

Global attractivity of positive periodic solution of a delayed Nicholson model with nonlinear density-dependent mortality term

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Abstract. This paper is concerned with the existence, uniqueness and global attractivity of positive periodic solution of a delayed Nicholson's blowflies model with nonlinear density-dependent mortality rate. By some comparison techniques via differential inequalities, we first establish sufficient conditions for the global uniform permanence and dissipativity of the model. We then utilize an extended version of the Lyapunov functional method to show the existence and global attractivity of a unique positive periodic solution of the underlying model. An application to the model with constant coefficients is also presented. Two numerical examples with simulations are given to illustrate the efficacy of the obtained results.

Keywords: Nicholson's blowflies model, positive periodic solution, attractivity, time-varying delays, nonlinear density-dependent mortality.


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1 Introduction

Mathematical models are important for describing dynamics of phenomena in the real world [7, 12, 25]. For example, in [24], Nicholson used the following delay differential equation

$$N'(t) = -\alpha N(t) + \beta N(t - \tau)e^{-\gamma N(t-\tau)}, \quad (1.1)$$

where α , β , γ are positive constants, to model the laboratory population of the Australian sheep-blowfly. In the biology interpretation of equation (1.1), $N(t)$ is the population size at time t , α is the per capita daily adult mortality rate, β is the maximum per capita daily egg production rate, $\frac{1}{\gamma}$ is the size at which the population reproduces at its maximum rate and $\tau \geq 0$ is the generation time (the time taken from birth to maturity). Model (1.1) is typically referred to the Nicholson's blowflies equation. It is interesting that when the maximum reproducing rate is not limited (i.e. $\frac{1}{\gamma} \rightarrow +\infty$) and the time τ is small which can be ignored,

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model (1.1) is reduced to a well-known model in population dynamics namely logistic growth model described as

$$N'(t) = -\alpha N(t) \left(1 - \frac{N(t)}{K} \right), \quad (1.2)$$

where $K = \frac{\alpha}{\beta}$ is a constant involving the environment capacity.

In the past few years, the qualitative theory for Nicholson model and its variants has been extensively studied and developed [1,2,9]. In particular, the problems associated with asymptotic behavior of positive periodic and almost periodic solutions of Nicholson-type models with delays were studied in [15,19,20,29]. Nicholson-type models with stochastic perturbations and harvesting terms were also investigated in [30] and [6,17,23,27], respectively. Very recently, in [3], the problems of stability and attractivity were studied for a class of n -dimensional Nicholson systems with constant coefficients and bounded time-varying delays. Based on some comparison techniques in the theory of monotone dynamical systems, delay-dependent sufficient conditions were derived for the existence and global exponential stability of a unique positive equilibrium.

Most of the existing works so far are devoted to Nicholson-type models with linear mortality terms. As discussed in [1], a model of linear density-dependent mortality rate will be most accurate for populations at low densities. According to marine ecologists, many models in fishery such as marine protected areas or models of B-cell chronic lymphocytic leukemia dynamics are suitably described by Nicholson-type delay differential equations with nonlinear density-dependent mortality rate of the form [1]

$$N'(t) = -D(N(t)) + \beta N(t - \tau) e^{-\gamma N(t - \tau)}, \quad (1.3)$$

where the function $D(N)$ might have one of the forms $D(N) = a - be^{-N}$ (type-I) or $D(N) = \frac{aN}{b+N}$ (type-II) with positive constants a and b . A natural extension of (1.3) to the case of variable coefficients and delays, which is more realistic in the theory of population dynamics [13,18], is given by

$$N'(t) = -D(t, N(t)) + \beta(t)N(t - \tau(t))e^{-\gamma(t)N(t - \tau(t))}, \quad (1.4)$$

where $D(t, N) = a(t) - b(t)e^{-N}$ or $D(t, N) = \frac{a(t)N}{b(t)+N}$. In model (1.4), $D(t, N)$ is the death rate of the population which depends on time t and the current population level $N(t)$, $B(t, N(t - \tau(t))) = \beta(t)N(t - \tau(t))e^{-\gamma(t)N(t - \tau(t))}$ is the time-dependent birth function which involves a maturation delay $\tau(t)$ and gets its maximum $\frac{\beta(t)}{\gamma(t)e}$ at rate $\frac{1}{\gamma(t)}$. Recently, Nicholson-type models with nonlinear density-dependent mortality terms have attracted considerable research attention. In particular, some results on the permanence property of certain types of Nicholson models with delays were established in [16,21]. The existence and exponential stability of positive periodic solutions, almost periodic solutions of various Nicholson-type models with both type-I and type-II nonlinear mortality terms were considered in [4,5,18] and [22,26,32,33], respectively. The author of [14] investigated the problem of global asymptotic stability of zero-equilibrium of the following special model

$$N'(t) = -a(t) + a(t)e^{-N(t)} + \sum_{j=1}^m \beta_j(t)N(t - \tau_j(t))e^{-\gamma_j(t)N(t - \tau_j(t))}. \quad (1.5)$$

It has been shown that under the restrictions

$$\sup_{t \geq 0} \gamma_j(t) \leq 1, \quad \sup_{t \geq 0} \sum_{j=1}^m \frac{\beta_j(t)}{a(t)\gamma_j(t)} < 1, \quad (1.6)$$

the equilibrium $N^* = 0$ of (1.5) is globally asymptotically stable with respect to an admissible phase space called \mathcal{C}_+ . In recent work [31], the effect of delay on the stability and attractivity was studied for the following model

$$N'(t) = -a + be^{-N(t)} + \sum_{j=1}^m \beta_j N(t - \tau_j(t)) e^{-\gamma_j N(t - \tau_j(t))}, \quad (1.7)$$

where a, b, β_j and γ_j are constants. Under the restrictions

$$\frac{1}{e} \sum_{j=1}^m \frac{\beta_j}{\gamma_j} < a, \quad \ln \frac{b}{a} > \frac{1}{\min_{1 \leq j \leq m} \gamma_j}, \quad (1.8)$$

which ensure the existence of at least one positive equilibrium \bar{N} , it was shown by utilizing a technique called fluctuation lemma [25, Lemma A.1] that such an equilibrium \bar{N} of (1.7) is globally attractive if the magnitude of delays satisfies the following condition

$$\max_{1 \leq j \leq m} \sup_{t \geq 0} \tau_j(t) \leq \frac{1}{a\gamma + \frac{1}{e} \sum_{j=1}^m \beta_j}, \quad \gamma = \max_{1 \leq j \leq m} \gamma_j. \quad (1.9)$$

Clearly, condition (1.9) can only be applied for models with small delays. This will restrict the applicability of the obtained results to practical models.

Motivated by the above literature review, in this paper we study the problem of existence and global attractivity of positive periodic solution of the following Nicholson model

$$N'(t) = -D(t, N(t)) + \sum_{k=1}^p \beta_k(t) N(t - \tau_k(t)) e^{-\gamma_k(t) N(t - \tau_k(t))}, \quad (1.10)$$

where $D(t, N) = a(t) - b(t)e^{-N}$. Main contributions and innovation points of this paper are three folds. First, improved conditions on global uniform permanence and dissipativity of time-varying Nicholson models with nonlinear density-dependent mortality term in the form of (1.10) are derived based on new comparison techniques via differential inequalities. We do not impose restrictions on the maximum reproducing rates $\frac{1}{\gamma_k}$ like (1.6), (1.8) and (1.9). Second, a novel approach to the problem of existence and global attractivity of a unique positive periodic solution of model (1.10) is presented. Third, as an application to Nicholson models with constant coefficients as (1.7), improved results on the existence, uniqueness and global attractivity of a positive equilibrium are obtained.

