Electronic Journal of Qualitative Theory of Differential Equations
2019, No. **3**, 1–6; https://doi.org/10.14232/ejqtde.2019.1.3 www.math.u-szeged.hu/ejqtde/

Period annulus of the harmonic oscillator with zero cyclicity under perturbations with a homogeneous polynomial field

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Received 2 August 2018, appeared 14 January 2019
Communicated by Alberto Cabada

Abstract. In this work we prove, using averaging theory at any order in the small perturbation parameter, that the period annulus of the harmonic oscillator has cyclicity zero (no limit cycles bifurcate) when it is perturbed by any fixed homogeneous polynomial field.

Keywords: averaging theory, periodic orbits, Poincaré map.

2010 Mathematics Subject Classification: 37G15, 37G10, 34C07.

1 Introduction and main result

Consider an arbitrary polynomial planar vector field

$$\dot{x} = -y + \varepsilon P(x,y,\varepsilon), \qquad \dot{y} = x + \varepsilon Q(x,y,\varepsilon), \tag{1.1}$$

where $P, Q \in \mathbb{R}\{\varepsilon\}[x,y]$ are polynomials in the state variables x and y with coefficients depending analytically on the small perturbation parameter $\varepsilon \in \mathbb{R}$. Here the dot denotes, as usual, derivative with respect to the time independent variable. The unperturbed system (1.1) with $\varepsilon = 0$ is the harmonic oscillator which has a period annulus $\mathcal{P}$ given by the punctured phase plane $\mathcal{P} = \mathbb{R}^2 \setminus \{(0,0)\}$.

Limit cycle bifurcations for the vector fields (1.1) can be produced either from the open set $\mathcal{P}$ or from its boundary $\partial\mathcal{P} = \{(0,0)\} \cup L_\infty$, where L_∞ is the line at infinity (equator of the Poincaré compactification). In this paper we do not pay attention to the Hopf bifurcations at the origin neither to the bifurcations at infinity (see Remark 1 of [5] for a simple example of limit cycle bifurcation at L_∞).

Let $\mathcal{X}_\varepsilon$ be the vector field associated to system (1.1). We denote by $\mathrm{Cycl}(\mathcal{X}_\varepsilon, \mathcal{P})$ the *cyclicity* of $\mathcal{P}$ under the perturbations (1.1) with $|\varepsilon| \ll 1$, that is, the maximum number of limit cycles of (1.1) bifurcating from the circles that foliates $\mathcal{P}$.

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Essentially there are two methods for finding limit cycles of (1.1) bifurcating from $\mathcal{P}$ which are averaging methods or Melnikov functions method. It is worth to emphasize that in [1] it is proved the equivalence between both methods.

In [5] the global upper bound $[k(n-1)/2]$ on $\mathrm{Cycl}(\mathcal{X}_\varepsilon, \mathcal{P})$ is given where n is defined as $n = \max\{\deg(P), \deg(Q)\}$ and k is the order of the first Melnikov function associated to (1.1) which is not identically zero. Also in [5] some values of $\mathrm{Cycl}(\mathcal{X}_\varepsilon, \mathcal{P})$ have been obtained for the values $1 \leq k \leq 6$ showing in most cases that the above upper bound is sharp. As far as we know, the bifurcation of limit cycles from $\mathcal{P}$ was first analyzed (with the alternative method based on the inverse integrating factor) in [3] under the assumption that (P, Q) has arbitrary degree $n \geq 1$ and it is independent of ε.

In this work we will compute $\mathrm{Cycl}(\mathcal{X}_\varepsilon, \mathcal{P})$ for any value of n and k but restricted to the special kind of deformations (1.1) having the perturbation field (P, Q) independent of ε and homogeneous in x and y. Thus we will analyze the perturbations of the form

$$\dot{x} = -y + \varepsilon P_n(x, y; \lambda), \qquad \dot{y} = x + \varepsilon Q_n(x, y; \lambda), \tag{1.2}$$

where the nonlinearities P_n and Q_n are arbitrary homogeneous polynomials in x and y and its coefficients are the components of the parameter vector λ, which does not depend on ε. Our main result is that no limit cycles bifurcate from $\mathcal{P}$ under deformations (1.2), which we restate as follows.

