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Existence and nonexistence of global solutions for doubly nonlinear diffusion equations with logarithmic nonlinearity

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Abstract. In this paper, we study an initial-boundary value problem for a doubly nonlinear diffusion equation with logarithmic nonlinearity. By using the potential well method, we give some threshold results on existence or nonexistence of global weak solutions in the case of initial data with energy less than or equal to potential well depth. In addition, the asymptotic behavior of solutions is also discussed.

Keywords: global existence, blow-up, asymptotic behavior, logarithmic nonlinearity.

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1 Introduction

In this paper, we will study the following doubly nonlinear diffusion equations with logarithmic nonlinearity

$$\begin{cases} u_t - \Delta_p \left(u^{(m-1)}\right) = f_q\left(u\right), & x \in \Omega, t > 0, \\ u(x,t) = 0, & x \in \partial\Omega, t > 0, \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$ $(n \geq 1)$ is an open bounded domain with smooth boundary $\partial\Omega$, $u^{(m-1)} := |u|^{m-2}u$, $\Delta_p(u) := \operatorname{div}\big((\nabla u)^{(p-1)}\big)$ the usual p-Laplacian operators and f_q is of the form of logarithmic term $f_q(s) = s^{(q-1)} \log|s|$.

Let us consider the following equation which is so-called doubly nonlinear parabolic equations

$$u_t - \Delta_p \left(u^{(m-1)}\right) = f\left(u\right), \tag{1.2}$$

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where $f(u)$ is a source if $f(u) \geq 0$, whereas $f(u)$ is called a sink. This equation generalizes many equations such as heat equation (as $m = 2$ and $p = 2$), the porous medium equation (as $m > 1$ and $p = 2$), and the p-Laplacian equation (as $m = 2$ and $p > 1$). The equation (1.2) can be divided into three following cases which are called the degenerate, critical and singular case, respectively.

$$(m-1)(p-1) > 1, \qquad (m-1)(p-1) = 1, \quad \text{and} \quad (m-1)(p-1) < 1 \tag{1.3}$$

In this paper, we merely consider the degenerate case, that is, the constants $m, p > 1$ satisfy

$$(m-1)(p-1) > 1. \tag{1.4}$$

The initial-boundary value problem for (1.2) has been studied by many mathematicians. For example, Tsutsumi (see [32]) studied the existence, uniqueness, regularity and large time behavior for weak, mild and strong solutions of an equivalent equation to (1.2) (after changing variables) with absorption $f(u) = -\lambda u^{\gamma}$, $\lambda > 0$ and initial value u_0 in some certain Lebesgue spaces. In [23], Matas et al. also studied the existence of weak solution to the equation (1.2) with inhomogeneous nonlinearities $f(u)$ in the degenerate case with initial value u_0 in Lebesgue spaces by using Galerkin method. The existence of weak solutions of Cauchy problem for equation (1.2) with $f(u) = 0$ has been studied by Ishige [14] for all three cases (1.3).

Regarding of the global existence and nonexistence results, there are some well-known methods to study the equation (1.2) depends on whether Ω is bounded or unbounded domain in $\mathbb{R}^n$. For example, in the case $\Omega = \mathbb{R}^n$, Fujita [9] studied the initial value problem for heat equations with power nonlinearity $f(u) = u^p$ and then Levine in the survey [18] has extended the results of Fujita for more general parabolic equations with nonlinear dissipative terms in unbounded domains. In [25], Pokhozaev and Mitidieri introduced the nonlinear capacity method in order to study of questions on the blow up of solutions of many nonlinear partial differential equations and inequalities. It is noting that although these methods are really powerful tools to treat the case of nonnegative nonlinearities $f(u)$, it cannot be applied to the case of sign-changing nonlinear terms. And therefore, this method cannot be applied to our problem.

On the other hand, in the case of bounded subdomain of $\mathbb{R}^n$, we refer the seminar papers of Kaplan [16], Levine [17] and Ball [3] in which the authors proved the blow up results under condition of non-positive initial energy. In [27], Payne and Sattinger developed the potential well method which is introduced by Lions [20] and Sattinger [30] to study the existence and nonexistence of global weak solutions to heat and wave equations with power like nonlinearity under condition of positive initial energy. More precisely, in [27] the authors show that if the initial energy $J(u_0) < d$, then weak solution $u(t)$ to equation (1.2) (for $m = p = 2$) is global provided that $u_0 \in \mathcal{W}$ (stable sets) and blows up in finite time provided that $u_0 \in \mathcal{U}$ (unstable sets). Afterward, analogous results have been studied extensively to various kind of equations. We refer to [6,10–13,17,21,33] for many heat and wave equations and [7,19] for porous medium equations.

In the case of p-Laplacian equation, Tsustumi [31], Fujii [8] and Ishii [15] studied the initial-boundary value problem for the equation

$$u_t - \Delta_p u = f(u), \tag{1.5}$$

where $f(u) = |u|^{q-2} u$, with $2 \leq q < p^* = \frac{np}{n-p}$. As for the existence and nonexistence of global weak solutions to (1.5), the following results are well-known:

(i) if $p > q$, (1.5) admits a global weak solution for any $u_0 \in W_0^{1,p}(\Omega)$;

(ii) if $p < q$, then weak solution $u(t)$ of (1.5) is global when initial data $u_0 \in W_0^{1,p}(\Omega)$ is in stable sets and blows up in finite time when $u_0 \in W_0^{1,p}(\Omega)$ is in unstable sets.

(iii) when $p = q$, Fujii [8] derived sufficient conditions on blow-up of solutions depending on first eigenvalue λ_1 of the operator $-\Delta_p$.

Although there is a lot of results of global existence and nonexistence of weak solutions to (1.2) in the case of power nonlinearities and its generalization, there is a little known about the one with logarithmic nonlinearity. We refer to [4,5,26] for a few recent papers involving logarithmic nonlinearity. In this paper, in the same spirit with previous works, we utilize the potential well method to study the existence and nonexistence of global weak solutions to (1.2) with logarithmic nonlinearities $f_q(u) = (u)^{(q-1)} \log |u|$, $q > 2$ and initial value $u_0^{(m-1)}$ belonging to Sobolev space $W_0^{1,p}(\Omega)$. Roughly speaking, our results are as follows:

(i) if $(m-1)(p-1) > q-1$, then (1.1) admits a global solution for each $u_0^{(m-1)} \in W_0^{1,p}(\Omega)$;

(ii) if $(m-1)(p-1) \leq q-1$, then there exists a weak solution to (1.1) which is global provided that u_0 belonging to stable sets, and blows up provided that u_0 belonging to unstable sets. In addition, decay estimates are also proved for the former case.

Define $\varphi : L^{m'}(\Omega) \to L^m(\Omega)$ as follows

$$\varphi(u) = u^{(m'-1)},$$

where $m' > 1$ is Hölder conjugate of m satisfying $\frac{1}{m} + \frac{1}{m'} = 1$, then one has $\varphi\big(u^{(m-1)}\big) = u$. Hence, by changing variable $w = u^{(m-1)}$, the equation (1.1) leads to the reformulated equation

$$\partial_t \varphi(w) - \Delta_p w = \left(m'-1\right) f_\gamma(w), \quad \text{where} \quad \gamma = \left(m'-1\right)(q-1) + 1. \tag{1.6}$$

It is also noticed that $f_\gamma(s)$ is nonhomogeneous and can change signs for $s \in (0, +\infty)$. In addition, since $\lim_{s \to 0^+} f_\gamma(s) = 0$, it can be extended continuously to the function $\tilde{f}_\gamma$ with $\tilde{f}_\gamma(0) = 0$.

In what follows, for the sake of brevity, we still denote $\tilde{f}_\gamma$ by f_γ with noticing that $f_\gamma(0) = f_\gamma(1) = 0$. The nonlinearity with such properties can be found in the paper [2]. Hence, instead of (1.1) we consider the following initial boundary value problems

$$\begin{cases} \partial_t \varphi(u) - \Delta_p(u) = (m'-1) f_\gamma(u), & x \in \Omega, t > 0, \\ u(x,t) = 0, & x \in \partial\Omega, t > 0, \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{1.7}$$

where $u_0 \in W_0^{1,p}(\Omega)$ and $\gamma = (m'-1)(q-1) + 1 > 1$, m' is Hölder conjugate of m.

The rest of this paper is organized as follows: Section 2 devotes to preliminaries in which we establish some properties of stationary problem associated to (1.7) and introduce the stable sets (potential well) and unstable sets as well as its properties; Section 3 states main results of this paper and their proofs are presented in the remaining sections.

2 Local minima and potential wells

In this section, we need the following logarithmic Gagliardo–Nirenberg inequality which was introduced by Merker [24].

Lemma 2.1 ([24], Logarithmic Gagliardo–Nirenberg inequalities)**.** *The inequalities*

$$\int \frac{|u|^q}{\|u\|_q^q} \log\left(\frac{|u|^q}{\|u\|_q^q}\right) dx \leq \frac{1}{1-q/p^*} \log\left(C_{n,p,q}^q \frac{\|\nabla u\|_p^q}{\|u\|_q^q}\right) \tag{2.1}$$

are valid for parameters $1 \leq p < +\infty$, $0 < q < p^*$ *and function* $u \in L^q(\mathbb{R}^n)$ *with* $\nabla u \in L^p(\mathbb{R}^n)$. *Hereby the constant C depends on n and p only in the case* $p < n$, *and on* n, p *and a finite upper bound of q in the case* $p \geq n$.