2 Preliminaries

For a given scalar $\omega > 0$, a function $f : \mathbb{R}^+ \rightarrow \mathbb{R}$ is said to be ω -periodic if $f(t + \omega) = f(t)$ for all $t \geq 0$. Let $\mathbb{P}_\omega(\mathbb{R}^+)$ denote the set of ω -periodic functions on \mathbb{R}^+ . Clearly, if $f : \mathbb{R}^+ \rightarrow \mathbb{R}$ is a continuous function and $f \in \mathbb{P}_\omega(\mathbb{R}^+)$ for some $\omega > 0$ then f is bounded on \mathbb{R}^+ . Hereafter, for a bounded function f on $[0, +\infty)$, we will denote

$$f^+ = \sup_{t \geq 0} f(t), \quad f^- = \inf_{t \geq 0} f(t).$$

Consider a Nicholson model with delays and nonlinear density-dependent mortality term of the form

$$N'(t) = -D(t, N(t)) + \sum_{k=1}^p \beta_k(t) N(t - \tau_k(t)) e^{-\gamma_k(t) N(t - \tau_k(t))}, \quad t \geq t_0, \quad (2.1)$$

$$N(t) = \varphi(t), \quad t \in [t_0 - \tau_M, t_0], \quad (2.2)$$

where the density-dependent mortality term $D(t, N)$ is of the form

$$D(t, N) = a(t) - b(t)e^{-N} \quad (2.3)$$

and $\tau_M = \max_{1 \leq k \leq p} \tau_k^+$ represents the upper bound of delays. For more detail on biological explanations of the coefficients of system (2.1)–(2.3), we refer the reader to [1, 3, 31].

Let $\mathcal{C} \triangleq \mathcal{C}([-\tau_M, 0], \mathbb{R})$ be the Banach space of continuous functions on $[-\tau_M, 0]$ endowed with the norm $\|\varphi\| = \sup_{t \in [-\tau_M, 0]} |\varphi(t)|$ and \mathcal{C}^+ be the cone of nonnegative functions in \mathcal{C} , that is,

$$\mathcal{C}^+ = \{\varphi \in \mathcal{C}([-\tau_M, 0], \mathbb{R}) : \varphi(t) \geq 0\}.$$

We write $\varphi \geq 0$ for $\varphi \in \mathcal{C}^+$. In addition, a function $\varphi \in \mathcal{C}$ is said to be positive, write as $\varphi > 0$, if $\varphi(t)$ is positive for all $t \in [-\tau_M, 0]$. Due to the biological interpretation, the set of admissible initial conditions in (2.2) is taken as

$$\mathcal{C}_0^+ = \{\varphi \in \mathcal{C}^+ : \varphi(0) > 0\}.$$

Let us first introduce the following assumptions and conditions.

Assumption (A):

(A1) $a, b, \gamma_k : [0, +\infty) \rightarrow (0, +\infty)$, $\beta_k : [0, +\infty) \rightarrow [0, +\infty)$ and $\tau_k : [0, +\infty) \rightarrow [0, \tau_M]$ are continuous bounded functions, where τ_M is some positive constant.

(A2) There exists an $\omega > 0$ such that the functions a, b, β_k, γ_k and τ_k belong to $\mathbb{P}_\omega(\mathbb{R}^+)$.

Condition (C):

(C1) a) $b(t) \geq a(t) \geq a^- > 0$; b) $\theta \triangleq \liminf_{t \rightarrow +\infty} \frac{b(t)}{a(t)} > 1$.

(C2) $\limsup_{t \rightarrow +\infty} \frac{1}{a(t)} \sum_{k=1}^p \frac{\beta_k(t)}{\gamma_k(t)} = \sigma$; $1 - \frac{\sigma}{e} > 0$.

(C3) $b^- - a^+ > 0$; $q \triangleq a^- - \frac{1}{e} \sum_{k=1}^p \frac{\beta_k^+}{\gamma_k} > 0$.

(C4) $\sum_{k=1}^p \beta_k^+ \max \left\{ \frac{1}{e^2}, \frac{1 - \gamma_k^- r_*}{e^{\gamma_k^- r_*}} \right\} < \frac{qb^-}{b^+}$; $r_* = \ln \left(\frac{b^-}{a^+} \right)$.

A preview of our main results is presented in the following table.

Conditions	Results
(A1), (C1)	Uniform permanence in \mathcal{C}_0^+ , $\liminf_{t \rightarrow +\infty} N(t, t_0, \varphi) \geq \ln(\theta)$.
(A1), (C1a), (C2)	Uniform dissipativity in \mathcal{C}_0^+ , $\limsup_{t \rightarrow +\infty} N(t, t_0, \varphi) \leq \ln \left(\frac{b^+}{a^-(1 - \frac{\sigma}{e})} \right)$.
(A1), (C3)	$\ln \left(\frac{b^-}{a^+} \right) \leq \liminf_{t \rightarrow +\infty} N(t, t_0, \varphi) \leq \limsup_{t \rightarrow +\infty} N(t, t_0, \varphi) \leq \ln \left(\frac{b^+}{q} \right)$.
(A1), (A2), (C3) and (C4)	There exists a unique positive ω -periodic solution $N^*(t)$ which is globally attractive in \mathcal{C}_0^+ .

For a biological interpretation of the proposed conditions, it is reasonable that when the population is absence the death rate is nonpositive (i.e. $D(t, 0) \leq 0$) and $D(t, N)$ is always positive when $N > 0$. This gives rise to condition (C1). On the other hand, in most of biological models, there typically exists a threshold related to the so-called carrying capacity. When the population size is very large, over the carrying capacity, the death rate can be bigger than the maximum birth rate. Similarly to model (1.4), the quantity $\sum_{k=1}^p \frac{\beta_k(t)}{\gamma_k(t)e}$ can be regarded

as the maximum birth rate of model (2.1). In addition, when N is large $D(t, N)$ is approximate to $a(t)$. By this observation, we make an assumption to ensure that $\sum_{k=1}^p \frac{\beta_k(t)}{\gamma_k(t)e} < a(t)$. This reveals the imposing of condition (C2) when considering long-time behavior of the model. (C3) is a testable condition derived from (C2) and (C1a) by taking into account the upper bound of the associated rates. While condition (C3) only guarantees non-extinction and non-blowup behavior, condition (C4) reveals that, by certain scaling coefficients, when maximum per capita daily egg production rates are smaller than the gap between the maximum death rate and birth rate (i.e. $\varrho = a^- - \frac{1}{e} \sum_{k=1}^p \frac{\beta_k^+}{\gamma_k^-}$), the population will be stable around a periodic trajectory (in the case of periodic coefficients) or a positive equilibrium (for time-invariant model).

In the remaining of this section, we present a local existence result of solutions of system (2.1)–(2.3). For $t > 0$, the function $N_t \in \mathcal{C}$ is defined as $N_t(\theta) = N(t + \theta)$, $\theta \in [-\tau_M, 0]$. Then, system (2.1)–(2.2) can be rewritten in the following abstract form

$$N'(t) = F(t, N_t), \quad t \geq t_0, \quad N_{t_0} = \varphi, \quad (2.4)$$

where the function $F : \mathbb{R}^+ \times \mathcal{C} \rightarrow \mathbb{R}$ is defined by

$$F(t, \varphi) = -D(t, \varphi(0)) + \sum_{k=1}^p \beta_k(t) \varphi(-\tau_k(t)) e^{-\gamma_k(t) \varphi(-\tau_k(t))}. \quad (2.5)$$

Proposition 2.1. *Under assumption (A1), for any $t_0 \geq 0$, $\varphi \in \mathcal{C}$, there exists a unique solution $N(t, t_0, \varphi)$ of system (2.1)–(2.2) defined on a maximal interval $[t_0, \eta(\varphi))$.*

Proof. Clearly, the function $F(\cdot, \cdot)$ defined in (2.5) is continuous and locally Lipschitz with respect to φ . Thus, the existence and uniqueness of a local solution of (2.1)–(2.2) is straightforward [8] and the proof is omitted here. \square

3 Permanence of global positive solutions

3.1 Global existence of positive solutions

Theorem 3.1. *Let assumption (A1) hold. Assume that $b(t) \geq a(t)$ for all $t \in [0, +\infty)$. Then, for any initial condition $\varphi \in \mathcal{C}_0^+$, the solution $N(t, t_0, \varphi)$ of system (2.1)–(2.3) is positive, $N(t, t_0, \varphi) > 0$, $t \in [t_0, \eta(\varphi))$, and $\eta(\varphi) = +\infty$.*

The following lemma will be used in the proof of Theorem 3.1.

