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Controllability of strongly degenerate parabolic problems with strongly singular potentials

Dedicated with esteem to Professor László Hatvani on the occasion of his 75th birthday

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Abstract. We prove a null controllability result for a parabolic Dirichlet problem with non smooth coefficients in presence of strongly singular potentials and a coefficient degenerating at an interior point. We cover the case of weights falling out the class of Muckenhoupt functions, so that no Hardy-type inequality is available; for instance, we can consider Coulomb-type potentials. However, through a cut-off function method, we recover the desired controllability result.

Keywords: strong degeneracy, strong singularity, non smooth coefficients, Coulomb potential, null controllability, cut-off functions.

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1 Introduction

This paper deals with null controllability for a class of degenerate and singular parabolic Dirichlet problems with interior degeneracy and singularity, whose prototype is

$$\begin{cases} u_t - (|x-x_0|u_x)_x - \dfrac{\lambda}{|x-x_0|}u = f\chi_\omega, & (t,x) \in Q_T := (0,T)\times(0,1), \\ u(t,0) = u(t,1) = 0, & t \in (0,T), \\ u(0,x) = u_0(x) \in L^2(0,1), & x \in (0,1). \end{cases} \tag{1.1}$$

Here $x_0 \in (0,1)$, $f \in L^2(0,1)$ denotes the control function, located in an open set ω compactly contained in $(0,1)$ and λ is a real parameter.

Of course, we shall consider more general operators of the form

$$u_t - (a(x)u_x)_x - \frac{\lambda}{b(x)}u, \tag{1.2}$$

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where functions a and b, possibly *non-smooth*, degenerate at the same interior point $x_0 \in (0,1)$. The fact that both a and b degenerate at x_0 is the most complicated situation, since, if they degenerate at different points, we can split the problem in a degenerate one and a singular one, so that known results apply separately.

Related problems have been studied before in [17], but not the case under consideration. Indeed, let us recall the following possibilities for degenerating functions or singular potentials:

- $a \in W^{1,1}(0,1)$ is said to be *weakly degenerate*, **WD** for short, if there exists $x_0 \in (0,1)$ such that $a(x_0) = 0$, $a > 0$ on $[0,1] \setminus \{x_0\}$ and there exists $K_a \in (0,1)$ such that $(x - x_0)a' \leq K_a a$ a.e. in $[0,1]$;
- $a \in W^{1,\infty}(0,1)$ is said to be *strongly degenerate*, **SD** for short, if there exists $x_0 \in (0,1)$ such that $a(x_0) = 0$, $a > 0$ on $[0,1] \setminus \{x_0\}$ and there exists $K_a \in [1,2)$ such that $(x - x_0)a' \leq K_a a$ a.e. in $[0,1]$.

Typical examples for the previous degeneracies are $a(x) = |x - x_0|^{K_a}$ with $0 < K_a < 2$. The restriction $K_a < 2$ is related to controllability and existence issues. In particular, if $a(x) = |x - x_0|^{K_a}$, $K_a \geq 2$ and $\lambda = 0$, by a standard change of variables (see [16]), the problem associated to the equation

$$u_t - (a(x)u_x)_x = f(t,x)\chi_\omega(x), \quad (t,x) \in Q_T,$$

is transformed in a non degenerate heat equation on an unbounded domain, while the control may remain distributed in a bounded domain: in this situation the lack of null controllability was already proved by Micu and Zuazua in [19]. Moreover, when $K_a > 2$, no characterization of the domain of the operator is available due to the strong degeneracy of a, and so some integrations by parts cannot be done, see for instance [8] or [18]. For this reasons, from now on, we will only consider coefficients $K_a, K_b < 2$.