This inequality can be reformulated in parametric form. Here, one introduces the following parametric form of logarithmic Gagliardo–Nirenberg inequalities

$$\int \frac{|u|^q}{\|u\|_q^q} \log\left(\frac{|u|^q}{\|u\|_q^q}\right) dx \leq \mu \frac{\|\nabla u\|_p^r}{\|u\|_q^r} + \frac{qp^*}{(p^*-q)\,r} \log\left(\frac{qp^* C_{n,p,q}^r}{(p^*-q) r\mu e}\right), \tag{2.2}$$

for all $\mu > 0$ where $0 < r \leq \min\{p,q\}$. By virtue of Young's inequality, one obtains the following proposition.

Proposition 2.2 (Parametric form of logarithmic Gagliardo–Nirenberg inequality)**.** *Let us suppose all assumptions in Lemma 2.1. Then we have*

$$\int |u|^q \log\left(\frac{|u|^q}{\|u\|_q^q}\right) dx + C_{n,p,q,\mu}^r \|u\|_q^q \leq \mu \frac{r}{p} \|\nabla u\|_p^p + \mu \frac{p-r}{p} \|u\|_q^{\frac{(q-r)}{p-r}p},$$

for all $\mu > 0$ *where* $0 < r \leq \min\{p,q\}$ *and* $C_{n,p,q,\mu}^r$ *is a constant given by*

$$C_{n,p,q,\mu}^r = \frac{qp^*}{(p^*-q)\,r} \log\left(\frac{(p^*-q) r\mu e}{qp^* C_{n,p,q}^r}\right).$$

Proposition 2.3 ([28,29], Parametric form of logarithmic Sobolev inequality)**.** *Let* $u \in W^{1,p}(\mathbb{R}^n)$, $u \neq 0$ *and* $\mu > 0$ *be any number. Then*

$$p \int_{\mathbb{R}^n} |u(x)|^p \log\left(\frac{|u(x)|}{\|u\|_p}\right) dx + \frac{n}{p} \log\left(\frac{p\mu e}{n\mathcal{L}_p}\right) \int_{\mathbb{R}^n} |u(x)|^p \, dx \leq \mu \int_{\mathbb{R}^n} |\nabla u(x)|^p \, dx,$$

where

$$\mathcal{L}_p = \frac{p}{n} \left(\frac{p-1}{e}\right)^{p-1} \pi^{-\frac{p}{2}} \left[\frac{\Gamma\left(\frac{n}{2}+1\right)}{\Gamma\left(n\frac{p-1}{p}+1\right)}\right]^{\frac{p}{n}}$$

and for $1 \leq p < +\infty$.

For $u \in W_0^{1,p}(\Omega)$, *we can define* $u(x) = 0$ *for* $x \in \mathbb{R}^n \backslash \Omega$. *Then* $u \in W^{1,p}(\mathbb{R}^n)$ *and therefore, we derive*

$$p \int_{\Omega} |u(x)|^p \log\left(\frac{|u(x)|}{\|u\|_p}\right) dx + \frac{n}{p} \log\left(\frac{p\mu e}{n\mathcal{L}_p}\right) \int_{\Omega} |u(x)|^p \, dx \leq \mu \int_{\Omega} |\nabla u(x)|^p \, dx,$$

for $u \in W_0^{1,p}(\Omega)$ *and* $\mu > 0$ *is any number.*

We now define the energy functional J and Nehari functional I on $W_0^{1,p}(\Omega)$ related to the problem (1.7) as follows

$$J(u) = \frac{1}{p}\int_\Omega |\nabla u|^p\,dx + \frac{m'-1}{\gamma^2}\int_\Omega |u|^\gamma\,dx - \frac{m'-1}{\gamma}\int_\Omega |u|^\gamma \log(|u|)\,dx, \tag{2.3}$$

$$I(u) = \int_\Omega |\nabla u|^p\,dx - (m'-1)\int_\Omega |u|^\gamma \log(|u|)\,dx. \tag{2.4}$$

It is clear that the functionals I and J are continuous on $W_0^{1,p}(\Omega)$ and

$$J(u) = \left(\frac{1}{p} - \frac{1}{\gamma}\right)\|\nabla u\|_p^p + \frac{m'-1}{\gamma^2}\|u\|_\gamma^\gamma + \frac{1}{\gamma}I(u). \tag{2.5}$$

We also define the Nehari manifold as follows

$$\mathcal{N} = \left\{u \in W_0^{1,p}(\Omega)\setminus\{0\} : I(u) = \langle J'(u), u\rangle = 0\right\}. \tag{2.6}$$

We shall see below (see Lemma 2.5) that each half line starting from the origin of the phase space $W_0^{1,p}(\Omega)$ intersects the Nehari manifold $\mathcal{N}$ exactly once.

It is also useful to understand the Nehari manifold $\mathcal{N}$ in terms of the critical points of the fibrering map $\lambda \mapsto J(\lambda u)$ for $\lambda > 0$ given by

$$J(\lambda u) = \frac{\lambda^p}{p}\|\nabla u\|_p^p + \frac{m'-1}{\gamma^2}\lambda^\gamma\|u\|_\gamma^\gamma - \frac{m'-1}{\gamma}\int_\Omega |\lambda u|^\gamma \log(|\lambda u|)\,dx, \qquad \lambda > 0.$$

Then we have

$$\frac{d}{d\lambda}J(\lambda u) = \langle J'(\lambda u), u\rangle = \frac{1}{\lambda}\langle J'(\lambda u), \lambda u\rangle = \frac{1}{\lambda}I(\lambda u), \quad \text{for } \lambda > 0. \tag{2.7}$$

This implies the following lemma immediately.

Lemma 2.4. *Let $u \in W_0^{1,p}(\Omega)\setminus\{0\}$ and $\lambda > 0$. Then $\lambda u \in \mathcal{N}$ if and only if λ is a critical point of the map $\lambda \mapsto J(\lambda u)$, that is, $\frac{d}{d\lambda}J(\lambda u) = 0$.*

Thanks to (2.7), in order to study the critical point of fibrering map, we need to study zero points of the map $\lambda \mapsto I(\lambda u)$ for $u \in W_0^{1,p}(\Omega)\setminus\{0\}$ given by $I(\lambda u) = \lambda^p K(\lambda u)$, $\lambda > 0$ where

$$K(\lambda u) = \|\nabla u\|_p^p - (m'-1)\lambda^{\gamma-p}\int_\Omega |u|^\gamma \log(|u|)\,dx - (m'-1)\|u\|_\gamma^\gamma \lambda^{\gamma-p}\log\lambda.$$

Lemma 2.5. *Let $2 \le p \le \gamma$ and $u \in W_0^{1,p}(\Omega)\setminus\{0\}$. Then we possess*

(i) there exists a unique $\lambda_* := \lambda_*(u) > 0$ such that $I(\lambda_* u) = 0$ and $I(\lambda u) > 0$ for $\lambda \in (0, \lambda_*)$, and $I(\lambda u) < 0$ for $\lambda > \lambda_*$;

(ii) the fibrering map $\lambda \mapsto J(\lambda u)$ attains its maximizer at $\lambda = \lambda_*$, that is,

$$\left.\frac{d}{d\lambda}J(\lambda u)\right|_{\lambda=\lambda_*} = 0 \quad \text{and} \quad \left.\frac{d^2}{d\lambda^2}J(\lambda u)\right|_{\lambda=\lambda_*} < 0.$$

Proof. In the case $p = \gamma$, it is not difficult to see that

$$\lambda_* := \lambda_* (u) = \exp \left\{ I(u) \, / \, (m' - 1) \, \|u\|_p^p \right\}$$

satisfies (i) and (ii). It remains to verify for the case $p < \gamma$. Indeed, if this is the case, it is not difficult to see that the function $\lambda \mapsto K(\lambda u)$ is continuous on $(0, +\infty)$ and

$$\lim_{\lambda \to 0^+} K(\lambda u) = \|\nabla u\|_p^p > 0 \quad \text{and} \quad \lim_{\lambda \to +\infty} K(\lambda u) = -\infty,$$

and $K(\lambda u)$ attains its unique maximizer at $\bar{\lambda} := \bar{\lambda}(u) > 0$

$$\bar{\lambda} = \exp \left\{ \frac{\|u\|_\gamma^\gamma + (\gamma - p) \int_\Omega |u|^\gamma \log(|u|) \, dx}{(p - \gamma) \|u\|_\gamma^\gamma} \right\}.$$

Hence, there must be a unique $\lambda_* > \bar{\lambda}$ such that $K(\lambda_* u) = 0$ and $K(\lambda u) > 0$ for $\lambda \in (0, \lambda_*)$, and $K(\lambda u) < 0$ for $\lambda > \lambda^*$. As a consequence, the proof follows from $I(\lambda u) = \lambda^p K(\lambda u)$ and (2.7). □

We now define the depth of potential well

$$d = \inf \left\{ \sup_{\lambda \geq 0} J(\lambda u) : u \in W_0^{1,p}(\Omega) \backslash \{0\} \right\},$$

which is also characterized as

$$0 < d = \inf_{u \in \mathcal{N}} J(u). \tag{2.8}$$

Lemma 2.6. *Let $p \geq 2$ and $p \leq \gamma < \frac{np}{n-p} := p^*$. Then there exists an extremal of the variational problem*

$$0 < d = \inf_{u \in \mathcal{N}} J(u).$$

Proof. The case $\gamma = p$ can be proved similarly to [26]. It remains to consider the case $\gamma > p$. Let $u \in \mathcal{N}$, then it follows by (2.5)

$$J(u) = \left(\frac{1}{p} - \frac{1}{\gamma} \right) \|\nabla u\|_p^p + \frac{m' - 1}{\gamma^2} \|u\|_\gamma^\gamma. \tag{2.9}$$