Lemma 3.2. *Let $a(t), b(t) \geq 0$, $t \in [0, +\infty)$, be given continuous functions, the unique solution of the initial value problem (IVP)*

$$x' = -a(t) + b(t)e^{-x}, \quad t \geq t_0 \geq 0, \quad x(t_0) = x_0, \quad (3.1)$$

is given by

$$x(t, t_0, x_0) = - \int_{t_0}^t a(s) ds + \ln \left(e^{x_0} + \int_{t_0}^t e^{\int_{t_0}^s a(\theta) d\theta} b(s) ds \right). \quad (3.2)$$

Proof. Define $\hat{x} = e^x$ then (3.1) is written as

$$\hat{x}' = -a(t)\hat{x} + b(t), \quad t \geq t_0, \quad \hat{x}(t_0) = e^{x_0}. \quad (3.3)$$

Observe that (3.3) is an IVP of linear differential equations. Thus, from (3.3) we have

$$\hat{x}(t) = e^{-\int_{t_0}^t a(s)ds} \left(\hat{x}(t_0) + \int_{t_0}^t e^{\int_{t_0}^s a(\theta)d\theta} b(s)ds \right) > 0, \quad \forall t \geq t_0,$$

which leads to (3.2). The proof is completed. \square

Proof. (of Theorem 3.1) Clearly, $N(t_0, t_0, \varphi) = \varphi(0) > 0$. Thus, for sufficiently small $t - t_0 > 0$, $N(t, t_0, \varphi) > 0$. We first show that $N(t, t_0, \varphi)$ is positive on $[t_0, \eta(\varphi))$. If this does not hold, then there exists a $t_1 > t_0$ such that $N(t_1, t_0, \varphi) = 0$ and $N(t, t_0, \varphi) > 0$ for all $t \in [t_0, t_1)$. In regard to initial condition $\varphi \in C_0^+$, we have $N(t, t_0, \varphi) \geq 0$, $t \in [t_0 - \tau_M, t_1]$. Thus, it follows from (2.1) that

$$N'(t) \geq -D(t, N(t)), \quad t \in [t_0, t_1]. \quad (3.4)$$

By the comparison principle, and by utilizing Lemma 3.2, from (3.4) we obtain

$$\begin{aligned} N(t, t_0, \varphi) &\geq -\int_{t_0}^t a(s)ds + \ln \left(e^{\varphi(0)} + \int_{t_0}^t e^{\int_{t_0}^s a(\theta)d\theta} b(s)ds \right) \\ &\geq -\int_{t_0}^t a(s)ds + \ln \left(e^{\varphi(0)} - 1 + e^{\int_{t_0}^t a(s)ds} \right). \end{aligned} \quad (3.5)$$

Let $t \uparrow t_1$, estimation (3.5) implies that

$$0 = N(t_1, t_0, \varphi) \geq \ln \left(1 + (e^{\varphi(0)} - 1)e^{-\int_{t_0}^{t_1} a(s)ds} \right) > 0,$$

which is obviously a contradiction. Therefore, $N(t, t_0, \varphi) > 0$ for all $t \in [t_0, \eta(\varphi))$.

Next, by using the fact $\sup_{x \geq 0} xe^{-\gamma x} = \frac{1}{\gamma e}$ for any $\gamma > 0$, it follows from (2.1) that

$$N'(t) \leq b(t)e^{-N(t)} - a(t) + \frac{1}{e} \sum_{k=1}^p \frac{\beta_k(t)}{\gamma_k(t)}, \quad t \geq t_0. \quad (3.6)$$

If $\eta(\varphi) < +\infty$ then $\lim_{t \uparrow \eta(\varphi)} N(t, t_0, \varphi) = +\infty$ [8, Theorem 3.2]. On the other hand, similarly to (3.2), from (3.6) we have

$$N(t, t_0, \varphi) \leq -\int_{t_0}^t \psi(s)ds + \ln \left(e^{\varphi(0)} + \int_{t_0}^t e^{\int_{t_0}^s \psi(\theta)d\theta} b(s)ds \right), \quad t \in [t_0, \eta(\varphi)), \quad (3.7)$$

where

$$\psi(t) = a(t) - \frac{1}{e} \sum_{k=1}^p \frac{\beta_k(t)}{\gamma_k(t)}.$$

Thus, $\lim_{t \uparrow \eta(\varphi)} N(t, t_0, \varphi) < +\infty$ as $\eta(\varphi) < +\infty$. This contradiction shows that $\eta(\varphi) = +\infty$. The proof is completed. \square

Remark 3.3. In general, the estimation (3.7) does not guarantee the boundedness of $N(t, t_0, \varphi)$ on interval $[0, +\infty)$. In order to get the boundedness of $N(t, t_0, \varphi)$, additional conditions should be imposed. For example, if there exists a positive constant M such that

$$\sup_{t \geq 0} \left\{ b(t)e^{-M} - \psi(t) \right\} < 0,$$

where $\psi(t)$ is defined as in (3.7), then $N(t, t_0, \varphi) < M$, $t \in [t_\varphi, +\infty)$, for some $t_\varphi > 0$ [5, 18]. For other types of conditions, we refer the reader to Theorem 3.7 in this paper.

Remark 3.4. To ensure the positiveness of solutions of (2.1)–(2.3) with initial conditions in C_0^+ , condition $b(t) \geq a(t)$ cannot be relaxed. For a counterexample, let $n = 1$ and assume that

$$\sup_{t \geq 0} \frac{b(t)}{a(t)} = \delta \in [0, 1), \quad \int_0^t a(s) ds \rightarrow +\infty, \quad t \rightarrow +\infty.$$

Then, it follows from (3.2) that

$$N(t, t_0, \varphi) \leq \ln \left(e^{\varphi(0) - \int_{t_0}^t a(s) ds} + \delta \left(1 - e^{-\int_{t_0}^t a(s) ds} \right) \right) \rightarrow \ln(\delta) < 0 \quad \text{as } t \rightarrow +\infty.$$

3.2 Uniform permanence

Theorem 3.5. Let assumption (A1) hold. Assume that $b(t) \geq a(t) \geq a^- > 0$ and

$$\liminf_{t \rightarrow +\infty} \frac{b(t)}{a(t)} \geq e^{\ell_m} > 1. \quad (3.8)$$

Then, for any $\varphi \in C_0^+$,

$$\liminf_{t \rightarrow +\infty} N(t, t_0, \varphi) \geq \ell_m > 0.$$

Proof. By Theorem 3.1, $N(t, t_0, \varphi) > 0$ for all $t \in [t_0, +\infty)$. On the other hand, for a sufficiently small $\epsilon > 0$, by (3.8), there exists a $T > t_0$ such that $b(t) \geq (e^{\ell_m} - \epsilon)a(t)$ for all $t \geq T$. Similarly to (3.5), we have

$$N(t, t_0, \varphi) \geq \ln \left(e^{N(T, t_0, \varphi) - \int_T^t a(s) ds} + (e^{\ell_m} - \epsilon) \left(1 - e^{-\int_T^t a(s) ds} \right) \right).$$

Note also that

$$\int_T^t a(s) ds \rightarrow +\infty \quad \text{as } t \rightarrow +\infty.$$

Thus, let $t \rightarrow +\infty$ and $\epsilon \downarrow 0$, we obtain

$$\liminf_{t \rightarrow +\infty} N(t, t_0, \varphi) \geq \ell_m.$$

The proof is completed. □

Remark 3.6. As a special case of (3.8), for bounded functions $a(t)$ and $b(t)$, if $b^- > a^+$ then the scalar ℓ_m in (3.8) can be chosen as

$$\ell_m = \ln \left(\frac{b^-}{a^+} \right).$$

Thus, Theorem 3.5 in this paper encompasses the result of Lemma 1 in [31].

The following result shows the uniform dissipativity of system (2.1)–(2.3) in C_0^+ [3] in the sense that there exists a constant $\ell_M > 0$ such that $\limsup_{t \rightarrow +\infty} N(t, t_0, \varphi) \leq \ell_M$.