Theorem 1.1. *The period annulus of the harmonic oscillator has cyclicity zero when it is perturbed by any fixed homogeneous polynomial field.*

2 Proof of Theorem 1.1

Introducing polar coordinates $x = r\cos\theta$, $y = r\sin\theta$, family (1.2) is written as

$$\dot{r} = \varepsilon\, r^n\, A(\theta; \lambda), \qquad \dot{\theta} = 1 + \varepsilon\, r^{n-1}\, B(\theta; \lambda), \tag{2.1}$$

where A and B are homogeneous trigonometric polynomials of degree $n+1$ with coefficients λ given by

$$\begin{aligned} A(\theta; \lambda) &= \cos\theta P_n(\cos\theta, \sin\theta; \lambda) + \sin\theta Q_n(\cos\theta, \sin\theta; \lambda) \\ B(\theta; \lambda) &= \cos\theta Q_n(\cos\theta, \sin\theta; \lambda) - \sin\theta P_n(\cos\theta, \sin\theta; \lambda). \end{aligned}$$

Here the perturbative parameter $\varepsilon \in I$ with $I \subset \mathbb{R}$ a small interval containing the origin. Therefore, for $|\varepsilon|$ sufficiently small, we can write system (2.1) into the analytic differential equation

$$\frac{dr}{d\theta} = \mathcal{F}(\theta, r; \lambda, \varepsilon) = \sum_{i \geq 1} \mathcal{F}_i(\theta, r; \lambda)\, \varepsilon^i, \tag{2.2}$$

with

$$\mathcal{F}_i(\theta, r; \lambda) = (-1)^{i+1} r^{i(n-1)+1}\, A(\theta; \lambda) B^{i-1}(\theta; \lambda), \quad \text{for } i \geq 1. \tag{2.3}$$

Notice that (2.2) is defined on the cylinder $\{(r, \theta) \in (\mathbb{R}^+ \cup \{0\}) \times \mathbb{S}^1\}$ with $\mathbb{S}^1 = \mathbb{R}/2\pi\mathbb{Z}$. Since, $\mathcal{F}(\theta, r; \lambda, 0) \equiv 0$ it follows that equation (2.2) is written in the standard form of the averaging theory with period 2π. The method of averaging is a classical tool that allows to study the dynamics of the periodic nonlinear differential systems. The reader can consult for example the book [7] or, for recent advances, the papers [2] and [6].

From the analyticity of (2.2) and the fact $\mathcal{F}(\theta, r; \lambda, 0) \equiv 0$ it follows that the solution $r(\theta; z, \lambda, \varepsilon)$ of (2.1) with initial condition $r(0; z, \lambda, \varepsilon) = z \in \mathbb{R}^+$ can be expanded into the convergent power series in ε as $r(\theta; z, \lambda, \varepsilon) = z + \sum_{j\geq 1} r_j(\theta, z, \lambda)\, \varepsilon^j$ where $r_j(\theta, z, \lambda)$ are real analytic functions such that $r_j(0, z, \lambda) = 0$. Therefore, from the results in [4] it follows that the recursive expressions of $r_j(\theta; z, \lambda)$ for $j \geq 1$ are given by

$$
\begin{aligned}
r_1(\theta, z, \lambda) &= \int_0^\theta \mathcal{F}_1(\tau, z; \lambda)\, d\tau, \\
r_k(\theta, z, \lambda) &= \int_0^\theta \left(\mathcal{F}_k(\tau, z; \lambda) + \sum_{\ell=1}^{k-1} \sum_{i=1}^{\ell} \frac{1}{i!} \frac{\partial^i \mathcal{F}_{k-\ell}}{\partial r^i}(\tau, z; \lambda) \sum_{j_1+j_2+\cdots+j_i=\ell} \prod_{p=1}^{i} r_{j_p}(\tau, z, \lambda) \right) d\tau.
\end{aligned}
\tag{2.4}
$$