On the other hand, by logarithmic Gagliardo–Nirenberg inequality, one has

$$\begin{aligned} \frac{1}{m' - 1} \|\nabla u\|_p^p &= \int_\Omega |u|^\gamma \log(|u|) \, dx \\ &\leq \frac{\mu r}{\gamma p} \|\nabla u\|_p^p + \mu \frac{p - r}{\gamma p} \|u\|_\gamma^{\frac{\gamma - r}{p - r} p} + \frac{1}{\gamma} C_{n,p,\gamma,\mu}^r \|u\|_\gamma^\gamma + \frac{1}{\gamma} \|u\|_\gamma^\gamma \log \left(\|u\|_\gamma^\gamma \right), \end{aligned}$$

where $r \in (0, p)$ is a constant and $C_{n,p,q,\mu}^r$ is a constant given by Proposition 2.2. By choosing $\mu = \frac{\gamma p}{(m'-1) r}$ then we get

$$\frac{p - r}{(m' - 1) r} \|u\|_\gamma^{\frac{\gamma - r}{p - r} p} + \frac{1}{\gamma} C_{n,p,q,\mu}^r \|u\|_\gamma^\gamma + \frac{1}{\gamma} \|u\|_\gamma^\gamma \log \left(\|u\|_\gamma^\gamma \right) \geq 0. \tag{2.10}$$

It is noticed that for $r \in (0, p)$ and $p < \gamma$ then $\frac{\gamma - r}{p - r} p > \gamma$. And therefore, we deduce from (2.10) that there exists a positive constant R independent of u such that $\|u\|_\gamma \geq R > 0$ which implies

$$\|\nabla u\|_p \geq \frac{1}{S_{p,\gamma}} \|u\|_\gamma \geq \frac{R}{S_{p,\gamma}}. \tag{2.11}$$

Here $S_{p,\gamma}$ stands for the best constant in the Sobolev embedding $W_0^{1,p}(\Omega) \hookrightarrow L^\gamma(\Omega)$ with $0 < \gamma \leq p^* = \frac{np}{n-p}$ $(p < n)$. Thus, the proof follows from (2.9) and (2.11). □

Denote the nontrivial stationary solution of problem (1.7) by

$$E = \left\{ u \in W_0^{1,p}(\Omega) \setminus \{0\} : -\Delta_p u = (m' - 1) f_\gamma(u) : u|_{\partial\Omega} = 0 \right\},$$
$$E_d = \{ u \in E : J(u) = d \}.$$

Then, by virtue of critical point theory, it is not difficult to see that if $u \in E$ (in weak sense) then u is a nontrivial critical point of $J(u)$. Hence, we get

$$E_d = \{ u \in \mathcal{N} : J(u) = d \}. \tag{2.12}$$

As a consequence of Lemma 2.6, E_d is a nonempty set.

We now define stable set $\mathcal{W}$ and unstable set $\mathcal{U}$ as in [15,27].

$$\mathcal{W} = \left\{ u \in W_0^{1,p}(\Omega) : J(u) < d, I(u) > 0 \right\} \cup \{0\}, \tag{2.13}$$
$$\mathcal{U} = \left\{ u \in W_0^{1,p}(\Omega) : J(u) < d, I(u) < 0 \right\}. \tag{2.14}$$

By continuity of the functionals I and J on $W_0^{1,p}(\Omega)$, one has

$$\overline{\mathcal{W}} = \left\{ u \in W_0^{1,p}(\Omega) : J(u) \leq d, I(u) \geq 0 \right\} \quad \text{and} \quad \overline{\mathcal{U}} = \left\{ u \in W_0^{1,p}(\Omega) : J(u) \leq d, I(u) \leq 0 \right\}.$$

Some properties of $\overline{\mathcal{W}}$ and $\overline{\mathcal{U}}$ are listed below.

Lemma 2.7.

(i) *$\mathcal{W}$ is a bounded neighborhood of 0 in $W_0^{1,p}(\Omega)$, that is, there exist $0 < r_1 < r_2$ such that $B(0, r_1) \subset \mathcal{W} \subset B(0, r_2)$;*

(ii) $0 \notin \overline{\mathcal{U}}$;

(iii) *$E_d \subset \mathcal{N}$ and $\overline{\mathcal{W}} \cap \overline{\mathcal{U}} = E_d$.*

Proof. (i) Let $u \in \mathcal{W}$ with $u \neq 0$, then it follows from the definition of $\mathcal{W}$ and (2.5) that

$$\|\nabla u\|_p^p < \frac{p\gamma}{\gamma - p} d \quad \text{and} \quad \|u\|_\gamma^\gamma < \frac{\gamma^2}{m' - 1} d, \tag{2.15}$$

for $\gamma > p$. In the case $\gamma = p$, we also deduce from (2.5) that

$$\|u\|_p^p < \frac{p^2}{m' - 1} d \quad \text{and} \quad I(u) < pd. \tag{2.16}$$

On the other hand, by virtue of logarithmic Sobolev inequality, we get

$$I(u) \geq \left(1 - \frac{\mu(m'-1)}{p}\right) \|\nabla u\|_p^p + \frac{n(m'-1)}{p^2} \log\left(\frac{p\mu e}{n\mathcal{L}_p}\right) \|u\|_p^p - \frac{m'-1}{p} \|u\|_p^p \log\left(\|u\|_p^p\right).$$

It follows that

$$\left(1 - \frac{\mu(m'-1)}{p}\right) \|\nabla u\|_p^p \leq I(u) + \frac{n(m'-1)}{p^2} \log\left(\frac{n\mathcal{L}_p}{p\mu e}\right) \|u\|_p^p + \frac{m'-1}{p} \|u\|_p^p \log\left(\|u\|_p^p\right). \tag{2.17}$$

By choosing $\mu < \frac{p}{m'-1}$, it follows from (2.15)–(2.17) that $\|\nabla u\|_p^p < C_d$, where C_d independent of u. The remain part of (i) can be prove similar to Lions [20]. Hence, we possess (i).

(ii) By contradiction, we assume that $0 \in \overline{\mathcal{U}}$. Then there exists a sequence $\{u_n\} \in \mathcal{U}$ such that $u_n \to 0$ in $W_0^{1,p}(\Omega)$ as $n \to \infty$. It follows from (i) that $u_n \in \mathcal{W}$ for n sufficiently large. This contradicts to the fact that $\mathcal{W} \cap \mathcal{U} = \emptyset$.

(iii) It is clear that $E_d \subset \mathcal{N}$. We now let $u \in \overline{W} \cap \overline{\mathcal{U}}$, then $I(u) = 0$ and $J(u) \leq d$. Since (ii), we get $u \neq 0$ and therefore $u \in \mathcal{N}$. On the other hand, by variational characterization of d, one has $J(u) = d$. Thus $u \in E_d$. Conversely, if $u \in E_d$, then it follows from (2.12) that $u \in W_0^{1,p}(\Omega) \setminus \{0\}$ satisfying $I(u) = 0$ and $J(u) = d$. This implies $u \in \overline{W} \cap \overline{\mathcal{U}}$. The lemma has been proven. □

3 Main results

Firsly, we introduce the definitions of weak solutions to (1.7) and maximal existence time.

Definition 3.1. A function u is said to be a weak solution of problem (1.7) on $[0,T)$ if $u \in L^\infty(0,T;W_0^{1,p}(\Omega))$ is such that $\varphi(u) \in L^\infty(0,T;L^m(\Omega))$ with

$$\partial_t\varphi(u) \in L^{p'}\left(0,T;W^{-1,p'}(\Omega)\right) \quad \text{and} \quad \partial_t\left(|u|^{\frac{m'}{2}}\right) \in L^2(Q_T),$$

satisfies the initial value $u(0) = u_0$ and the equation (1.7) in a generalized sense, that is,

$$\int_0^T \langle \partial_t\varphi(u), v\rangle\, dt + \int_0^T \left\langle (\nabla u)^{(p-1)}, \nabla v\right\rangle dt = (m'-1) \int_0^T \langle f_\gamma(u), v\rangle\, dt, \tag{3.1}$$

for all $v \in L^p(0,T;W_0^{1,p}(\Omega))$.

Definition 3.2 (Maximal existence time)**.** Let u be a weak solution to problem (1.7). Then we define the maximal existence time $T_{\max}$ of u as follows:

- if $u := u(t)$ exists on $[0,T)$ for all $T > 0$, then $T_{\max} = +\infty$. In this case, we say that u is a global solution of (1.7);
- if there is $T > 0$ such that $u := u(t)$ exists on $[0,T)$, but it does not exist at $t = T$, then $T_{\max} = T$. In this case, we say that u is blow up at $t = T$.

We now give the existence and nonexistence of global weak solutions to (1.7) depending on parameters m, p and q.

Theorem 3.3. *Let $T > 0$, $u_0 \in W_0^{1,p}(\Omega)$ and let m, p be constants satisfying (1.4). Then we possess the following statements.*

(i) If $(m'-1)(q-1) < p-1$, then there exists a weak solution u to problem (1.7) on $[0, T_{\max})$ for $T_{\max} = T$ which satisfies $\partial_t\left(|u|^{\frac{m'}{2}}\right) \in L^2(Q_T)$ and the energy inequality

$$\int_0^t \|U_\tau(\tau)\|_2^2 \, d\tau + J(u(t)) \leq J(u_0), \quad \text{a.e. in } [0, T_{\max}), \tag{3.2}$$

where $U(t) = \frac{2\sqrt{m'-1}}{m'} |u(t)|^{\frac{m'}{2}}$.

(ii) If $(m'-1)(q-1) \geq p-1$ and $q > 2$ such that

$$(m'-1)(q-1) + 1 < p\left(1 + \frac{m'}{n}\right) \quad \text{if } p < n.$$

then there exists a weak solution u satisfying (3.2) on $[0, T_{\max})$ with $0 < T_{\max} < T$.