Theorem 3.7. Assume assumption (A1) and the following conditions hold

$$b^+ \geq b(t) \geq a(t) \geq a^- > 0, \quad t \in [0, +\infty), \quad (3.9)$$

$$\limsup_{t \rightarrow +\infty} \frac{1}{a(t)} \sum_{k=1}^p \frac{\beta_k(t)}{\gamma_k(t)} = \sigma, \quad 1 - \frac{\sigma}{e} > 0. \quad (3.10)$$

Then, system (2.1)–(2.3) is uniformly dissipative in \mathcal{C}_0^+ . More precisely, for any initial condition $\varphi \in \mathcal{C}_0^+$, the corresponding solution $N(t, t_0, \varphi)$ of (2.1)–(2.3) satisfies

$$\limsup_{t \rightarrow +\infty} N(t, t_0, \varphi) \leq \ell_M \triangleq \ln \left(\frac{b^+}{a^-(1 - \frac{\sigma}{e})} \right).$$

Proof. By similar lines used in the proof of Theorem 3.1, from (3.7), we have

$$\begin{aligned} N(t, t_0, \varphi) &\leq - \int_{t_0}^t \psi(s) ds + \ln \left(e^{\varphi(0)} + b^+ \int_{t_0}^t e^{\int_{t_0}^s \psi(\theta) d\theta} ds \right) \\ &= \ln \left(e^{\varphi(0)} e^{-\int_{t_0}^t \psi(s) ds} + b^+ e^{-\int_{t_0}^t \psi(s) ds} \int_{t_0}^t e^{\int_{t_0}^s \psi(\theta) d\theta} ds \right), \quad t \in [t_0, +\infty). \end{aligned} \quad (3.11)$$

On the other hand, by (3.10), there exists a $T > 0$ such that

$$1 - \frac{1}{e} \sum_{k=1}^p \frac{\beta_k(t)}{a(t)\gamma_k(t)} \geq 1 - \frac{\sigma}{e} > 0, \quad t \geq T.$$

Therefore,

$$\begin{aligned} \psi(t) &= a(t) \left(1 - \frac{1}{e} \sum_{k=1}^p \frac{\beta_k(t)}{a(t)\gamma_k(t)} \right) \\ &\geq a^-(1 - \frac{\sigma}{e}) > 0, \quad t \geq T. \end{aligned}$$

By this, $\int_{t_0}^t \psi(s) ds \rightarrow +\infty$ as $t \rightarrow +\infty$, and thus

$$\limsup_{t \rightarrow +\infty} e^{-\int_{t_0}^t \psi(s) ds} \int_{t_0}^t e^{\int_{t_0}^s \psi(\theta) d\theta} ds = \limsup_{t \rightarrow \infty} \frac{1}{\psi(t)} \leq \frac{1}{a^-(1 - \frac{\sigma}{e})}.$$

Let $t \rightarrow +\infty$, from (3.11) we obtain $\limsup_{t \rightarrow +\infty} N(t, t_0, \varphi) \leq \ell_M$. The proof is completed. \square

The following result is obtained as a consequence of Theorems 3.5 and 3.7.

Corollary 3.8. *Let assumption (A1) hold, where a, b, β_k and γ_k are bounded functions, $\gamma_k^- > 0$. Assume that*

$$\varrho \triangleq a^- - \frac{1}{e} \sum_{k=1}^p \frac{\beta_k^+}{\gamma_k^-} > 0, \quad (3.12a)$$

$$b^- - a^+ > 0. \quad (3.12b)$$

Then, for any $\varphi \in \mathcal{C}_0^+$, it holds that

$$\ln \left(\frac{b^-}{a^+} \right) \leq \liminf_{t \rightarrow +\infty} N(t, t_0, \varphi) \leq \limsup_{t \rightarrow +\infty} N(t, t_0, \varphi) \leq \ln \left(\frac{b^+}{\varrho} \right). \quad (3.13)$$

Remark 3.9. For bounded coefficients a, b, β_k, γ_k , it follows from (2.5) that the function $F(t, \varphi)$ maps any bounded set $\mathcal{B} \subset \mathcal{C}$ into a bounded set $F(t, \mathcal{B})$ in \mathbb{R} . Thus, by the assumptions of Corollary 3.8, for any $\varphi \in \mathcal{C}_0^+$, $F(t, N_t)$ is bounded. Consequently, the corresponding solution $N(t, t_0, \varphi)$ is uniformly continuous on $[0, +\infty)$.

4 Global attractivity of positive periodic solution

The following lemmas will be used in the proof of our results in this section.

Lemma 4.1 ([10]). *Let $u : \mathbb{R}^+ \rightarrow \mathbb{R}^n$ be a uniformly continuous function. If the limit $\lim_{t \rightarrow +\infty} \int_0^t u(s) ds$ exists and is finite, then $\lim_{t \rightarrow +\infty} \|u(t)\| = 0$.*

Proof. A detailed proof was presented in [10]. Let us omit it here. \square

Lemma 4.2 ([10]). *Let $u : \mathbb{R}^+ \rightarrow \mathbb{R}^n$ be a bounded uniformly continuous function. If there exists an $\omega > 0$ such that*

$$\int_0^{+\infty} \|u(t + \omega) - u(t)\| dt < +\infty,$$

then there exists a continuous ω -periodic function $u^(t)$ satisfying $\lim_{t \rightarrow +\infty} \|u(t) - u^*(t)\| = 0$.*

This lemma was stated in [10]. To make it easier to follow, in this paper, we will also reconduct the following proof.

Proof. Since u is bounded, there exists a constant $u_\infty > 0$ satisfying $\|u(t)\| \leq u_\infty$ for all $t \geq 0$. Besides that for a given $\epsilon > 0$ there exists a $\delta = \delta(\epsilon) > 0$ such that $\|u(t_1) - u(t_2)\| < \epsilon$ whenever $|t_1 - t_2| < \delta$ due to the uniform continuity of u . We now defined a sequence of functions $u_k : \mathbb{R}^+ \rightarrow \mathbb{R}^n$, $k \in \mathbb{N}_0$, by $u_k(t) = u(t + k\omega)$. Then, we have $\|u_k(t)\| = \|u(t + k\omega)\| \leq u_\infty$ for all $t \geq 0$, $k \in \mathbb{N}_0$, which shows the uniform boundedness of the sequence $\{u_k\}$. On the other hand,

$$\|u_k(t_1) - u_k(t_2)\| = \|u(t_1 + k\omega) - u(t_2 + k\omega)\| < \epsilon, \quad \forall k \in \mathbb{N}_0,$$

whenever $|t_1 - t_2| < \delta$. Thus, the sequence $\{u_k\}$ is uniformly equicontinuous. By the Arzelà-Ascoli Theorem, there exists a subsequence $\{u_{k_p}\}$ that converges uniformly on the interval $[0, \omega]$ to a continuous function denoted as $u^*(t)$. Hence, for a given $\epsilon > 0$,

$$\|u(t + k_{p_\epsilon}\omega) - u^*(t)\| < \frac{\epsilon}{2\omega}, \quad \forall t \in [0, \omega], p \geq p_\epsilon,$$

for some $p_\epsilon \in \mathbb{N}$, which yields

$$\int_0^\omega \|u(t + k_{p_\epsilon}\omega) - u^*(t)\| dt < \frac{\epsilon}{2}.$$

On the other hand, it follows from the assumption that

$$\sum_{n=0}^{\infty} \int_0^\omega \|u(t + (n+1)\omega) - u(t + n\omega)\| dt = \int_0^\infty \|u(t + \omega) - u(t)\| dt < \infty.$$

This ensures $\sum_{n=k}^{\infty} \int_0^\omega \|u_n(t + \omega) - u_n(t)\| dt \rightarrow 0$ as $k \rightarrow \infty$. Thus, we can assume without lost of generality that

$$\sum_{n=k_{p_\epsilon}}^{\infty} \int_0^\omega \|u_n(t + \omega) - u_n(t)\| dt < \frac{\epsilon}{2}.$$

Now, for any $k > k_{p_\epsilon}$, we have

$$\begin{aligned} \int_0^\omega \|u(t + k\omega) - u^*(t)\| dt &\leq \int_0^\omega \|u(t + k\omega) - u(t + k_{p_\epsilon}\omega)\| dt + \int_0^\omega \|u(t + k_{p_\epsilon}\omega) - u^*(t)\| dt \\ &\leq \sum_{n=k_{p_\epsilon}}^{k-1} \int_0^\omega \|u_n(t + \omega) - u_n(t)\| dt + \frac{\epsilon}{2} \\ &< \epsilon. \end{aligned}$$