This paper is in some sense a completion of the previous works [14] and [17], where we considered well-posedness and null controllability for the following problem via suitable Hardy–Poincaré inequalities and Carleman estimates:

$$\begin{cases} u_t - \mathcal{A}u = f(t,x)\chi_\omega(x), & (t,x) \in Q_T, \\ u(t,0) = u(t,1) = 0, & t \in (0,T), \\ u(0,x) = u_0(x) \in X, & x \in (0,1). \end{cases} \tag{1.3}$$

Here

$$\mathcal{A}u := (au_x)_x + \lambda \frac{u}{b} \quad \text{or} \quad \mathcal{A}u := au_{xx} + \lambda \frac{u}{b},$$

X is a suitable Hilbert space and $f \in L^2(0,T;X)$. In both papers a key assumption was that $K_a + K_b \leq 2$, and the case $K_a = K_b = 1$ was treated only in the non divergence case in [14] under additional assumptions (see below). Hence, the general situation for strongly degenerate a and b was completely open. For this reason, in this paper we complete the description of the evolution system

$$\begin{cases} u_t - (a(x)u_x)_x - \dfrac{\lambda}{b(x)}u = f(t,x)\chi_\omega(x), & (t,x) \in Q_T, \\ u(t,0) = u(t,1) = 0, & t \in (0,T), \\ u(0,x) = u_0(x), & x \in (0,1) \end{cases} \tag{1.4}$$

when $K_a, K_b \geq 1$. In particular, we aim at showing null controllability results for (1.4), that is: for every $u_0 \in L^2(0,1)$ there exists $f \in L^2(Q_T)$ such that the solution u of (1.4) satisfies $u(T,x) = 0$ for every $x \in [0,1]$ and $\|f\|^2_{L^2(Q_T)} \leq C\|u_0\|^2_{L^2(0,1)}$ for some universal positive constant C.

The originality of this paper is that in the previous papers the controllability issue was a consequence of Carleman and observability inequalities. However, the last inequalities were obtained by the Hardy–Poincaré type inequality with interior degeneracy

$$\int_0^1 \frac{u^2}{b} dx \leq C \int_0^1 a(u')^2 dx, \tag{1.5}$$

which was obtained as a corollary of the inequality

$$\frac{(1-\alpha)^2}{4} \int_0^1 \frac{u^2}{|x-x_0|^{2-\alpha}} dx \leq \int_0^1 |x-x_0|^\alpha (u')^2 dx, \tag{1.6}$$

valid for every $u \in H^1_{|x-x_0|^\alpha, |x-x_0|^{2-\alpha}}(0,1)$ (see below for the definition of this space) and for every $\alpha \in \mathbb{R}$.

It is clear that inequality (1.6) above fails to be interesting for $\alpha = 1$, in agreement with the celebrated characterization of Muckenhoupt [20]. For this reason, in order to obtain the controllability result, we cannot follow the approach used so far and we need a completely different one. Indeed, we will prove the null controllability result, also when $K_a = K_b = 1$, *only using cut-off functions*. This technique can be applied also to the non divergence case generalizing the result given in [14].

We conclude this introduction recalling that null controllability for problems like (1.4) has been a mainstream in recent years, especially when $\lambda = 0$ (we recall, for example, [1], [2–4], [5–8], [13], [15], [16], [18] and [10] for the nonlinear case). If $\lambda \neq 0$, the first results in this direction are obtained in [22] for the *non degenerate* heat operator with singular potential

$$u_t - u_{xx} - \lambda \frac{1}{x^{K_b}} u, \quad (t,x) \in Q_T, \tag{1.7}$$

and Dirichlet boundary conditions. In particular, in [22], Carleman estimates (and consequently null controllability properties) are established for (1.7) when $\lambda \leq 1/4$. On the contrary, if $\lambda > 1/4$, in [9] it is proved that null controllability fails.

To our best knowledge, the first paper coupling a degenerate diffusion coefficient and a singular potential is [21]. In particular, the author establishes Carleman estimates (and thus null controllability results) for the operator

$$u_t - (x^{K_a} u_x)_x - \lambda \frac{1}{x^{K_b}} u, \quad (t,x) \in Q_T,$$

under suitable conditions on λ and assuming $K_a + K_b \leq 2$, but excluding $K_a = K_b = 1$. In this way, she combines the results of [8] and [22] for the purely degenerate operator and the purely singular one, respectively. Her result is then extended in [11] and in [12] for operators of the form

$$u_t - (a(x)u_x)_x - \lambda \frac{1}{x^{K_b}} u, \quad (t,x) \in Q_T, \tag{1.8}$$

where $a(x) \sim x^{K_a}$.

However, all the previously cited papers deal with a degenerate/singular operator with degeneracy or singularity at the boundary of the domain.