Next, we give similar results as in [15,27,31] on the existence and nonexistence of global solution when the initial data u_0 is in stable set $\mathcal{W}$ and unstable set $\mathcal{U}$.

Theorem 3.4 (Global existence for $J(u_0) < d$)**.** *Let m, p satisfy (1.4) and $q > 2$ such that*

$$p \leq (m'-1)(q-1) + 1 < p^* = \frac{np}{n-p} \quad \text{as } p < n, \tag{3.3}$$

and $u_0 \in \mathcal{W}$. Then the problem (1.7) admits a global weak solution $u \in L^\infty\left(0, T; W_0^{1,p}(\Omega)\right)$ with $\partial_t U \in L^2(Q_T)$ and $u(t) \in \overline{\mathcal{W}}$ for $t \in [0, T)$ for any $T > 0$. In addition, we have the following decay estimates:

$$\|\nabla u(t)\|_p \leq \|\nabla u_0\|_p \left(\frac{p}{m' + \omega(p-m')t}\right)^{\frac{1}{p-m'}} \quad \text{for } t \geq 0, \tag{3.4}$$

for some $\omega > 0$.

Theorem 3.5 (Blow up for $J(u_0) < d$)**.** *Let m, p satisfy (1.4) and $q > 2$ such that*

$$p \leq (m'-1)(q-1) + 1 < p^* \quad \text{as } p < n, \tag{3.5}$$

and $u_0 \in \mathcal{U}$. Then weak solution u of the problem (1.7) blows up in finite time, that is, there is T_* such that

$$\lim_{t \to T_*} \|\nabla u(t)\|_p^p = +\infty.$$

Remark 3.6. The results of Theorem 3.4 and 3.5 are still valid if we replace the initial value u_0 by $u(t_0)$ for some $t_0 \in [0, T_{\max})$. In Theorem 3.4, by assumption $u_0 \in \mathcal{W}$, we can relax the constraint on $\gamma < p\left(1 + \frac{m'}{n}\right)$ by $\gamma < p^*$.

Remark 3.7 (Sharp condition for $J(u_0) < d$)**.** Let m, p satisfy (1.4) and $q > 2$ such that

$$p \leq (m'-1)(q-1) + 1 < p^*,$$

and $u_0 \in W_0^{1,p}(\Omega) \setminus \{0\}$ with $J(u_0) < d$. Then problem 1.7 admits a global weak solution provided that $I(u_0) > 0$ and does not admit any global weak solution provided that $I(u_0) < 0$.

Finally, we have a threshold result on the existence and non-existence of global weak solution to (1.7) in the case $J(u_0) = d$.

Theorem 3.8. *Let m, p satisfy (1.4) and $q > 2$ such that*

$$p \le (m' - 1)(q - 1) + 1 < p^* = \frac{np}{n-p} \quad \text{as } p < n, \tag{3.6}$$

and $u_0 \in W_0^{1,p}(\Omega) \setminus \{0\}$ with $J(u_0) = d$. Then the local weak solution u of (1.7) is global provided that $I(u_0) > 0$ and blows up in finite time provided that $I(u_0) < 0$. Moreover, in the former case, there exists a positive constant ω_1 such that

$$\|\nabla u(t)\|_p \le \|\nabla u(t_1)\|_p \left(\frac{p}{m' + \omega_1 (p - m') t} \right)^{\frac{1}{p-m'}}, \quad \text{for } t \ge t_1, \tag{3.7}$$

for some $t_1 > 0$.

4 Proof of Theorem 3.3

In this section we prove the existence of weak solutions by Faedo–Galerkin method. The proof comprises of several steps in which we use the following well-known Gronwall–Bellman–Bihari integral inequality [1, p. 53].

Lemma 4.1 (Gronwall–Bellman–Bihari)**.** *Let $S(t)$ be a nonnegative continuous function such that*

$$S(t) \le C_1 + C_2 \int_0^t S^\kappa(s) ds,$$

where C_1, C_2 are positive constants. Then we get

(i) $S(t) \le \left[C_1^{1-\kappa} + (1-\kappa) C_2 t \right]^{\frac{1}{1-\kappa}}$ *for* $0 < \kappa < 1$;

(ii) $S(t) \le C_1 \exp\{C_2 t\}$ *for* $\kappa = 1$;

(iii) $S(t) \le C_1 \left[1 - (\kappa - 1) C_2 C_1^{\kappa-1} t \right]^{-\frac{1}{\kappa-1}}$ *for* $\kappa > 1$.

Step 1: Finite-dimensional approximations

Let $\{w_j\}$ be a system of basis functions in $W_0^{1,p}(\Omega)$ and define

$$V_k = \{w_1, w_2, \dots, w_k\}.$$

Let u_{0k} be an element of V_k such that

$$u_{0k} = \sum_{j=1}^{k} a_{jk} w_j \to u_0, \quad \text{in } W_0^{1,p}(\Omega), \tag{4.1}$$

as $k \to \infty$. We shall construct the approximate solutions $u_k(x,t)$ of the problem (1.1) by the form

$$u_k(t) = \sum_{j=1}^{k} \alpha_{kj}(t) w_j, \quad \text{for } k = 1, 2, \dots, \tag{4.2}$$

where the coefficients α_{kj} $(1 \leq j \leq k)$ satisfies the system of integro-differential equations

$$\left\langle \left(m'-1\right) |u_k(t)|^{m'-2} u_{kt}(t), w_i \right\rangle + \left\langle (\nabla u_k(t))^{(p-1)}, \nabla w_i \right\rangle \\ = \left(m'-1\right) \left\langle (u_k(t))^{(\gamma-1)} \log\left(|u_k(t)|\right), w_i \right\rangle, \tag{4.3}$$

for $i = 1, 2, \ldots, k$, with the initial conditions

$$\alpha_{kj}(0) = a_{kj}, \qquad j = 1, 2, \ldots, k. \tag{4.4}$$

In order to recognize that the system (4.3)–(4.4) has a local solution, for $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k) \in \mathbb{R}^k$, we set

- $\psi(\alpha) = (\psi_1(\alpha), \ldots, \psi_k(\alpha))^T$, with

$$\psi_i(\alpha) = \int_\Omega \left\{ \left(\sum_{j=1}^k \alpha_j w_j \right)^{(m'-1)} w_i \right\} dx;$$

- $\mathcal{B}(\alpha) = (\mathcal{B}_1(\alpha), \ldots, \mathcal{B}_k(\alpha))^T$, with

$$\mathcal{B}_i(\alpha) = \int_\Omega \left\{ \left(\sum_{j=1}^k \alpha_j \nabla w_j \right)^{(p-1)} \nabla w_i \right\} dx;$$

- $\mathcal{F}(\alpha) = (\mathcal{F}_1(\alpha), \ldots, \mathcal{F}_k(\alpha))^T$, with

$$\mathcal{F}_i(\alpha) = \left(m'-1\right) \int_\Omega \left\{ \left(\sum_{j=1}^k \alpha_j w_j \right)^{(\gamma-1)} \log\left(\left| \sum_{j=1}^k \alpha_j w_j \right| \right) \right\} w_i dx.$$

Then it is obvious that the system (4.3)–(4.4) can be rewritten as

$$\frac{d}{dt}\psi\left(\alpha_k(t)\right) + \mathcal{B}\left(\alpha_k(t)\right) = \mathcal{F}\left(\alpha_k(t)\right), \tag{4.5}$$

which is also equivalent to the integral equation

$$\psi\left(\alpha_k(t)\right) = \psi\left(\alpha_k(0)\right) - \int_0^t \left[-\mathcal{B}\left(\alpha_k(s)\right) + \mathcal{F}\left(\alpha_k(s)\right)\right] ds, \tag{4.6}$$

where $\alpha_k(t) = \left(\alpha_{k1}(t), \alpha_{k2}(t), \ldots, \alpha_{kk}(t)\right)^T$. The standard theory of ordinary differential and integral equations yields that there exists a positive $0 < T_k \leq T$ such that $\alpha_{kj} \in C^1([0, T_k])$, and therefore $u_k \in C^1([0, T_k]; W_0^{1,p}(\Omega))$.

Step 2: The fundamental priori estimates

In order to obtain the boundedness of the approximate solutions $\{u_k\}$, we need the following inequality.

Lemma 4.2. *Let $1 < m' < p < \infty$ and r be a constant such that*

$$p \le r < p\left(1+\frac{m'}{n}\right) \quad \text{if } p < n \quad \text{and} \quad p \le r \quad \text{if } p \ge n.$$

Then for each $\varepsilon > 0$, there exists a positive constant C_ε such that

$$\|v\|_r^r \le \varepsilon \|\nabla v\|_p^p + C_\varepsilon \left(\|v\|_{m'}^{m'}\right)^\kappa, \tag{4.7}$$

for all $v \in W_0^{1,p}(\Omega)$, where

$$\kappa = \frac{(1-\theta)\, r}{(1-\alpha)\, m'} > 1, \qquad \theta = \left(\frac{1}{m'} - \frac{1}{r}\right)\left(\frac{1}{m'} - \frac{1}{p^*}\right)^{-1}, \qquad \alpha = \frac{\theta r}{p}.$$

Proof. By virtue of Gagliardo–Nirenberg inequality, we have

$$\|v\|_r \le C \|\nabla v\|_p^\theta \|v\|_{m'}^{1-\theta}, \qquad \forall v \in W_0^{1,p}(\Omega),$$

where

$$\theta = \left(\frac{1}{m'} - \frac{1}{r}\right)\left(\frac{1}{m'} - \frac{1}{p^*}\right)^{-1}.$$