We can assume without loss of generality that the function $r(\cdot; z, \lambda, \varepsilon)$ is defined on the interval $[0, 2\pi]$ provided that ε is close enough to 0. So we can define the *displacement map* as $d : \mathbb{R}^+ \times \mathbb{R}^p \times I \to \mathbb{R}^+$ with $d(z, \lambda, \varepsilon) = r(2\pi; z, \lambda, \varepsilon) - z$. Clearly, the isolated positive zeros $z_0 \in \mathbb{R}^+$ of $d(\cdot, \lambda, \varepsilon)$ are initial conditions for the 2π-periodic solutions of the differential equation (2.2) and they are in one-to-one correspondence with the limit cycles of system (1.2) bifurcating from the circle $x^2 + y^2 = z_0^2$.

The displacement map d is analytic at $\varepsilon = 0$, hence it can be expressed via the following convergent series expansion

$$
d(z, \lambda, \varepsilon) = \sum_{i\geq 1} f_i(z; \lambda)\, \varepsilon^i. \tag{2.5}
$$

We call the coefficient functions $f_i(z; \lambda)$ the *averaged functions* (they are also called Melnikov functions) which are clearly given by

$$
f_i(z; \lambda) = r_i(2\pi, z, \lambda). \tag{2.6}
$$

We say that a *branch of limit cycles* bifurcates from the point $z_0 \in \mathbb{R}^+$ if there is a function $z^*(\lambda, \varepsilon)$ (may be only defined in a half-neighborhood of zero) such that $z^*(\lambda, 0) = z_0$ and $d(z^*(\lambda, \varepsilon), \lambda, \varepsilon) \equiv 0$. It is well known, see [7] for example, that under these conditions it follows that z_0 must be a zero of the function $f_\ell(\cdot; \lambda)$ where ℓ is the first subindex such that $f_\ell(z; \lambda) \not\equiv 0$.

Let $\mathbb{N}$ be the set of non-negative integers. We recall that given the pair $(i, j) \in \mathbb{N}^2$, the function $\int_0^\theta \sin^i(\tau) \cos^j(\tau)\, d\tau$ is a trigonometric polynomial (that is, a function in $\mathbb{R}[\sin(\theta), \cos(\theta)]$) plus an eventual linear term $\alpha\theta$ where $\alpha \neq 0$ only in case that both i and j are even numbers. More generally, when $(i, j, k) \in \mathbb{N}^3$, the function $\int_0^\theta \tau^k \sin^i(\tau) \cos^j(\tau)\, d\tau$ belongs to $\mathbb{R}[\theta][\sin(\theta), \cos(\theta)]$, the set of trigonometric polynomials with real polynomial coefficients in θ. The fact that $\mathbb{R}[\theta][\sin(\theta), \cos(\theta)]$ is closed under sums, products and quadratures will be key in what follows.

We claim that

$$
r_k(\theta, z, \lambda) = R_k(\theta, \lambda) z^{k(n-1)+1} \tag{2.7}
$$

where $R_k \in \mathbb{R}[\theta, \lambda][\sin(\theta), \cos(\theta)]$ is a trigonometric polynomial with polynomial coefficients in $\mathbb{R}[\theta, \lambda]$. We will prove the claim by induction over k. From (2.2) we have $\mathcal{F}_1(\theta, z; \lambda) = z^n\, A(\theta; \lambda)$, hence from the first equation in (2.4) we get

$$
r_1(\theta, z, \lambda) = \int_0^\theta \mathcal{F}_1(\tau, z; \lambda)\, d\tau = R_1(\theta, \lambda) z^n,
$$

where $R_1(\theta,\lambda) = \int_0^\theta A(\tau;\lambda)\,d\tau$. Therefore the claim is true for $k=1$.