To our best knowledge, [3], [4], [15], [16] and [18] are the first papers where purely degenerate operators are treated from the point of view of well-posedness and Carleman estimates (and, thus, null controllability) when the degeneracy is at an interior point of the space domain. In particular, [16] is the first paper that deals with a *non smooth* degenerate function a. On the contrary, if $\lambda \neq 0$, we refer to [14] and [17] for operators with a degeneracy and a singularity both occurring in the interior of the domain (we refer to [14] and [17] for other references on this subject).

A final comment on the notation: by C we shall denote universal positive constants, which are allowed to vary from line to line.

2 The controllability results

2.1 The divergence case

Let us start introducing the functional setting from [17]. First of all, define the weighted Hilbert spaces

$$H^1_a(0,1) := \left\{u \in W^{1,1}_0(0,1) \,:\, \sqrt{a}u' \in L^2(0,1)\right\}$$

and

$$H^1_{a,b}(0,1) := \left\{u \in H^1_a(0,1) \,:\, \frac{u}{\sqrt{b}} \in L^2(0,1)\right\},$$

endowed with the inner products

$$\langle u,v\rangle_{H^1_a(0,1)} := \int_0^1 au'v'dx + \int_0^1 uv\,dx,$$

and

$$\langle u,v\rangle_{H^1_{a,b}(0,1)} = \int_0^1 au'v'dx + \int_0^1 uv\,dx + \int_0^1 \frac{uv}{b}dx,$$

respectively. Finally, introduce the Hilbert space

$$H^2_{a,b} := \left\{u \in H^1_a(0,1) \,:\, au' \in H^1(0,1) \text{ and } Au \in L^2(0,1)\right\},$$

where

$$Au := \left(au'\right)' + \frac{\lambda}{b}u \quad \text{with } D(A) = H^2_{a,b}(0,1).$$

We assume:

(H): a and b are **SD** and $\lambda < 0$.

As a particular case of [17, Theorem 2.22], we have the following well-posedness result.

Theorem 2.1. *Assume* ***(H)****. Then, for every $u_0 \in L^2(0,1)$ and $f \in L^2(Q_T)$ there exists a unique solution of problem (1.4). In particular, the operator $A : D(A) \to L^2(0,1)$ is non positive and self-adjoint in $L^2(0,1)$ and it generates an analytic contraction semigroup of angle $\pi/2$. Moreover, if $u_0 \in D(A)$, then*

$$f \in W^{1,1}(0,T;L^2(0,1)) \Rightarrow u \in C^1(0,T;L^2(0,1)) \cap C([0,T];D(A)),$$
$$f \in L^2(Q_T) \Rightarrow u \in H^1(0,T;L^2(0,1)).$$

We remark that Theorem 2.1 is based on [17, Proposition 2.18] which holds if $\lambda < 0$; otherwise, i.e. if $\lambda > 0$, we had to require the additional condition $K_a + K_b \leq 2$ with K_a and K_b not simultaneously equal to 1 and λ small.

On the control set ω we assume:
$(\mathcal{O})$: either

$$\omega = (\alpha, \beta) \subset (0,1) \text{ is such that } x_0 \in \omega, \tag{2.1}$$

or

$$\omega = \omega_1 \cup \omega_2, \tag{2.2}$$

where

$$\omega_i = (\alpha_i, \beta_i) \subset (0,1),\ i = 1,2, \quad \text{and} \quad \beta_1 < x_0 < \alpha_2.$$

The main result is the following.

Theorem 2.2. *Assume **(H)** and $(\mathcal{O})$. Then, given $u_0 \in L^2(0,1)$, there exists $f \in L^2(Q_T)$ such that the solution u of (1.4) satisfies*

$$u(T,x) = 0 \quad \textit{for every } x \in [0,1].$$

Moreover,

$$\int_{Q_T} f^2 dxdt \leq C \int_0^1 u_0^2 dx \tag{2.3}$$

for some universal positive constant C.