This implies

$$\|v\|_r^r \le C \left(\|\nabla v\|_p^p\right)^\alpha \left(\|v\|_{m'}^{m'}\right)^{\frac{(1-\theta) r}{m'}}, \quad \text{with } \alpha = \frac{\theta r}{p}.$$

Since $p \le r < p\,(1+m'/n)$, we get $\theta \le \alpha = \theta r/p < 1$. By virtue of Young's inequality, one has

$$\|v\|_r^r \le \varepsilon \|\nabla v\|_p^p + C_\varepsilon \left(\|v\|_{m'}^{m'}\right)^\kappa,$$

where $\kappa = \frac{(1-\theta) r}{(1-\alpha) m'} > 1$. The proof is complete. □

Multiplying both sides of (4.3) by $\alpha_{ki}(t)$ and taking the sum over $i = 1, 2, \ldots, k$, and then integrating with respect to time variable from 0 to t, one has

$$\|u_k(t)\|_{m'}^{m'} = \|u_{0k}\|_{m'}^{m'} - \int_0^t I\left(u_k(\tau)\right) d\tau, \tag{4.8}$$

where

$$I\left(u_k(t)\right) = \|\nabla u_k(t)\|_p^p - \left(m'-1\right) \int_\Omega |u_k(t)|^\gamma \log\left(|u_k(t)|\right) dx. \tag{4.9}$$

We now estimate $I\left(u_k(t)\right)$. By elementary inequality, we get the following estimate for $\beta > 0$ sufficiently small

$$\begin{aligned}\int_\Omega |u_k(t)|^\gamma \log|u_k(t)|\, dx &= \int_{\{|u_k(t)|\le 1\}} |u_k(t)|^\gamma \log|u_k(t)|\, dx + \int_{\{|u_k(t)|>1\}} |u_k(t)|^\gamma \log|u_k(t)|\, dx \\ &\le e^{-1}|\Omega| + \frac{1}{\beta}\int_\Omega |u_k(t)|^{\gamma+\beta} dx. \end{aligned} \tag{4.10}$$

We now consider the two following cases:
CASE 1: $(m'-1)(q-1) < p-1$. In this case, we have $\gamma < p$. By virtue of Young inequality and Poincaré inequality, we get

$$\int_\Omega |u_k(t)|^\gamma \log |u_k(t)|\, dx \le \varepsilon \|\nabla u_k(t)\|_p^p + C(\Omega,\varepsilon), \tag{4.11}$$

with $\varepsilon > 0$. It follows from (4.9) and (4.11) that

$$I(u_k(t)) \ge \big(1-(m'-1)\varepsilon\big) \|\nabla u_k(t)\|_p^p - C(\Omega,\varepsilon). \tag{4.12}$$

By choosing $\varepsilon = \frac{p-1}{p(m'-1)}$, we deduce from (4.1), (4.8) and above inequality that

$$S_k(t) := \|u_k(t)\|_{m'}^{m'} + \frac{1}{p}\int_0^t \|\nabla u_k(\tau)\|_p^p\, d\tau \le C_T, \qquad \forall t \in [0,T],\ \forall k \in \mathbb{N}. \tag{4.13}$$

CASE 2: $(m'-1)(q-1) \ge p-1$ and $(m'-1)(q-1)+1 < p\big(1+\frac{m'}{n}\big)$. If this is the case, then we have $p \le \gamma < p\big(1+\frac{m'}{n}\big)$. By Lemma 4.2 and (4.10), we derive that

$$\int_\Omega |u_k(t)|^\gamma \log |u_k(t)|\, dx \le \varepsilon \|\nabla u_k(t)\|_p^p + C(\varepsilon)\left(\|u_k(t)\|_{m'}^{m'}\right)^\kappa + C(\Omega,\varepsilon), \tag{4.14}$$

where $\kappa > 1$ and $\varepsilon > 0$, which implies

$$I(u_k(t)) \ge \big(1-(m'-1)\varepsilon\big) \|\nabla u_k(t)\|_p^p - C(\varepsilon)\left(\|u_k(t)\|_{m'}^{m'}\right)^\kappa - C(\Omega,\varepsilon). \tag{4.15}$$

By choosing $\varepsilon = \frac{p-1}{p(m'-1)}$, it follows from (4.1), (4.8) and (4.15) that

$$S_k(t) \le C_1 + C_2\int_0^t S_k^\kappa(\tau)d\tau, \qquad \forall t \in [0,T], \tag{4.16}$$

where $\kappa > 1$ and C_1, C_2 are positive constants independence of k, and

$$S_k(t) = \|u_k(t)\|_{m'}^{m'} + \frac{1}{p}\int_0^t \|\nabla u_k(\tau)\|_p^p\, d\tau. \tag{4.17}$$

By virtue of Gronwall–Bellman–Bihari integral inequality, Lemma 4.1, there exists a constant $T^* = 1/2(\kappa-1)C_2C_1^{\kappa-1} \in (0,T)$ such that

$$S_k(t) \le C_{T^*}, \qquad \forall t \in [0,T^*],\ \forall k \in \mathbb{N}. \tag{4.18}$$

Now, by multiplying the i^{th} equation of (4.3) by $\alpha'_{ki}(t)$, summing up with respect to i and integrating with respect to time variable from 0 to t, we obtain

$$\int_0^t \|U_{k\tau}(\tau)\|_{L^2(\Omega)}^2\, d\tau + J(u_k(t)) \le J(u_{0k}), \qquad \forall t \in [0,T], \tag{4.19}$$

where $U_k(t) = \frac{2\sqrt{m'-1}}{m'}|u_k(t)|^{\frac{m'}{2}}$. Thanks to (4.1) and the continuity of J, there is a positive constant C such that

$$J(u_{0k}) \le C, \qquad \forall k \in \mathbb{N}. \tag{4.20}$$

We now estimate $J(u_k(t))$. It is worth noting that

$$J(u_k(t)) = \left(\frac{1}{p} - \frac{1}{\gamma}\right) \|\nabla u_k(t)\|_p^p + \frac{m'-1}{\gamma^2} \|u_k(t)\|_\gamma^\gamma + \frac{1}{\gamma} I(u_k(t)).$$

On the other hand, it follows from (4.12)–(4.13) and (4.15)–(4.18) that

$$I(u_k(t)) \geq \left(1 - \varepsilon(m'-1)\right) \|\nabla u_k(t)\|_p^p - C,$$

for sufficiently small $\varepsilon > 0$. Hence, we get

$$J(u_k(t)) \geq \left(\frac{1}{p} - \frac{\varepsilon(m'-1)}{\gamma}\right) \|\nabla u_k(t)\|_p^p + \frac{m'-1}{\gamma^2} \|u_k(t)\|_\gamma^\gamma - C. \tag{4.21}$$

It follows from (4.19)-(4.21) that

$$\int_0^t \|U_{k\tau}(\tau)\|_2^2 d\tau + \left(\frac{1}{p} - \frac{\varepsilon(m'-1)}{\gamma}\right) \|\nabla u_k(t)\|_p^p + \frac{m'-1}{\gamma^2} \|u_k(t)\|_\gamma^\gamma \leq C, \tag{4.22}$$

for sufficiently small $\varepsilon > 0$.

Step 3: Passage to the limit

In this section, we use some compactness results which is given by Matas and Merker [23].

Lemma 4.3 ([23]). *Let m, p satisfy (1.4), then we have*

(i) the map $\varphi : W_0^{1,p}(\Omega) \cap L^{m'}(\Omega) \to L^m(\Omega')$ defined by $\varphi(u) = u^{(m'-1)}$ is compact for any arbitrary bounded subdomain $\Omega' \subset \Omega$.

(ii) Let $\{u_k\} \subset L^p\left(0, T; W_0^{1,p}(\Omega)\right) \cap L^\infty\left(0, T; L^{m'}(\Omega)\right)$ be the sequence of weak solutions of projected equations. Then $\{\varphi(u_k)\}$ is relatively compact in $L^1(0, T; L^m(\Omega))$.

From the priori estimates devired above (see (4.13), (4.18) and (4.22)), we deduce a subsequence that still denotes as $\{u_k\}$ such that

$$u_k \to u \quad \text{weakly in } L^p\left(0, T; W_0^{1,p}(\Omega)\right), \tag{4.23}$$

$$u_k \to u \quad \text{weakly star in } L^\infty\left(0, T; W_0^{1,p}(\Omega) \cap L^{m'}(\Omega)\right), \tag{4.24}$$

$$\frac{d}{dt} U_k \to \left(\frac{d}{dt} U\right)_{ex} \quad \text{weakly in } L^2\left(0, T; L^2(\Omega)\right), \tag{4.25}$$

$$\varphi(u_k) \to (\varphi(u))_{ex} \quad \text{weakly star in } L^\infty\left(0, T; L^m(\Omega)\right), \tag{4.26}$$

$$\Delta_p u_k \to (\Delta_p u)_{ex} \quad \text{weakly in } L^{p'}\left(0, T; W^{-1,p'}(\Omega)\right). \tag{4.27}$$

It is obviously to deduce from (4.23)–(4.25) that $\left(\frac{d}{dt} U\right)_{ex} = \frac{d}{dt} U$. By virtue of Lemma 4.3, it follows from (4.23)–(4.24) that $\varphi(u_k)$ is bounded in $L^\infty\left(0, T; L^m(\Omega)\right)$ and is relatively compact in $L^1\left(0, T; L^m(\Omega)\right)$. By monotone operator theory, using similar arguments as in [23], we get $(\varphi(u))_{ex} = \varphi(u)$ and

$$\varphi(u_k) \longrightarrow \varphi(u) \quad \text{strongly in } L^1\left(0, T; L^m(\Omega)\right) \text{ and a.e. in } Q_T = \Omega \times [0, T]. \tag{4.28}$$