This shows that $\int_0^\omega \|u(t+k\omega) - u^*(t)\| dt \rightarrow 0$ as $k \rightarrow \infty$. Then, it can be verified by similar arguments used in [10] that $u_k \rightrightarrows u^*$ on $[0, \omega]$ as $k \rightarrow \infty$. It is clear that for any $t \geq 0$ there exist a unique $k \in \mathbb{N}_0$ and $s \in [0, \omega)$ such that $t = k\omega + s$. We now span the function u^* by defining $u^*(t) = u^*(s)$ then u^* is a continuous and ω -periodic function on $[0, \infty)$ satisfying $\|u(t) - u^*(t)\| = \|u_k(s) - u^*(s)\| \rightarrow 0$ as $t = k\omega + s \rightarrow \infty$. The proof is completed. \square

Remark 4.3. For positive scalars $\theta_1 \leq \theta_2$ and γ , we have

$$\max_{\theta_1 \leq x \leq \theta_2} |1 - \gamma x| e^{-\gamma x} \leq \max \left\{ \frac{1}{e^2}, (1 - \gamma \theta_1) e^{-\gamma \theta_1} \right\}.$$

In our latter derivations, γ is typically time-varying with a lower bound namely $\gamma \geq \gamma^- > 0$. Then, for a given $\theta_1 > 0$, $\max \left\{ \frac{1}{e^2}, (1 - \gamma \theta_1) e^{-\gamma \theta_1} \right\} = \frac{1}{e^2}$ whenever $\gamma \theta_1 \geq 1$. In addition, the function $(1 - \gamma \theta_1) e^{-\gamma \theta_1}$ is decreasing in $\gamma \in (0, \frac{1}{\theta_1})$. Thus, the estimation

$$|1 - \gamma x| e^{-\gamma x} \leq \max \left\{ \frac{1}{e^2}, \frac{1 - \gamma^- \theta_1}{e^{\gamma^- \theta_1}} \right\} \quad (4.1)$$

holds for all $x \in [\theta_1, \theta_2]$ and $\gamma \geq \gamma^- > 0$.

Let f be a differentiable function. Similarly to [28], we define a generalized sign-function σ_f as follows

$$\sigma_f(t) = \begin{cases} 1 & \text{if } [f(t) > 0] \vee [f(t) = 0 \wedge f'(t) > 0], \\ 0 & \text{if } [f(t) = 0 \wedge f'(t) = 0], \\ -1 & \text{if } [f(t) < 0] \vee [f(t) = 0 \wedge f'(t) < 0]. \end{cases}$$

Then, it is clear that $|f(t)| = f(t)\sigma_f(t)$. Moreover, by similar lines used in the proof of Lemma 3.1 in [28], we have the following lemma.

Lemma 4.4. For a differentiable function f , it holds that

$$D^+ |f(t)| \triangleq \limsup_{h \rightarrow 0^+} \frac{|f(t+h)| - |f(t)|}{h} = f'(t)\sigma_f(t),$$

where D^+ is the upper-right Dini derivative.

In the following, we assume that assumptions (A1), (A2) and conditions (3.12a)–(3.12b) are satisfied. For convenience, we denote

$$r_* = \ln \left(\frac{b^-}{a^+} \right), \quad r^* = \ln \left(\frac{b^+}{\varrho} \right), \quad v_k = \max \left\{ \frac{1}{e^2}, \frac{1 - \gamma_k^- r_*}{e^{\gamma_k^- r_*}} \right\}.$$

Note that, by (3.12b), $\frac{b^-}{a^+} > 1$ and hence $r_* > 0$. In addition, since the condition $r_* < \frac{1}{\max \gamma_k^-}$ is not imposed, $1 - \gamma_k^- r_*$ can be positive, negative or zero. For $1 \leq k \leq p$ that $1 - \gamma_k^- r_* \leq 0$, $v_k = \frac{1}{e^2}$.

We are now in a position to present the existence, uniqueness and global attractivity of a positive periodic solution of system (2.1)–(2.3) as in the following theorem.

Theorem 4.5. *Let assumptions (A1), (A2), conditions (3.12a), (3.12b) and the following ones are satisfied*

$$\inf_{t \geq 0} \{1 - \tau_k'(t)\} = \mu > 0, \quad (4.2)$$

$$\sum_{k=1}^p v_k \beta_k^+ < \mu \frac{qb^-}{b^+}, \quad (4.3)$$

where q is the constant defined in (3.12a). Then, system (2.1)–(2.3) has a unique positive ω -periodic solution $N^*(t)$ which is globally attractive in \mathcal{C}_0^+ .

Proof. We divide the proof of Theorem 4.5 into the following three steps.

Step 1. Since condition (4.3) can be written as $\sum_{k=1}^p v_k \beta_k^+ - \mu e^{-r^*} b^- < 0$, for a given $\epsilon > 0$ such that $r_* - \epsilon > 0$, we have

$$\sum_{k=1}^p v_k^\epsilon \beta_k^+ - \mu e^{-(r_* + \epsilon)} b^- < 0,$$

where

$$v_k^\epsilon = \max \left\{ \frac{1}{e^2}, \frac{1 - \gamma_k^-(r_* - \epsilon)}{e^{\gamma_k^-(r_* - \epsilon)}} \right\}.$$

Let $N(t) = N(t, t_0, \varphi)$ be a solution of system (2.1)–(2.3) with initial condition $\varphi \in \mathcal{C}_0^+$. Then, by Corollary 3.8, there exists a $T > t_0$ such that

$$r_* - \epsilon \leq N(t) \leq r^* + \epsilon, \quad \forall t \geq T - \tau_M.$$

We define the function $N_\omega(t) = N(t + \omega) - N(t)$ and consider the following Lyapunov-like functional

$$V(t) = \underbrace{|N_\omega(t)|}_{V_1(t)} + \sum_{k=1}^p \frac{v_k^\epsilon \beta_k^+}{\mu} \int_{t - \tau_k(t)}^t |N_\omega(s)| ds. \quad (4.4)$$

By virtue of the periodicity of τ_k , γ_k and other coefficients, and by utilizing Lemma 4.4, the upper-right Dini derivative of $V_1(t)$ is computed and estimated as follows

$$\begin{aligned} D^+ V_1(t) &= \sigma_{N_\omega}(t) (N'(t + \omega) - N'(t)) \\ &= \sigma_{N_\omega}(t) \left\{ b(t) \left(e^{-N(t+\omega)} - e^{-N(t)} \right) \right. \\ &\quad \left. + \sum_{k=1}^p \beta_k(t) \left[N(t + \omega - \tau_k(t + \omega)) e^{-\gamma_k(t+\omega)N(t+\omega - \tau_k(t+\omega))} \right. \right. \\ &\quad \left. \left. - N(t - \tau_k(t)) e^{-\gamma_k(t)N(t - \tau_k(t))} \right] \right\} \\ &\leq \sigma_{N_\omega}(t) b(t) \left(e^{-N(t+\omega)} - e^{-N(t)} \right) \\ &\quad + \sum_{k=1}^p \beta_k(t) \left| N(t + \omega - \tau_k(t)) e^{-\gamma_k(t)N(t+\omega - \tau_k(t))} - N(t - \tau_k(t)) e^{-\gamma_k(t)N(t - \tau_k(t))} \right|. \quad (4.5) \end{aligned}$$

By mean-value theorem,

$$b(t) \sigma_{N_\omega}(t) \left(e^{-N(t+\omega)} - e^{-N(t)} \right) = -b(t) \sigma_{N_\omega}(t) N_\omega(t) e^{-\zeta(t)},$$

where $\zeta(t)$ is some value between $N(t)$ and $N(t + \omega)$. Therefore,

$$-b(t)\sigma_{N_\omega}(t) \left(e^{-N(t+\omega)} - e^{-N(t)} \right) \leq -b^- e^{-(r^*+\epsilon)} |N_\omega(t)|, \quad \forall t \geq T. \quad (4.6)$$