Proof. First, assume (2.1). Consider $0 < r' < r$ with $(x_0 - r, x_0 + r) \subset \omega$. Then, given an initial condition $u_0 \in L^2(0,1)$, by classical controllability results in the non degenerate and non singular case, there exist two control functions $h_1 \in L^2((0,T) \times (0, x_0 - r'))$ and $h_2 \in L^2((0,T) \times (x_0 + r', 1))$, such that the corresponding solutions v_1 and v_2 of the parabolic problems

$$\begin{cases} u_t - (a(x)u_x)_x - \dfrac{\lambda}{b(x)} u = h_1(t,x)\chi_{\omega \cap (\alpha, x_0 - r)}(x), & (t,x) \in (0,T) \times (0, x_0 - r'), \\ u(t,0) = u(t, x_0 - r') = 0, & t \in (0,T), \\ u(0,x) = u_0(x), & x \in (0, x_0 - r'), \end{cases} \tag{2.4}$$

and

$$\begin{cases} u_t - (a(x)u_x)_x - \dfrac{\lambda}{b(x)} u = h_2(t,x)\chi_{\omega \cap (x_0 + r, \beta)}(x), & (t,x) \in (0,T) \times (x_0 + r', 1), \\ u(t, x_0 + r') = u(t,1) = 0, & t \in (0,T), \\ u(0,x) = u_0(x), & x \in (x_0 + r', 1), \end{cases} \tag{2.5}$$

respectively, satisfy $v_1(T,x) = 0$ for all $x \in (0, x_0 - r')$ and $v_2(T,x) = 0$ for all $x \in (x_0 + r', 1)$ with

$$\int_0^T \int_0^{x_0 - r'} h_1^2 dxdt \leq C \int_0^T \int_0^{x_0 - r'} u_0^2 dxdt \tag{2.6}$$

and

$$\int_0^T \int_{x_0 + r'}^1 h_2^2 dxdt \leq C \int_0^T \int_{x_0 + r'}^1 u_0^2 dxdt \tag{2.7}$$

for some constant C. Now, let u_3 be the solution of the problem

$$\begin{cases} u_t - (a(x)u_x)_x - \dfrac{\lambda}{b(x)}u = 0, & (t,x) \in (0,T) \times (0,1), \\ u(t,0) = u(t,1) = 0, & t \in (0,T), \\ u(0,x) = u_0(x), & x \in (0,1). \end{cases} \tag{2.8}$$

Denote by u_1 and u_2, f_1 and f_2 the trivial extensions of v_1 and v_2, h_1 and h_2 in $[x_0 - r', 1]$ and $[0, x_0 + r']$, respectively. Then take some cut-off functions $\phi_i \in C^\infty([0,1])$, $i = 0, 1, 2$, with

$$\phi_1(x) := \begin{cases} 0, & x \in [x_0 - r', 1], \\ 1, & x \in [0, x_0 - r], \end{cases} \qquad \phi_2(x) := \begin{cases} 0, & x \in [0, x_0 + r'], \\ 1, & x \in [x_0 + r, 1], \end{cases}$$

and $\phi_0 = 1 - \phi_1 - \phi_2$. Finally, take

$$u(t,x) = \phi_1(x)u_1(t,x) + \phi_2(x)u_2(t,x) + \frac{T-t}{T}\phi_0(x)u_3(t,x). \tag{2.9}$$

Then, $u(T,x) = 0$ for all $x \in [0,1]$ and u satisfies problem (1.4) in the domain Q_T with

$$\begin{aligned} f = \phi_1 f_1 \chi_{(\alpha, x_0 - r)} + \phi_2 f_2 \chi_{(x_0 + r, \beta)} - \frac{1}{T}\phi_0 u_3 - \phi_1' a u_{1,x} - \phi_2' a u_{2,x} \\ - \phi_0' \frac{T-t}{T} a u_{3,x} - \left(\phi_1' a u_1 + \phi_2' a u_2 + \phi_0' \frac{T-t}{T} a u_3\right)_x. \end{aligned}$$

Since a belongs to $W^{1,\infty}(0,1)$, one has that $f \in L^2(Q_T)$, as required. Moreover, it is easy to see that the support of f is contained in ω.