This implies

$$u_k(x,t) \to u(x,t) \text{ a.e. in } Q_T \quad \text{which implies} \quad f_\gamma(u_k(x,t)) \to f_\gamma(u(x,t)) \text{ a.e. in } Q_T. \tag{4.29}$$

On the other hand, direct computation gives us

$$\begin{aligned}\int_\Omega |f_\gamma(u_k(t))|^{\gamma'} dx &= \int_{|u_k(t)|\le 1} |f_\gamma(u_k(t))|^{\gamma'} dx + \int_{|u_k(t)|>1} |f_\gamma(u_k(t))|^{\gamma'} dx \\ &\le e^{-\gamma'} |\Omega| + C \int_\Omega |u_k(t)|^{\gamma+\varepsilon\gamma'} dx.\end{aligned}$$

By the Poincaré inequality and Lemma 4.2, it is not difficult to see that

$$\int_\Omega |f_\gamma(u_k(t))|^{\gamma'} dx \le C_{T,\Omega} \quad \text{for } \gamma > 1. \tag{4.30}$$

Combining (4.29) and (4.30), we get

$$f_\gamma(u_k) \longrightarrow f_\gamma(u) \quad \text{weakly star in } L^\infty\left(0,T;L^{\gamma'}(\Omega)\right). \tag{4.31}$$

Let $k \to \infty$ in (4.2)–(4.3), we obtain

$$\int_0^T \langle \partial_t \varphi(u), w\rangle\, dt + \int_0^T \left\langle (\nabla u)^{p-1}, w\right\rangle dt = (m'-1) \int_0^T \langle f_\gamma(u), w\rangle\, dt, \tag{4.32}$$

for all $w \in L^p(0,T;W_0^{1,p}(\Omega))$.

Moreover, if $m' \ge 2$, then we have $1 < \frac{m'}{m'-1} \le 2$ and $\partial_t\varphi(u) \in L^{\frac{m'}{m'-1}}(Q_T)$, since

$$\begin{aligned}\int_\Omega |\partial_t\varphi(u(t))|^{\frac{m'}{m'-1}} dx &= \int_\Omega \left((m'-1)\,|u(t)|^{m'-2} u_t(t)\right)^{\frac{m'}{m'-1}} dx \\ &= \left(\frac{m'\sqrt{m'-1}}{2}\right)^{\frac{m'}{m'-1}} \int_\Omega |U_t(t)|^{\frac{m'}{m'-1}} |u(t)|^{\frac{m'(m'-2)}{2(m'-1)}} dx \\ &\le \left(\frac{m'\sqrt{m'-1}}{2}\right)^{\frac{m'}{m'-1}} \left(\int_\Omega |U_t(t)|^2 dx\right)^{\frac{m'}{2(m'-1)}} \left(\int_\Omega |u(t)|^{m'} dx\right)^{\frac{m'-2}{2(m'-1)}} \\ &= \left(\frac{m'\sqrt{m'-1}}{2}\right)^{\frac{m'}{m'-1}} \left(\|U_t(t)\|_2^2\right)^{\frac{m'}{2(m'-1)}} \left(\|u(t)\|_{m'}^{m'}\right)^{\frac{m'-2}{2(m'-1)}} \\ &\le C\left(\|U_t(t)\|_2^2 + \|u(t)\|_{m'}^{m'}\right).\end{aligned}$$

Here we use the well-known Hölder and Young inequalities. If $1 < m' < 2$, then we have $u_t \in L^{m'}(Q_T)$,

$$\begin{aligned}\int_\Omega |u_t(t)|^{m'} dx &= \left(\frac{m'}{2\sqrt{m'-1}}\right)^{m'} \int_\Omega (|U_t(t)|)^{m'} |u(t)|^{\frac{(2-m')m'}{2}} dx \\ &\le \left(\frac{m'}{2\sqrt{m'-1}}\right)^{m'} \left(\int_\Omega |U_t(t)|^2 dx\right)^{\frac{m'}{2}} \left(\int_\Omega |u(t)|^{m'} dx\right)^{\frac{2-m'}{2}} \\ &\le \left(\frac{m'}{2\sqrt{m'-1}}\right)^{m'} \left(\frac{m'}{2}\int_\Omega |U_t(t)|^2 dx + \frac{2-m'}{2}\int_\Omega |u(t)|^{m'} dx\right).\end{aligned}$$

Step 4: Energy estimate

Similar to the method in [20, 26, 31], let Θ be the function which lies in $C[0,T]$ and is non-negative. We deduce from (4.19) that

$$\int_0^T \Theta(t)\int_0^t \|U_{k\tau}(\tau)\|_2^2\,d\tau dt + \int_0^T J\left(u_k(t)\right)\Theta(t)dt = \int_0^T J\left(u_{0k}\right)\Theta(t)dt.$$

Let $k\to\infty$, the right-hand side of this equality tends to $\int_0^T J\left(u_0\right)\Theta(t)dt$ and using the lower semi-continuous with respect to the weak topology of $L^p\left(0,T;W_0^{1,p}(\Omega)\right)$ and $L^2\left(Q_T\right)$, we get

$$\int_0^T \Theta(t)\int_0^t \|U_{\tau}(\tau)\|_2^2\,d\tau dt + \int_0^T J\left(u(t)\right)\Theta(t)dt \le \int_0^T J\left(u_0\right)\Theta(t)dt.$$

Since Θ is arbitrary, this inequality implies (3.2).

5 Proof of Theorem 3.4

Step 1: Global existence

As in the proof of Theorem 3.3, since $u_0\in\mathcal{W}$, we can find a sequence of Faedo–Galerkin approximation solutions $\{u_k\}$ such that

$$u_k(0) = u_{0k}\to u_0 \quad \text{strongly in } W_0^{1,p}\left(\Omega\right), \tag{5.1}$$

and satisfies the following identities

$$\frac{d}{dt}\|u_k(t)\|_{m'}^{m'} = -I(u_k(t)) \quad \text{and} \quad \int_0^t \|U_{k\tau}(\tau)\|_2^2\,d\tau + J(u_k(t)) = J(u_{0k}), \quad 0\le t<T_{\max}. \tag{5.2}$$

From (5.1) and the continuity of J, it follows from (5.2) that

$$\int_0^t \|U_{k\tau}\left(\tau\right)\|_2^2\,d\tau + J\left(u_k(t)\right) = J\left(u_{0k}\right) < d, \qquad 0\le t<T_{\max}. \tag{5.3}$$

Next, we shall show that $u_k(t)\in\mathcal{W}$ for all $t\in[0,T_{\max})$ for k sufficiently large. Indeed, by contradiction, we assume that there exists $t_0\in(0,T_{\max})$ such that $u_k(t)\in\mathcal{W}$ for all $t\in[0,t_0)$ and $u_k(t_0)\in\partial\mathcal{W}$, that is,

$$J\left(u_k(t_0)\right) = d \quad \text{or} \quad I\left(u_k(t_0)\right) = 0.$$

On the other hand, thanks to (5.3), we must have $I\left(u_k(t_0)\right)=0$ which implies $u_k(t_0)\in\mathcal{N}$ and therefore

$$J\left(u_k(t_0)\right) \ge \inf_{u\in\mathcal{N}} J(u) = d.$$

This contradicts to (5.3). Hence, we get $u_k(t)\in\mathcal{W}$ for all $t\in[0,T_{\max})$. From this fact and (5.2), we arrive at

$$\frac{d}{dt}\|u_k(t)\|_{m'}^{m'} = -I\left(u_k(t)\right)\le 0.$$

On the other hand, by virtue of Lemma 2.7, one has

$$u_k \quad \text{is bounded in} \quad L^\infty\left(0, T_{\max}; W_0^{1,p}(\Omega)\right). \tag{5.4}$$

In addition, since $I(u_k(t)) \geq 0$, we deduce from (2.5) that

$$J(u_k(t)) \geq \left(\frac{1}{p} - \frac{1}{\gamma}\right) \|\nabla u_k(t)\|_p^p + \frac{m'-1}{\gamma^2} \|u_k(t)\|_\gamma^\gamma.$$

Hence, (5.3) leads to

$$\int_0^t \|U_{k\tau}(\tau)\|_2^2 d\tau + \left(\frac{1}{p} - \frac{1}{\gamma}\right) \|\nabla u_k(t)\|_p^p + \frac{m'-1}{\gamma^2} \|u_k(t)\|_\gamma^\gamma < d.$$

This inequality allows us to take $T_{\max} = T$ for arbitrary $T > 0$. The rest of the proof is similar to the proof of Theorem 3.3. Hence, u is a global solution of (1.7) and $u(t) \in \overline{\mathcal{W}}$ for $t \geq 0$.

Step 2: Decay estimates

We shall need the following lemma.

Lemma 5.1 (see [22]). *Let $f : \mathbb{R}^+ \to \mathbb{R}^+$ be a nonincreasing function and σ is a nonnegative constant such that*

$$\int_t^{+\infty} f^{1+\sigma}(s) ds \leq \frac{1}{\omega} f^\sigma(0) f(t), \qquad \forall t \geq 0.$$

Then we have

(a) $f(t) \leq f(0)e^{1-\omega t}$, for all $t \geq 0$, whenever $\sigma = 0$,

(b) $f(t) \leq f(0)\left(\frac{1+\sigma}{1+\omega\sigma t}\right)^{1/\sigma}$, for all $t \geq 0$, whenever $\sigma > 0$.

We first construct subsets of $\overline{\mathcal{W}}$ which are invariant under the flow of (1.7). For any $\varepsilon \in (0, d)$, let

$$\overline{\mathcal{W}}_\varepsilon := \left\{u \in \overline{\mathcal{W}} : J(u) \leq d - \varepsilon\right\}.$$

Since the boundedness of $\mathcal{W}$, we get immediately that for any $\varepsilon \in (0, d)$, the set $\overline{\mathcal{W}}_\varepsilon$ is closed and bounded. In addition, the invariant of $\overline{\mathcal{W}}_\varepsilon$ under the flow of (1.7) is given by the following lemma which its proof is just a consequence of Step 1.

