{d}x\mathrm{d}y$$
$$\leq A\bigg(\int_{\mathbb{C}} |a_{1,k-(l+2)}(z)|^2 e^{-\alpha(\beta|z|)} \mathrm{d}x\mathrm{d}y + \int_{\mathbb{C}} \left|\frac{f_1^{(l+1)}(z)}{f_1(z)}\right|^2 e^{-\alpha(\beta|z|)} \mathrm{d}x\mathrm{d}y$$
$$+ \sum_{m=1}^{l} \left(\int_{\mathbb{C}} \left|a_{k-(l+1)+m}(z)\right|^2 e^{-\alpha(\beta|z|)} \mathrm{d}x\mathrm{d}y\right)^{1/2}$$
$$\times \left(\int_{\mathbb{C}} \left|\frac{f_1^{(m)}(z)}{f_1(z)}\right|^4 e^{-\alpha(\beta|z|)} \mathrm{d}x\mathrm{d}y\right)^{1/2}\bigg).$$

Thus, by (7), (16) and (18), we know that (14) holds for j=1, ..., k.

For $a_0(z)$, from

$$a_0(z) = -\frac{f_1^{(k)}}{f_1} - a_{k-1}\frac{f_1^{(k-1)}}{f_1} - \dots - a_1\frac{f_1'}{f_1},$$

the conclusion follows from Lemma 3.3.

□

Our result on the inverse problem is as follows.

Theorem 3.1. *Suppose that $\alpha(r)$ is a nonnegative and nondecreasing continuous function of r satisfying (7) for some $\tau_0 > 4k$ and $\beta > \beta_0 > 1$. Let $a_j(z)(j = 0, ..., k-1)$ denote the coefficients of (1). If all of the solutions of (1) belong to $F_{e^{\alpha(r)}}$, then for any nonnegative integer $m \leq 2$ and $\beta > \beta_0 > 1$, $(a_j(z))^m (j = 0, ..., k-1)$ belong to $F_{\alpha(\beta r)}$.*

Proof. When $k = 1$, the equation (1) has the following form

$$f' + a_0(z)f = 0.$$

Let f be a nontrivial entire function solution of (1). By (8) in Lemma 3.3, it is obvious that for any nonnegative integer $m \leq 2k$ and $\beta > 1$, $(a_0(z))^m \in F_{\alpha(\beta r)}$.

Suppose that $k \geq 2$. After $k-1$ order reduction steps, we obtain the differential equation

$$\nu_{k-1}^{'} + a_{k-1,0}(z)\nu_{k-1} = 0.$$

Thus

$$a_{k-1,0} = -\frac{\nu_{k-1}^{'}(z)}{\nu_{k-1}(z)},$$

where $\nu_{k-1}(z)$ is the meromorphic function defined in (6). Combine (9) in Lemma 3.3 with Lemma 3.4, we conclude $a_{k-1}(z) \in F_{\alpha(\beta r)}$, proving Theorem 3.1

□

4. Acknowledgement

The authors would like to thank the referee for many valuable suggestions and detailed criticisms on the original version of the paper. Their thanks also go to Prof. J. Heittokangas for sending papers [7] and [8].

References

[1] V. Bargmann, *On a Hilbert space of analytic functions and an associated integral transform,* Part I Comm. Pure Appl. Math. **14** (1961), 187-214.

[2] V. Bargmann, *On a Hilbert space of analytic functions and an associated integral transform,* Part II Comm. Pure Appl. Math. **20** (1967), 1-101.

[3] A. A. Borichev, *The polynomial approximation property in Fock-type spaces,* Mathematica Scandinavica **82** (1998), 256-264.

[4] P. Duren, *Theory of H^p Spaces.* Academic Press, New York, 1970.

[5] G. Gundersen, *Estimates for the logarithmic derivative of a meromorphic funtion, plus similar estimates,* J. London Math. Soc. **37** (1988) 88-104.

[6] J. Heittokangas, R. Korhonen and J. Rättyä, *Growth estimates for solutions of complex differential equations,* Annales Academiæ Scientiarum Fennicæ **29** (2004) 233-246.

[7] J. Heittokangas, R. Korhonen and J. Rättyä, *Linear differential equations with solutions in the Dirichlet type subspace of the Hardy space,* Nagoya Math. J. **187** (2007) 91-113.

[8] J. Heittokangas, R. Korhonen and J. Rättyä, *Linear differential equations with coefficients in weighted Bergman and Hardy spaces,* Trans. Amer. Math. Soc. **360** (2008) 1035-1055.

[9] W. K. Hayman, *Meromorphic Functions.* Oxford University Press, Amen House, London, 1964.

[10] J. R. Klauder and E. C. Sudarshan, *Fundamentals of Quantum Optics.* Benjamin, New York, 1968.

[11] J. R. Klauder and B. Skagerstam, *Coherent states, Application in physics and Mathematical physiscs.* World Scientific Press, 1985.

[12] I. E. Segal, *Mathematical problems of Relativistic Physics.* American Mathematical Society, Providence, Rhode Island, 1963.

(Received November 30, 2010)

(Yang) Department of Mathematics, Kunming University of Science and Technology, 650093 Kunming, China

E-mail address: yangsddp@126.com

(Tu) College of Mathematics and Information Sciences, Jiangxi Normal University, 330022 Nanchang, China

E-mail address: tujin2008@sina.com