Now, we prove (2.3). To this aim, consider the equation satisfied by v_1 and multiply it by v_1. Then, integrating over $(0, x_0 - r')$, we have

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|v_1(t)\|^2_{L^2(0,x_0-r')} + \|\sqrt{a} v_{1,x}(t)\|^2_{L^2(0,x_0-r')} - \lambda \left\| \frac{v_1}{\sqrt{b}} \right\|^2_{L^2(0,x_0-r')} \\ \le \frac{1}{2}\|v_1(t)\|^2_{L^2(0,x_0-r')} + \frac{1}{2}\|h_1\|^2_{L^2(\omega \cap (\alpha, x_0 - r))}. \end{aligned}$$

Using the fact that $\lambda < 0$, we get

$$\begin{aligned} \frac{d}{dt}\|v_1(t)\|^2_{L^2(0,x_0-r')} &\le \frac{d}{dt}\|v_1(t)\|^2_{L^2(0,x_0-r')} + 2\|\sqrt{a} v_{1,x}(t)\|^2_{L^2(0,x_0-r')} \\ &\le \|v_1(t)\|^2_{L^2(0,x_0-r')} + \|h_1(t,\cdot)\|^2_{L^2(\omega \cap (\alpha, x_0 - r))}. \end{aligned}$$

Integrating the previous inequality, we get

$$\|v_1(t)\|^2_{L^2(0,x_0-r')} \le e^T \left(\|u_0\|^2_{L^2(0,x_0-r')} + \int_0^t \|h_1(s,\cdot)\|^2_{L^2(\omega \cap (\alpha, x_0 - r))} ds \right) \quad \text{for all } t \in [0,T],$$

and so

$$\|v_1\|^2_{L^2((0,x_0-r')\times[0,T])} \le C \left(\|u_0\|^2_{L^2(Q_T)} + \|h_1\|^2_{L^2((0,x_0-r')\times[0,T])} \right). \tag{2.10}$$

Now, integrating over $(0,T)$ the inequality

$$\frac{d}{dt}\|v_1(t)\|^2_{L^2(0,x_0-r')} + 2\|\sqrt{a} v_{1,x}(t)\|^2_{L^2(0,x_0-r')} \le \|v_1(t)\|^2_{L^2(0,x_0-r')} + \|h_1(t,\cdot)\|^2_{L^2(\omega \cap (\alpha, x_0 - r))},$$

by using (2.10), we immediately find

$$\|\sqrt{a}v_{1,x}\|^2_{L^2((0,x_0-r')\times[0,T])} \leq C\left(\|u_0\|^2_{L^2(Q_T)} + \|h_1\|^2_{L^2((0,x_0-r')\times[0,T])}\right) \tag{2.11}$$

for some $C > 0$.

Now, let us note that, since $a \in W^{1,\infty}(0,1)$, then

$$\|(av_1)_x\|_{L^2((0,x_0-r')\times[0,T])} \leq C\left(\|v_1\|_{L^2((0,x_0-r')\times[0,T])} + \|\sqrt{a}v_{1,x}\|_{L^2((0,x_0-r')\times[0,T])}\right).$$

By using (2.10) and (2.11) in the previous inequality, we get

$$\|(av_1)_x\|_{L^2((0,x_0-r')\times[0,T])} \leq C\left(\|u_0\|^2_{L^2(Q_T)} + \|h_1\|^2_{L^2((0,x_0-r')\times[0,T])}\right) \tag{2.12}$$

for some $C > 0$.

An estimate analogous to (2.12) holds for v_2 with h_2 replacing h_1, and for v_3 only in terms of u_0.

In conclusion, by (2.10), (2.11), (2.12), from the very definition of f and by (2.6) and (2.7), inequality (2.3) follows immediately.

Now, assume (2.2). Take $r > 0$ such that $\beta_1 < x_0 - r$ and $x_0 + r < \alpha_2$. As before, given an initial condition $u_0 \in L^2(0,1)$, by classical controllability results in the non degenerate and non singular case, there exist two control functions $h_4 \in L^2((0,T)\times(0,x_0-r))$ and $h_5 \in L^2((0,T)\times(x_0+r,1))$, such that the corresponding solutions v_4 and v_5 of the parabolic problems

$$\begin{cases} u_t - (a(x)u_x)_x - \dfrac{\lambda}{b(x)}u = h_1(t,x)\chi_{(\alpha_1,\beta_1)}(x), & (t,x) \in (0,T)\times(0,x_0-r), \\ u(t,0) = u(t,x_0-r) = 0, & t \in (0,T), \\ u(0,x) = u_0(x), & x \in (0,x_0-r), \end{cases} \tag{2.13}$$