Lemma 5.2. *Suppose parameters m, p and q satisfy conditions in Theorem 3.4. Furthermore, assume that $\varepsilon \in (0, d)$ and $u_0 \in \overline{\mathcal{W}}_\varepsilon$. Then the local solution $u(t)$ of (1.7) is global and $u(t) \in \overline{\mathcal{W}}_\varepsilon$ for $t \geq 0$.*

Since $u_0 \in \mathcal{W} \subset \overline{\mathcal{W}}_\varepsilon$ for $\varepsilon > 0$, it follows from Lemma 5.2 that

$$u(t) \in \overline{\mathcal{W}}_\varepsilon \quad \text{for } t \geq 0.$$

It is worth noting that if there exists first $T > 0$ such that $I(u(T)) = 0$, then we get a contradiction

$$d = \inf_{u \in \mathcal{N}} J(u) \leq J(u(T)) \leq d - \varepsilon.$$

Hence, one must have $I(u(t)) > 0$ for $t > 0$. On the other hand, by Lemma 2.5, there exists $\lambda_* > 1$ such that $I(\lambda_* u(t)) = 0$ which implies

$$\begin{aligned} d \le J(\lambda_* u(t)) &= \left(\frac{1}{p} - \frac{1}{\gamma}\right) \|\nabla (\lambda_* u(t))\|_p^p + \frac{1}{\gamma^2} \|\lambda_* u(t)\|_\gamma^\gamma \\ &\le \lambda_*^\gamma \left(\left(\frac{1}{p} - \frac{1}{\gamma}\right) \|\nabla u(t)\|_p^p + \frac{1}{\gamma^2} \|u(t)\|_\gamma^\gamma \right). \end{aligned} \tag{5.5}$$

While $I(u(t)) > 0$ also implies

$$\left(\frac{1}{p} - \frac{1}{\gamma}\right) \|\nabla u(t)\|_p^p + \frac{1}{\gamma^2} \|u(t)\|_\gamma^\gamma \le J(u(t)) \le J(u_0) < d. \tag{5.6}$$

Combining (5.5) and (5.6), we obtain $\lambda_*^\gamma \ge d/J(u_0) > 1$. By this fact and the following identity

$$\begin{aligned} \lambda_*^p I(u(t)) &= I(\lambda_* u(t)) + \left(\lambda_*^\gamma - \lambda_*^p\right) \|\nabla u(t)\|_p^p + \left(m' - 1\right) \lambda_*^\gamma \log \lambda_* \|u(t)\|_\gamma^\gamma \\ &= \left(\lambda_*^\gamma - \lambda_*^p\right) \|\nabla u(t)\|_p^p + \left(m' - 1\right) \lambda_*^\gamma \log \lambda_* \|u(t)\|_\gamma^\gamma , \end{aligned}$$

we get

$$\begin{aligned} I(u(t)) &\ge \left(\lambda_*^{\gamma-p} - 1\right) \|\nabla u(t)\|_p^p + \left(m' - 1\right) \lambda_*^{\gamma-p} \log \lambda_* \|u(t)\|_\gamma^\gamma \\ &\ge \left[\left(\frac{d}{J(u_0)}\right)^{\frac{\gamma-p}{\gamma}} - 1 \right] \|\nabla u(t)\|_p^p + \left(m' - 1\right) \left(\frac{d}{J(u_0)}\right)^{\frac{\gamma-p}{\gamma}} \log \left(\frac{d}{J(u_0)}\right)^{\frac{1}{\gamma}} \|u(t)\|_\gamma^\gamma . \end{aligned} \tag{5.7}$$

On the other hand, by virtue of parametric form of logarithmic Gagliardo–Nirenberg inequality, we obtain

$$\begin{aligned} I(u(t)) \ge &\left[1 - (m' - 1)\frac{\mu r}{\gamma p}\right] \|\nabla u(t)\|_p^p \\ &- \frac{m' - 1}{\gamma} \|u(t)\|_\gamma^\gamma \left[\mu \frac{p - r}{p} \left(\|u(t)\|_\gamma^\gamma\right)^{\frac{r(\gamma-p)}{\gamma(p-r)}} + C^r_{m,p,\gamma,\mu} + \log \left(\|u(t)\|_\gamma^\gamma\right) \right], \end{aligned} \tag{5.8}$$

with noting that $\gamma \ge p$ since $(m' - 1)(q - 1) \ge p - 1$. By choosing $\mu = \gamma p / 2r(m' - 1)$ and $r = p/2$, we derive from (5.6) and (5.8) that

$$I(u(t)) \ge \frac{1}{2} \|\nabla u(t)\|_p^p - \frac{m' - 1}{\gamma} C_1 \|u(t)\|_\gamma^\gamma . \tag{5.9}$$

It follows from (5.7) and (5.9) that there is a positive constant C such that

$$I(u(t)) \ge C \left(\|\nabla u(t)\|_p^p + \|u(t)\|_\gamma^\gamma \right). \tag{5.10}$$

From the energy identity

$$\int_t^T I(u(\tau))\, d\tau = \|u(t)\|_{m'}^{m'} - \|u(T)\|_{m'}^{m'} ,$$

and Poincaré's inequality, we deduce from (5.10) that

$$\int_t^T \|\nabla u(\tau)\|_p^p \, d\tau \le \frac{1}{\omega} \|\nabla u_0\|_p^{p-m'} \|\nabla u(t)\|_p^{m'} \quad \text{for} \quad 0 \le t \le T,$$

where $\omega = CS_{p,m'}^{-m'}\|\nabla u_0\|_p^{p-m'} > 0$. Let $T \to +\infty$ and apply the Lemma 5.1 with $\sigma = (p-m')/m'$ and $f(t) = \|\nabla u(t)\|_p^{m'}$, we get

$$f(t) \le f(0)\left(\frac{p}{m' + \omega(p-m')t}\right)^{\frac{m'}{p-m'}} \quad \text{for } t \ge 0.$$

The proof is complete.

6 Proof of Theorem 3.5

First, we need the following lemma which its proof is similar to [15,26,27,31]. So we omit it.

Lemma 6.1. *Let m, p and q satisfy conditions in Theorem 3.5 and $u_0 \in \mathcal{U}$, then weak solution $u(t)$ to problem (1.7) satisfies*

$$u(t) \in \mathcal{U}, \quad \text{for } t \in [0, T_{\max}).$$

We next give the proof of Theorem 3.5. By contradiction arguments, we assume that $u(t)$ is global solution, that is, $T_{\max} = +\infty$. Then we define the function $F : [0, +\infty) \to \mathbb{R}^+$ by

$$F(t) = \int_0^t \|u(\tau)\|_{m'}^{m'} d\tau. \tag{6.1}$$

A direct computation yields

$$F'(t) = \int_\Omega |u(t)|^{m'} dx, \quad \text{and} \quad F''(t) = m'\int_\Omega |u(t)|^{m'-2} u(t)u_t(t)dx = -\frac{m'}{m'-1} I(u(t)). \tag{6.2}$$

Since $\gamma \ge p$ and $u_0 \in \mathcal{U}$, by Lemma 6.1, we get $u(t) \in \mathcal{U}$ for all $t \ge 0$ which implies $u(t) \ne 0$ and $I(u(t)) < 0$ due to Lemma 2.7. On the other hand, by virtue of Lemma 2.5, there is $\lambda_* \in (0,1)$ such that $I(\lambda_* u(t)) = 0$. As a consequence, one has

$$\begin{aligned} d \le J(\lambda_* u(t)) &= \left(\frac{1}{p} - \frac{1}{\gamma}\right)\lambda_*^p \|\nabla u(t)\|_p^p + \frac{m'-1}{\gamma^2}\lambda_*^\gamma \|u(t)\|_\gamma^\gamma \\ &< \left(\frac{1}{p} - \frac{1}{\gamma}\right)\|\nabla u(t)\|_p^p + \frac{m'-1}{\gamma^2}\|u(t)\|_\gamma^\gamma, \quad \text{when } \gamma \ge p. \end{aligned} \tag{6.3}$$

Combining (2.5), (3.2), (6.2) and (6.3), we obtain

$$\begin{aligned} F''(t) &= -\frac{m'\gamma}{m'-1} J(u(t)) + \frac{m'\gamma}{m'-1}\left[\left(\frac{1}{p} - \frac{1}{\gamma}\right)\|\nabla u(t)\|_p^p + \frac{m'-1}{\gamma^2}\|u(t)\|_\gamma^\gamma\right] \\ &\ge \frac{m'\gamma}{m'-1}\int_0^t \|U_\tau(\tau)\|_2^2 d\tau + \frac{m'\gamma}{m'-1}(d - J(u_0)). \end{aligned} \tag{6.4}$$

Since $J(u_0) \le d$, this implies that $F'(t)$ is an increasing function. In addition, we have

$$\begin{aligned} 0 \le F'(t) - F'(0) &= \|u(t)\|_{L^{m'}}^{m'} - \|u_0\|_{L^{m'}}^{m'} = m'\int_0^t \int_\Omega |u(\tau)|^{m'-2} u(\tau)u_\tau(\tau)dxd\tau \\ &= \frac{m'}{\sqrt{m'-1}}\int_0^t \int_\Omega |u(\tau)|^{\frac{m'}{2}} U_\tau(\tau)\, dxd\tau \\ &\le \frac{m'}{\sqrt{m'-1}}\int_0^t \|u(\tau)\|_{m'}^{\frac{m'}{2}} \|U_\tau(\tau)\|_2 d\tau \\ &\le \frac{m'}{\sqrt{m'-1}}\left(\int_0^t \|u(\tau)\|_{m'}^{m'} d\tau\right)^{\frac{1}{2}}\left(\int_0^t \|U_\tau(\tau)\|_2^2 d\tau\right)^{\frac{1}{2}}. \end{aligned} \tag{6.5}$$