In regard to (4.1), we also have

$$\left| x e^{-\gamma_k(t)x} - y e^{-\gamma_k(t)y} \right| \leq v_k^\epsilon |x - y|, \quad t \geq 0, x, y \in [r_* - \epsilon, r^* + \epsilon].$$

The above estimation gives

$$\left| N(t + \omega - \tau_k(t)) e^{-\gamma_k(t)N(t+\omega-\tau_k(t))} - N(t - \tau_k(t)) e^{-\gamma_k(t)N(t-\tau_k(t))} \right| \leq v_k^\epsilon |N_\omega(t - \tau_k(t))|. \quad (4.7)$$

Combining (4.5)–(4.7) we then obtain

$$D^+ V_1(t) \leq -b^- e^{-(r^*+\epsilon)} |N_\omega(t)| + \sum_{k=1}^p \beta_k^+ v_k^\epsilon |N_\omega(t - \tau_k(t))|. \quad (4.8)$$

Similarly to (4.5), for

$$V_2(t) = \sum_{k=1}^p \frac{v_k^\epsilon \beta_k^+}{\mu} \int_{t-\tau_k(t)}^t |N_\omega(s)| ds,$$

we have

$$\begin{aligned} D^+ V_2(t) &= \sum_{k=1}^p \frac{v_k^\epsilon \beta_k^+}{\mu} (|N_\omega(t)| - (1 - \tau_k'(t)) |N_\omega(t - \tau_k(t))|) \\ &\leq \sum_{k=1}^p v_k^\epsilon \beta_k^+ \left(\frac{1}{\mu} |N_\omega(t)| - |N_\omega(t - \tau_k(t))| \right). \end{aligned} \quad (4.9)$$

It follows from (4.8) and (4.9) that

$$D^+ V(t) \leq \underbrace{\left[\sum_{k=1}^p v_k^\epsilon \frac{\beta_k^+}{\mu} - b^- e^{-(r^*+\epsilon)} \right]}_{-\rho} |N_\omega(t)|. \quad (4.10)$$

Since $\rho > 0$, it follows from (4.10) that

$$\int_T^{+\infty} |N_\omega(t)| dt \leq \frac{V(T)}{\rho} < +\infty.$$

Note also that $|N'(t)|$ is bounded since the right-hand side of (2.1) is bounded. Consequently, $N(t)$ is a uniformly continuous function. By Lemma 4.2, there exists an ω -periodic function $N^*(t)$ such that $|N(t) - N^*(t)| \rightarrow 0$ as $t \rightarrow +\infty$.

Step 2. It follows from (2.4) that

$$N(t) = N(0) + \int_0^t F(\lambda, N_\lambda) d\lambda.$$

Thus, for any $n \in \mathbb{N}$,

$$\begin{aligned} N(t + n\omega) &= N(0) + \int_0^{t+n\omega} F(\lambda, N_\lambda) d\lambda \\ &= N(n\omega) + \int_{n\omega}^{t+n\omega} F(\lambda, N_\lambda) d\lambda \\ &= N(n\omega) + \int_0^t F(\lambda, N_{\lambda+n\omega}) d\lambda \end{aligned} \quad (4.11)$$

due to the periodicity of the functions a, b, β_k and γ_k . Since $N^*(t)$ is an ω -periodic function, when n tends to infinity, we have

$$\begin{aligned} N(n\omega) &= N(n\omega) - N^*(n\omega) + N^*(0) \rightarrow N^*(0), \\ N_{\lambda+n\omega} &= N_{\lambda+n\omega} - N_{\lambda+n\omega}^* + N_{\lambda}^* \rightarrow N_{\lambda}^*. \end{aligned}$$

Let $n \rightarrow +\infty$ in (4.11) we obtain

$$N^*(t) = N^*(0) + \int_0^t F(\lambda, N_{\lambda}^*) d\lambda, \quad t \in [0, \omega].$$

This means that $N^*(t)$ is an ω -periodic solution of (2.1), which is also a positive solution of (2.1) according to Corollary 3.8.

Step 3. We now prove that such an ω -periodic solution $N^*(t)$ of (2.1) is unique. Assume in contrary that $\hat{N}^*(t)$ is also an ω -periodic solution of (2.1). Consider the following functional

$$W(t) = |N^*(t) - \hat{N}^*(t)| + \sum_{k=1}^p \frac{v_k^\epsilon \beta_k^+}{\mu} \int_{t-\tau_k(t)}^t |N^*(s) - \hat{N}^*(s)| ds. \quad (4.12)$$

Similarly to (4.10), we have

$$\begin{aligned} D^+W(t) &\leq -b^- e^{-(r+\epsilon)} |N^*(t) - \hat{N}^*(t)| + \sum_{k=1}^p v_k^\epsilon \frac{\beta_k^+}{\mu} |N^*(t) - \hat{N}^*(t)| \\ &\leq -\rho |N^*(t) - \hat{N}^*(t)|, \quad t \geq T. \end{aligned}$$

This leads to

$$\int_0^{+\infty} |\hat{N}^*(t) - N^*(t)| dt < +\infty.$$

Since $N^*(t)$ and $\hat{N}^*(t)$ are uniformly continuous, which is deduced from Corollary 3.8, by Lemma 4.1, $\lim_{t \rightarrow +\infty} |\hat{N}^*(t) - N^*(t)| = 0$. Thus, for any $\delta > 0$ there exists a $T_* > 0$ such that $|\hat{N}^*(t) - N^*(t)| < \delta$ for all $t \geq T_*$. For any $t \geq 0$, with $n \in \mathbb{N}$ satisfying $t + n\omega > T_*$, we have

$$|\hat{N}^*(t) - N^*(t)| = |\hat{N}^*(t + n\omega) - N^*(t + n\omega)| < \delta. \quad (4.13)$$

Let $\delta \downarrow 0$ in (4.13), we obtain $\hat{N}^*(t) = N^*(t)$.

Finally, for any solution $N(t, t_0, \varphi)$ of (2.1)–(2.3), it can be deduced from the arguments used in Steps 1 and 3 that $|N(t, t_0, \varphi) - N^*(t)| \rightarrow 0$ as $t \rightarrow +\infty$. This shows the global attractivity of $N^*(t)$. The proof is completed. \square

Remark 4.6. Conditions (4.2) and (4.3) are involved a scalar $\mu > 0$ related to the rate of change of delay functions $\tau_k(t)$. However, this scalar can be relaxed and conditions (4.2), (4.3) are reduced to the following one

$$\sum_{k=1}^p v_k \beta_k^+ < \frac{\rho b^-}{b^+}. \quad (4.14)$$

More precisely, we state that in the following theorem.

Theorem 4.7. *Under assumptions (A1) and (A2), assume that conditions (3.12a), (3.12b) and (4.14) are satisfied. Then, system (2.1)–(2.3) has a unique positive ω -periodic solution $N^*(t)$ which is globally attractive in C_0^+ .*

Proof. The proof is similar to that of Theorem 4.5. Specifically, let $\tilde{V}(t) = |N(t + \omega) - N(t)|$, then, similarly to (4.8), we have

$$\begin{aligned} D^+ \tilde{V}(t) &\leq -b^- e^{-(r^* + \epsilon)} |N_\omega(t)| + \sum_{k=1}^p \beta_k^+ v_k^\epsilon |N_\omega(t - \tau_k(t))| \\ &\leq -b^- e^{-(r^* + \epsilon)} \tilde{V}(t) + \sum_{k=1}^p \beta_k^+ v_k^\epsilon \sup_{-\tau_M \leq s \leq 0} \tilde{V}(t + s). \end{aligned} \quad (4.15)$$

Since $0 \leq \sum_{k=1}^p \beta_k^+ v_k^\epsilon < b^- e^{-(r^* + \epsilon)}$, by Halanay inequality [11, Corollary 3.1], there exists a $\lambda > 0$ such that

$$\tilde{V}(t) \leq \tilde{V}(T) e^{-\lambda(t-T)}, \quad t \geq T. \quad (4.16)$$

It follows from (4.16) that

$$\int_T^{+\infty} |N_\omega(t)| dt \leq \frac{\tilde{V}(T)}{\lambda}$$

which yields

$$\int_0^{+\infty} |N_\omega(t)| dt < +\infty.$$

The remainder of the proof can be proceeded similarly to that of Theorem 4.5. \square

Remark 4.8. Based on the coincidence degree theory, it can be shown using the method of [4, Theorem 2.1] that model (2.1) has a unique positive periodic solution $N^*(t)$ if (A1), (A2) and the conditions $\frac{b^-}{a^+} > 0$ and $a^- > \frac{1}{e} \sum_{k=1}^p \frac{\beta_k^+}{\gamma_k}$ hold. The latter conditions are clearly same as those in (3.12). However, the method of [4] could not show the attractivity of $N^*(t)$. Thus, Theorems 4.5, 4.7 in this paper complement the result of [4].