and

$$\begin{cases} u_t - (a(x)u_x)_x - \dfrac{\lambda}{b(x)}u = h_2(t,x)\chi_{(\alpha_2,\beta_2)}(x), & (t,x) \in (0,T)\times(x_0+r,1), \\ u(t,x_0+r) = u(t,1) = 0, & t \in (0,T), \\ u(0,x) = u_0(x), & x \in (x_0+r,1), \end{cases} \tag{2.14}$$

respectively, satisfy $v_4(T,x) = 0$ for all $x \in (0,x_0-r)$ and $v_5(T,x) = 0$ for all $x \in (x_0+r,1)$ with

$$\int_0^T\int_0^{x_0-r} h_4^2\,dxdt \leq C\int_0^T\int_0^{x_0-r} u_0^2\,dxdt \tag{2.15}$$

and

$$\int_0^T\int_{x_0+r}^1 h_5^2\,dxdt \leq C\int_0^T\int_{x_0+r}^1 u_0^2\,dxdt \tag{2.16}$$

for some constant C. As before, let u_4 and f_4, u_5 and f_5 be the trivial extensions of v_4 and h_4, v_5 and h_5 in $[x_0-r,1]$ and $[0,x_0+r]$, respectively.

Then, define cut-off functions $\varphi_i \in C^\infty([0,1])$, $i = 0,1,2$, such that

$$\varphi_1(x) := \begin{cases} 0, & x \in [\beta_1,1], \\ 1, & x \in [0,\alpha_1], \end{cases} \qquad \varphi_2(x) := \begin{cases} 0, & x \in [0,\alpha_2], \\ 1, & x \in [\beta_2,1], \end{cases}$$

and $\varphi_0 = 1 - \varphi_1 - \varphi_2$. Finally, set

$$u(t,x) = \varphi_1(x)u_4(t,x) + \varphi_2(x)u_5(t,x) + \frac{T-t}{T}\varphi_0(x)u_3(t,x), \tag{2.17}$$

where u_3 is the solution of (2.8).

As before, $u(T,x) = 0$ for all $x \in [0,1]$ and u satisfies problem (1.4) in the domain Q_T with

$$\begin{aligned} f = {} & \varphi_1 f_4 \chi_{(\alpha_1,\beta_1)} + \varphi_2 f_5 \chi_{(\alpha_2,\beta_2)} - \frac{1}{T}\varphi_0 u_3 - \varphi_1' a u_{4,x} - \varphi_2' a u_{5,x} \\ & - \varphi_0' \frac{T-t}{T} a u_{3,x} - \left(\varphi_1' a u_4 + \varphi_2' a u_5 + \varphi_0' \frac{T-t}{T} a u_3 \right)_x . \end{aligned}$$

Again $f \in L^2(Q_T)$, as required and the support of f is contained in ω. In order to conclude we have to prove (2.3) for the control function f, but such an estimate can be proved as above, and the result is proved. □

Remark 2.3. We strongly remark that if a is **WD**, the previous approach *does not work*. Indeed, the function f found in the previous proof *is not* in $L^2(Q_T)$, since a is only of class $W^{1,1}(0,1)$.

Remark 2.4. If a is **SD** and b is **WD** the technique above, and so the controllability result, still works provided that there exists a solution of (1.4), for example if $\lambda < 0$ or $\lambda > 0$ small enough and $K_a + K_b \leq 2$ (see [17, Theorem 2.22]). Thus, we re-obtain the controllability result in [17].

The importance of Theorem 2.2 is clarified also in the following.

Remark 2.5. The null controllability result in Theorem 2.2 *cannot be obtained* by results already known in literature. Indeed, one may think to consider

$$\begin{cases} u_t - (a(x)u_x)_x - \dfrac{\lambda}{b(x)}u = f(t,x)(t,x)\chi_{(\alpha_1\beta_1)}(x), & (t,x) \in (0,T) \times (0,x_0), \\ u(t,0) = u(t,x_0) = 0, & t \in (0,T), \\ u(0,x) = u_0(x)_{|_{[0,x_0)}}, \end{cases} \tag{2.18}$$

and

$$\begin{cases} u_t - (a(x)u_x)_x - \dfrac{\lambda}{b(x)}u = f(t,x)\chi_{(\alpha_2,\beta_2)}(x), & (t,x) \in (0,T) \times (x_0,1), \\ u(t,x_0) = u(t,1) = 0, & t \in (0,T), \\ u(0,x) = u_0(x)_{|_{(x_0,1]}}, \end{cases} \tag{2.19}$$