We deduce from (6.1), (6.4) and (6.5) that

$$\begin{aligned} F(t)F''(t) &\geq \frac{\gamma m'}{m'-1}\int_0^t \|U_\tau(\tau)\|_2^2\,d\tau \int_0^t \|u(\tau)\|_{m'}^{m'}\,d\tau + \frac{\gamma m'}{m'-1}\left(d - J\left(u_0\right)\right)F(t) \\ &\geq \frac{\gamma}{m'}\left(F'(t)-F'(0)\right)^2 + \frac{\gamma m'}{m'-1}\left(d-J\left(u_0\right)\right)F(t). \end{aligned} \tag{6.6}$$

In addition, it follows from (6.5) that

$$F(t) \geq t\,\|u_0\|_{m'}^{m'} \quad \text{for all } t \geq 0. \tag{6.7}$$

We now fix $t_0 > 0$ and define the function

$$G(t) = F(t) + (T-t)\,\|u_0\|_{m'}^{m'} \quad \text{for } t \in [0,T].$$

Here T is chosen sufficiently large. Then we can derive form (6.6)–(6.7) that

$$G(t)G''(t) - \frac{\gamma}{m'}\left(G'(t)\right)^2 \geq \frac{\gamma m'}{m'-1}\left(d - J(u_0)\right)\|u_0\|_{m'}^{m'}\,t_0, \qquad \forall t \in [t_0,T]\,. \tag{6.8}$$

Since $m' < p \leq \gamma$, we have $\gamma/m' > 1$. By setting $y(t) = G(t)^{-\frac{\gamma-m'}{m'}}$, this inequality implies y to be a concave function on $[t_0,T]$. Hence, $y(t)$ reaches zero in finite time, that is, there is $T_* > 0$ such that $\lim_{t\to T_*^-} G(t) = +\infty$. As a consequence, we get

$$\lim_{t\to T_*^-} \|u(t)\|_{m'}^{m'} = +\infty \quad \text{and} \quad \lim_{t\to T_*^-} \|\nabla u(t)\|_p^p = +\infty.$$

The proof is complete.

7 Proof of Theorem 3.8

We shall need the following lemma.

Lemma 7.1. *Let $u_0 \in W_0^{1,p}(\Omega)\setminus\{0\}$ with $J(u_0) = d$. Then weak solution u to problem (1.7) satisfies:*

(i) $I\left(u(t)\right) > 0$ for $0 \leq t < T_{\max}$ provides that $I(u_0) > 0$;

(ii) $I\left(u(t)\right) < 0$ for $0 \leq t < T_{\max}$ provides that $I(u_0) < 0$.

Proof. (i) By contradiction, assume that there exists $t_1 \in (0, T_{\max})$ such that

$$I\left(u(t)\right) > 0 \quad \text{for } 0 \leq t < t_1 \quad \text{and} \quad I\left(u(t_1)\right) = 0.$$

By the fact that

$$\left\langle \frac{d}{dt}\varphi(u(t)), u(t)\right\rangle = -I\left(\left(u(t)\right)\right) < 0 \quad \text{for } 0 \leq t < t_1.$$

and the estimate

$$\begin{aligned} \left|\left\langle \frac{d}{dt}\varphi(u(t)), u(t)\right\rangle\right| &= \frac{m'\sqrt{m'-1}}{2}\int_\Omega |U_t(t)|\,|u(t)|^{\frac{m'}{2}}\,dx \\ &\leq \frac{m'\sqrt{m'-1}}{2}\,\|U_t(t)\|_2\,\|u(t)\|_{m'}^{\frac{m'}{2}}, \end{aligned}$$

we obtain

$$\|u(t)\|_{m'} > 0 \quad \text{and} \quad \|U_t(t)\|_2 > 0, \qquad 0 \le t < t_1.$$

Hence, $\int_0^t \|U_\tau(\tau)\|_2^2 \, d\tau$ is strictly positive for $0 < t \le t_1$ and

$$J(u(t)) < J(u_0) - \int_0^t \|U_\tau(\tau)\|_2^2 \, d\tau < d, \quad \text{for } 0 < t \le t_1. \tag{7.1}$$

On the other hand, by analogous arguments as in the proof of Lemma 2.6, we deduce from $I(u(t_1)) = 0$ that $\|\nabla u(t_1)\|_p > 0$, that is, $u(t_1) \in \mathcal{N}$ which implies a contradiction with (7.1)

$$d = \inf_{u \in \mathcal{N}} J(u) \le J(u(t_1)).$$

(ii) Also using contradiction arguments, suppose that there exists $t_2 \in (0, T_{\max})$ such that

$$I(u(t)) < 0 \quad \text{for } 0 \le t < t_2 \quad \text{and} \quad I(u(t_2)) = 0.$$

Similar to (i), we get $\int_0^t \|U_\tau(\tau)\|_2^2 \, d\tau$ is strictly positive for $0 < t \le t_2$ and

$$J(u(t)) < J(u_0) - \int_0^t \|U_\tau(\tau)\|_2^2 \, d\tau < d, \quad \text{for } 0 < t \le t_2.$$

Using again the implementation $I(u(t_2)) = 0$ implies $\|\nabla u(t_2)\|_p > 0$, we get $u(t_2) \in \mathcal{N}$ and a contradiction

$$d = \inf_{u \in \mathcal{N}} J(u) \le J(u(t_2)) < d.$$

Hence, the proof is complete. □

We now prove the Theorem 3.8 by two following steps.

Step 1: Global existence and decay estimate for $J(u_0) = d$ and $I(u_0) > 0$

Take a sequence of real numbers $\{\lambda_k\} \subset (0,1)$ satisfying $\lim_{k\to\infty} \lambda_k = 1$. By Lemma 2.5, there exists $\lambda_* := \lambda_*(u_0) \ge 1$ such that $I(\lambda_* u_0) = 0$ and

$$I(\lambda_k u_0) > 0 \quad \text{and} \quad J(\lambda_k u_0) < J(u_0) \le d.$$

So, by putting $u_{0k} = \lambda_k u_0 \in W_0^{1,p}(\Omega) \setminus \{0\}$, then we get $u_{0k} \in \mathcal{W}$. For each k, we now consider the initial-boundary value problem

$$\begin{cases} \partial_t \varphi(u) - \Delta_p u = (m'-1) f_\gamma(u), & x \in \Omega, t > 0, \\ u(x,t) = 0, & x \in \partial\Omega, t > 0, \\ u(x,0) = u_{0k}(x), & x \in \Omega. \end{cases} \tag{7.2}$$

By virtue of Lemma 5.2, we obtain a sequence of global weak solutions $\{u_k\}$ to (7.2) satisfying $u_k(t) \in \overline{\mathcal{W}}$ and

$$\frac{d}{dt} \|u_k(t)\|_{m'}^{m'} = -I(u_k(t)) \quad \text{and} \quad \int_0^t \|U_{k\tau}(\tau)\|_2^2 \, d\tau + J(u_k(t)) \le J(u_{0k}).$$

As in the proof of Theorem 3.4, we obtain a global weak solution u such that $u(t) \in \overline{\mathcal{W}}$ for $t \ge 0$.

It remains to prove the decay of solution $u(t)$. Since $u_0 \in W_0^{1,p}(\Omega)\setminus\{0\}$ with $J(u_0) = d$ and $I(u_0) > 0$, by virtue of Lemma (7.1), we get

$$I(u(t)) > 0, \quad \text{for } t \geq 0.$$

This implies $\|u(t)\|_{m'} > 0$ and $\|U_t(t)\|_2 > 0$ for $t \geq 0$. As a results, $\int_0^t \|U_\tau(\tau)\|_2^2 \, d\tau$ is strictly positive for all $t \geq 0$. Taking $t_1 > 0$, then $I(u(t_1)) > 0$ and

$$J(u(t_1)) \leq J(u_0) - \int_0^{t_1} \|U_\tau(\tau)\|_2^2 \, d\tau < d.$$

Hence, $u(t_1) \in \mathcal{W}$. If we take $t = t_1$ as the initial time, then by analogous arguments in Step 2 in the proof of Theorem 3.4, we possess (3.7).

Step 2: Blow up for $J(u_0) = d$ and $I(u_0) < 0$

Let $u_0 \in W_0^{1,p}(\Omega)\setminus\{0\}$ with $J(u_0) = d$ and $I(u_0) < 0$. Then, by virtue of Lemma 7.1, we get

$$I(u(t)) < 0 \quad \text{for} \quad 0 \leq t < T_{\max}.$$

This implies $\|u(t)\|_{m'} > 0$ and $\|U_t(t)\|_2 > 0$ for $0 \leq t < T_{\max}$. As a results, $\int_0^t \|U_\tau(\tau)\|_2^2 \, d\tau$ is strictly positive for all $0 \leq t < T_{\max}$. Taking $t_2 \in (0, T_{\max})$, then $I(u(t_2)) > 0$ and

$$J(u(t_2)) \leq J(u_0) - \int_0^{t_2} \|U_\tau(\tau)\|_2^2 \, d\tau < d.$$

Hence, $u(t_2) \in \mathcal{U}$. If we take $t = t_2$ as the initial time, then by using similar arguments as in the proof of Theorem 3.5, we imply that weak solution $u(t)$ of the probelm (1.7) blows up in finite time.

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