5 Attractivity of positive equilibrium

In this section, we apply our results presented in the preceding sections to the following Nicholson model

$$N'(t) = -D(N(t)) + \sum_{k=1}^p \beta_k N(t - \tau_k(t)) e^{-\gamma_k N(t - \tau_k(t))}, \quad t \geq t_0 \geq 0, \quad (5.1)$$

where $\beta_k \geq 0$, $\gamma_k > 0$ are known coefficients, $\sum_{k=1}^p \beta_k > 0$. The nonlinear density-dependent mortality term is given by $D(N) = a - b e^{-N}$, $a > 0$, $b > 0$. Time-varying delays $\tau_k(t)$ are continuous and bounded in the range $[0, \tau_M]$.

For model (5.1), conditions (3.12a), (3.12b) are reduced to the following coupled condition

$$\frac{1}{e} \sum_{k=1}^p \frac{\beta_k}{\gamma_k} < a < b. \quad (5.2)$$

Proposition 5.1. *Let condition (5.2) hold. Then, for any $\varphi \in C_0^+$, it holds that*

$$\ln \left(\frac{b}{a} \right) \leq \liminf_{t \rightarrow +\infty} N(t, t_0, \varphi) \leq \limsup_{t \rightarrow +\infty} N(t, t_0, \varphi) \leq \ln \left(\frac{b}{a - \frac{1}{e} \sum_{k=1}^p \frac{\beta_k}{\gamma_k}} \right). \quad (5.3)$$

Proof. The proof is deduced from (3.13). \square

Theorem 5.2. Assume that

$$\sum_{k=1}^q \beta_k \left(\hat{v}_k + \frac{1}{e\gamma_k} \right) < a < b, \quad (5.4)$$

where

$$\hat{v}_k = \max \left\{ \frac{1}{e^2}, \frac{1 - \gamma_k \ln(\frac{b}{a})}{e\gamma_k \ln(\frac{b}{a})} \right\}.$$

Then, model (5.1) has a unique positive equilibrium N^* which is globally attractive in C_0^+ .

Proof. An equilibrium of (5.1) is a solution of the following nonlinear equation

$$\Phi(N) \triangleq -D(N) + \sum_{k=1}^p \beta_k N e^{-\gamma_k N} = 0. \quad (5.5)$$

According to (5.3), any equilibrium point of (5.1) should be within the range $[\theta_1, \theta_2]$, where

$$\theta_1 = \ln \left(\frac{b}{a} \right), \quad \theta_2 = \ln \left(\frac{b}{\hat{\varrho}} \right) \quad \text{and} \quad \hat{\varrho} = a - \frac{1}{e} \sum_{k=1}^p \frac{\beta_k}{\gamma_k}.$$

Since $\Phi(N)$ is continuous on $[\theta_1, \theta_2]$, $\Phi(\theta_1) = \sum_{k=1}^p \beta_k \theta_1 e^{-\gamma_k \theta_1} > 0$ and

$$\Phi(\theta_2) = \sum_{k=1}^p \beta_k \left(\theta_2 e^{-\gamma_k \theta_2} - \frac{1}{e\gamma_k} \right) < 0,$$

there exists an $N^* \in (\theta_1, \theta_2)$ such that $\Phi(N^*) = 0$.

For any $N_1, N_2 \in (\theta_1, \theta_2)$, we have

$$\begin{aligned} \Phi(N_1) - \Phi(N_2) &= b \left(e^{-N_1} - e^{-N_2} \right) + \sum_{k=1}^p \beta_k \left(N_1 e^{-\gamma_k N_1} - N_2 e^{-\gamma_k N_2} \right) \\ &= \left[-b e^{-\xi_1} + \sum_{k=1}^p \beta_k e^{-\gamma_k \xi_2} (1 - \gamma_k \xi_2) \right] (N_1 - N_2), \end{aligned}$$

where ξ_1, ξ_2 are mean values between N_1 and N_2 . Note that

$$-b e^{-\xi_1} + \sum_{k=1}^p \beta_k e^{-\gamma_k \xi_2} (1 - \gamma_k \xi_2) < -b e^{-\ln(\frac{b}{\hat{\varrho}})} + \sum_{k=1}^p \hat{v}_k \beta_k = -a + \sum_{k=1}^p \beta_k \left(\hat{v}_k + \frac{1}{e\gamma_k} \right) < 0.$$

Thus, $\Phi(N_1) = \Phi(N_2)$ if and only if $N_1 = N_2$. This shows that the equilibrium N^* is unique. In addition, by similar arguments used in the proof of Theorem 4.7, it can be concluded that

$$\lim_{t \rightarrow +\infty} |N(t, t_0, \varphi) - N^*| = 0$$

for any $\varphi \in C_0^+$, which shows the global attractivity of N^* in C_0^+ . The proof is completed. \square

Remark 5.3. When $\gamma_k = 1$, condition (5.4) is reduced to the following condition

$$\left(\hat{v}_0 + \frac{1}{e} \right) \beta^* < a < b,$$

where $\hat{v}_0 = \max \left\{ \frac{1}{e^2}, \frac{a}{b} (1 - \ln \frac{b}{a}) \right\}$ and $\beta^* = \sum_{k=1}^p \beta_k$.

Remark 5.4. By the methods proposed in [5, 18], model (2.1) has a unique positive periodic solution $N^*(t)$, which is globally attractive, if there exists a positive constant M such that

$$\gamma_k^+ \leq \frac{\tilde{\kappa}}{M}, \quad k = 1, 2, \dots, p, \quad (5.6a)$$

$$\left. \begin{aligned} \sup_{t \geq 0} \left\{ -a(t) + b(t)e^{-M} + \frac{1}{e} \sum_{k=1}^p \frac{\beta_k(t)}{\gamma_k(t)} \right\} < 0 \\ \inf_{t \geq 0, s \in [0, \kappa]} \left\{ -a(t) + b(t)e^{-s} + \sum_{k=1}^p \frac{\beta_k(t)}{\gamma_k(t)} s e^{-s} \right\} > 0 \end{aligned} \right\}, \quad (5.6b)$$

$$\sup_{t \geq 0} \left\{ -b(t)e^{-M} + \frac{1}{e^2} \sum_{k=1}^p \beta_k(t) \right\} < 0, \quad (5.6c)$$

where $\kappa \in (0, 1)$ and $\tilde{\kappa} > 0$ are constants satisfying $\frac{1-\kappa}{e^\kappa} = \frac{1}{e^2}$ and $\kappa e^{-\kappa} = \tilde{\kappa} e^{-\tilde{\kappa}}$. It is clear that condition (5.6b) implies both (C1) and (C2), where $\frac{\sigma}{e} \leq 1 - e^{-M} \left(\frac{b}{a}\right)^-$. In addition, by taking into account the upper bound of coefficients, testable conditions derived from (5.6b) and (5.6c) are given as

$$\begin{cases} b^+ e^{-M} + \frac{1}{e} \sum_{k=1}^p \frac{\beta_k^+}{\gamma_k^+} < a^- \\ b^- - a^+ > 0 \\ \frac{1}{e^2} \sum_{k=1}^p \beta_k^+ < b^- e^{-M}. \end{cases}$$

Consequently, $\frac{b^-}{b^+} \left(a^- - \frac{1}{e} \sum_{k=1}^p \frac{\beta_k^+}{\gamma_k^+} \right) > \frac{1}{e^2} \sum_{k=1}^p \beta_k^+$. Thus, conditions (C3) and (C4) are fulfilled. The above verification shows that, for the existence of positive solutions, uniform permanence, uniform dissipativity and global attractivity of positive periodic solution of (2.1), the method presented in this paper can give less conditions with simpler proofs than existing ones in the literatures [5, 18].