and say that u is a solution of (1.4) if and only if the restrictions of u to $[0,x_0)$ and to $(x_0,1]$, are solutions to (2.18) and (2.19), respectively. Thanks to the characterization of the space $H^1_{a,b}(0,1)$ (see [17, Lemma 2.11]), if ω satisfies (2.2) and the initial datum is more regular, this can actually be done. Hence, we have two problems with degeneracy and singularity at the boundary. However, in this case the only available results are, for instance for (2.18), when $b(x) \sim x^{K_b}$ ([11] and [12]) or $a(x) = x^{K_a}$, $b(x) = x^{K_b}$ ([21]), provided that $K_a + K_b \leq 2$, excluding the case $K_a = K_b = 1$. Hence, we can not deduce null controllability for (1.4) by known results. Moreover, if u_0 is only of class $L^2(0,1)$, the solution is not sufficiently regular to verify the additional condition at x_0 established in [17, Lemma 2.11], and this procedure cannot be pursued.

2.2 The non divergence case

The technique used in the proof of Theorem 2.2 can be applied also for the problem in *non divergence* form

$$\begin{cases} u_t - a(x)u_{xx} - \dfrac{\lambda}{b(x)}u = f(t,x)\chi_\omega(x), & (t,x) \in Q_T, \\ u(t,0) = u(t,1) = 0, & t \in (0,T), \\ u(0,x) = u_0(x), & x \in (0,1). \end{cases} \tag{2.20}$$

The null controllability for (2.20) was studied in [14] requiring additional assumptions: for example, if $\lambda < 0$ then one has to ask that $(x-x_0)b'(x) \ge 0$ in $[0,1]$. However, using the technique used in the proof of Theorem 2.2, in order to prove the global controllability result, one has to require only the conditions for the existence theorem (see [14, Hypothesis 3.1]). Indeed, proceeding as in the proof of Theorem 2.2 but with problems written in non divergence form, the control function f of (2.20) is given by

$$f = \phi_1 f_1 \chi_{(\alpha,x_0-r)} + \phi_2 f_2 \chi_{(x_0+r,\beta)} - \frac{1}{T}\phi_0 u_3 - 2\phi_1' a u_{1,x} - \phi_1'' a u_1 - 2\phi_2' a u_{2,x} \\ - \phi_2'' a u_2 - \phi_0' \frac{T-t}{T} a u_{3,x} - a\frac{T-t}{T}\left(\phi_0' u_3\right)_x,$$

if ω satisfies (2.1) or

$$f = \varphi_1 f_4 \chi_{(\alpha_1,\beta_1)} + \varphi_2 f_5 \chi_{(\alpha_2,\beta_2)} - \frac{1}{T}\varphi_0 u_3 - 2\varphi_1' a u_{4,x} - \varphi_1'' a u_4 - 2\phi_2' a u_{5,x} \\ - \varphi_2'' a u_5 - \varphi_0' \frac{T-t}{T} a u_{3,x} - a\frac{T-t}{T}\left(\varphi_0' u_3\right)_x,$$

if ω satisfies (2.2). In every case f belongs to the $L^2_{\frac{1}{a}}(Q_T)$ as required (for the definition of the space see, e.g., [14]). Hence, the next theorem holds.

Theorem 2.6. *Assume [14, Hypothesis 3.1] and* $(\mathcal{O})$. *Then, given* $u_0 \in L^2_{\frac{1}{a}}(0,1)$, *there exists* $f \in L^2_{\frac{1}{a}}(Q_T)$ *such that the solution* u *of* (2.20) *satisfies*

$$u(T,x) = 0 \ \text{ for every } \ x \in [0,1].$$

Moreover

$$\int_{Q_T} \frac{f^2}{a} dxdt \le C \int_0^1 \frac{u_0^2}{a} dx, \tag{2.21}$$

for some universal positive constant C.

We remark that [14, Hypothesis 3.1] is just an assumption ensuring that problem (2.20) is well posed. Hence, the previous theorem generalizes the result given in [14] in the sense that here we prove the controllability result under weaker assumptions. This is due to the fact that in [14] it is proved via Carleman estimates and observability inequality, while here we use only cut-off functions.

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