6 Simulations

In this section, we give two examples to illustrate the effectiveness of the obtained results.

Example 6.1. Consider model (2.1) with the following parameters

$$\begin{aligned} a(t) &= 1 + \lambda_a \sin^2(\omega_0 t), \quad b(t) = 2 + \lambda_b \cos^2(\omega_0 t), \\ \beta_1(t) &= \beta_1 |\sin(\omega_0 t)|, \quad \beta_2(t) = \beta_2 |\sin(2\omega_0 t)|, \\ \gamma_k(t) &= 1 + 0.1 |\cos(\omega_0 t)|, \quad \tau_k(t) = \tau_M \sin^2(\omega_0 t), \quad k = 1, 2, \end{aligned}$$

where $\omega_0 > 0$, $\tau_M > 0$ and nonnegative constants λ_a , λ_b , β_1 , β_2 are given. Clearly, assumptions (A1) and (A2) are satisfied. Moreover, we have

$$\begin{aligned} a^- &= 1, & a^+ &= 1 + \lambda_a, & b^- &= 2, & b^+ &= 2 + \lambda_b, \\ \beta_k^+ &= \beta_k, & \gamma_k^- &= 1, & & & k &= 1, 2. \end{aligned}$$

Thus, the derived conditions in Theorem 4.5 are satisfied if and only if it holds that

$$0 \leq \lambda_a < 1, \quad \frac{\beta_1 + \beta_2}{e} < 1, \quad (6.1a)$$

$$v(\beta_1 + \beta_2) < (1 - \omega_0 \tau_M) \left(1 - \frac{\beta_1 + \beta_2}{e} \right) \frac{2}{2 + \lambda_b}, \quad (6.1b)$$

where

$$\nu = \max \left\{ \frac{1}{e^2}, \frac{1 + \lambda_a}{2} \left(1 - \ln \frac{2}{1 + \lambda_a} \right) \right\}.$$

Let $\tau_M = 1$, $\omega_0 = 0.5$ and $\beta_1 + \beta_2 = 1$. It can be verified that conditions (6.1a) and (6.1b) are satisfied with $\lambda_a = 0.1$, $\lambda_b = 0.25$. By Theorem 4.5, system (2.1)–(2.3) has a unique positive 2π -periodic solution $N^*(t)$, which is globally attractive in C_0^+ .

The simulation result presented in Figure 6.1 is taken with $\beta_1 = \beta_2 = 0.5$ and various initial values in the range $[0.5, 1.5]$. It can be seen in Figure 6.1 that all state trajectories of system (2.1)–(2.3) are eventually bounded within the range $[r_*, r^*]$ and converge to the unique positive periodic solution. This illustrates the results in Corollary 3.8 and Theorem 4.5.

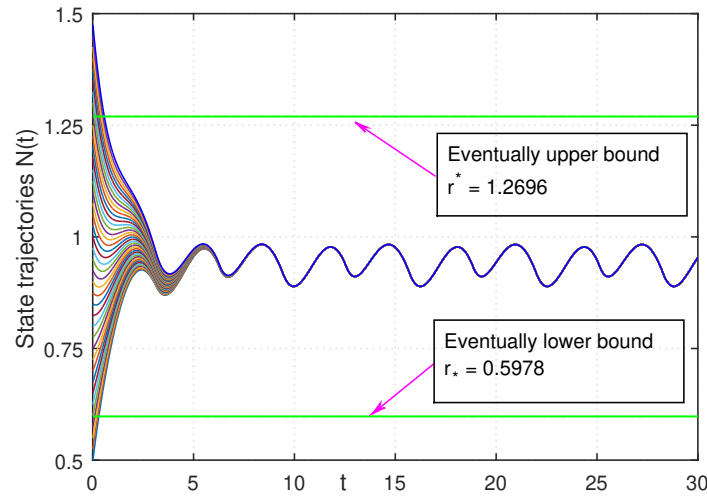


Figure 6.1: Attractivity of the unique positive periodic solution

Example 6.2. Consider model (5.1) with $\gamma_k = 1$ and $\sum_{k=1}^p \beta_k = 1$. Then, condition (5.4) is satisfied if and only if

$$\frac{1}{e} + \max \left\{ \frac{1}{e^2}, \frac{a}{b} \left(1 - \ln \frac{b}{a} \right) \right\} < a < b. \quad (6.2)$$

Let κ_* be the unique positive solution of the following equation

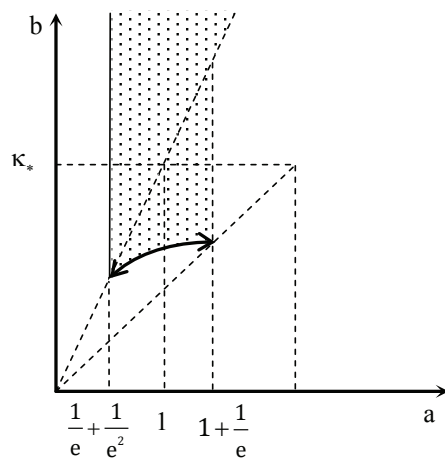
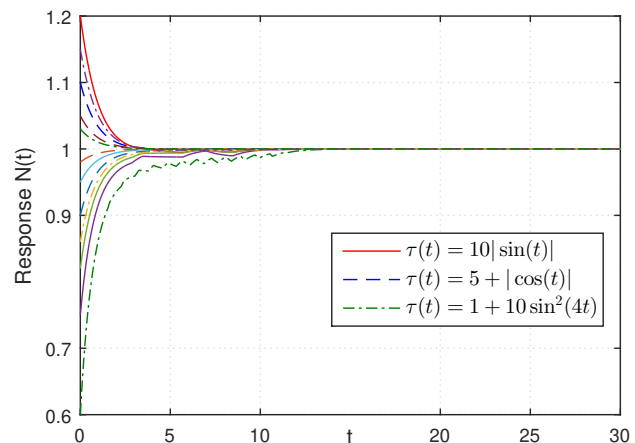
$$\frac{1 - \ln \kappa}{\kappa} = \frac{1}{e^2}.$$

Then, $1 < \kappa_* < e$ ($\kappa_* \simeq 2.0576$) and condition (6.2) holds if and only if $b = \kappa a$, where

$$a > \begin{cases} \frac{1}{e} + \frac{1 - \ln \kappa}{\kappa} & \text{if } \kappa \in (1, \kappa_*) \\ \frac{1}{e} + \frac{1}{e^2} & \text{if } \kappa \geq \kappa_*. \end{cases}$$

The admissible region of (a, b) is illustrated in Figure 6.2.

For illustrative purpose, let $a = \frac{4}{e}$, $b = 3$. By Theorem 5.2, the unique equilibrium $N^* = 1$ of (5.1) is globally attractive for any bounded delays $\tau_k(t) \in [0, \tau_M]$. The simulation result given in Figure 6.3 is taken with various common delay $\tau_k(t) = \tau(t)$. It can be seen that all the corresponding state trajectories of model (5.1) converge to N^* , which validates the theoretical results.

Figure 6.2: Admissible region of (a, b) Figure 6.3: Global attractivity of N^*

It is noted also that for this example conditions (1.8) and (1.9) are reduced to

$$\frac{1}{a + \frac{1}{e}} \geq \tau_M, \quad a > \frac{1}{e}, \quad \ln\left(\frac{b}{a}\right) > 1.$$

Thus, the method of [31, Theorem 2.6] does not give any feasible solution when $\tau_M > \frac{e}{2}$ or $1 < \frac{b}{a} \leq e$. This shows that our conditions are competitive with those of [31].

7 Conclusion

A delayed Nicholson's blowflies model with nonlinear density-dependent mortality has been studied in this paper. Based on a comparison technique via differential inequalities, sufficient conditions that ensure global uniform permanence and dissipativity of the model have been first derived. The obtained results on uniform permanence and dissipativity have been then utilized to deal with the existence and global attractivity of a unique positive periodic solution of the underlying model. An application to the case of model with constant coefficients has also been presented. Two numerical examples with simulations have been given to illustrate the efficacy of the obtained results.

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