Assume now by induction hypothesis that $r_j(\theta,z,\lambda) = R_j(\theta,\lambda)z^{j(n-1)+1}$ where by definition $R_j \in \mathbb{R}[\theta,\lambda][\sin(\theta),\cos(\theta)]$ for all $1 \le j \le k-1$. Then, since all integer subindex j_p appearing in (2.4) satisfy $1 \le j_p \le \ell \le k-1$, it follows that

$$r_{j_p}(\theta,z,\lambda) = R_{j_p}(\theta,\lambda)\,z^{j_p(n-1)+1}$$

with $R_{j_p} \in \mathbb{R}[\theta,\lambda][\sin(\theta),\cos(\theta)]$. Hence

$$\prod_{p=1}^{i} r_{j_p}(\tau,z,\lambda) = \hat{R}_{j_1,\ldots,j_i}(\tau,\lambda)\,z^{(j_1+\cdots+j_i)(n-1)+i}$$

with $\hat{R}_{j_1,\ldots,j_i}(\theta,\lambda) = \prod_{p=1}^{i} R_{j_p}(\theta,\lambda) \in \mathbb{R}[\theta,\lambda][\sin(\theta),\cos(\theta)]$. Thus

$$\sum_{j_1+\cdots+j_i=\ell}\ \prod_{p=1}^{i} r_{j_p}(\tau,z,\lambda) = R^*_{i\ell}(\theta,\lambda)\,z^{\ell(n-1)+i} \tag{2.8}$$

with $R^*_{i\ell}(\theta,\lambda) = \sum_{j_1+\cdots+j_i=\ell} \hat{R}_{j_1,\ldots,j_i}(\theta,\lambda) \in \mathbb{R}[\theta,\lambda][\sin(\theta),\cos(\theta)]$.

On the other hand, equation (2.3) yield

$$\frac{\partial^i \mathcal{F}_{k-\ell}}{\partial r^i}(\theta,z;\lambda) = (-1)^{k-\ell} z^{(k-\ell)(n-1)+1-i} A(\theta;\lambda)B^{k-\ell-1}(\theta;\lambda). \tag{2.9}$$

Therefore, using (2.3), (2.8) and (2.9) we rewrite (2.4) like (2.7) with

$$R_k(\theta,\lambda) = \int_0^\theta \left[(-1)^{k+1} A(\tau;\lambda)B^{k-1}(\tau;\lambda) + \sum_{\ell=1}^{k-1}\sum_{i=1}^{\ell} \frac{(-1)^{k-\ell}}{i!} A(\tau;\lambda)B^{k-\ell-1}(\tau;\lambda)R^*_{i\ell}(\tau,\lambda)\right] d\tau,$$

so that $R_k \in \mathbb{R}[\theta,\lambda][\sin(\theta),\cos(\theta)]$ proving the claim.

Once the claim (2.7) is proved we get that $R_k(2\pi,\lambda) \in \mathbb{R}[\lambda]$ for all $k \in \mathbb{N}$ and therefore, from (2.6), that the averaged functions are $f_k(z;\lambda) = P_k(\lambda)z^{k(n-1)+1}$ where $P_k \in \mathbb{R}[\lambda]$ for all $k \in \mathbb{N}$. Hence it is clear that the only finite point from where 2π-periodic orbit bifurcation can occur in the differential equation (2.2) is just from the initial condition $z_0 = 0$ which corresponds to the singularity located at the origin of the vector field (1.2). So no periodic orbit bifurcation appear in the period annulus and the theorem is proved.

3 Some remarks

Theorem 1.1 is not true if the perturbation field (P_n, Q_n) is not homogeneous, see for example [3]. One of the most simple counterexamples is given by the van der Pol differential equation $\ddot{x} + x = \varepsilon(1-x^2)\dot{x}$ which is a perturbation of the harmonic oscillator with associated vector field $\dot{x} = y$, $\dot{y} = -x + \varepsilon(1-x^2)y$. Computations show that the first averaged function is $f_1(z) = \frac{1}{4}\pi z(z^2-4)$ so that from the circle $x^2+y^2 = z_0^2 = 4$ the van der Pol limit cycle bifurcates.

Theorem 1.1 is no longer valid if the homogeneous perturbation field (P_n, Q_n) has coefficients depending on the perturbation parameter ε, that is, for systems of the form

$\dot{x} = -y + \varepsilon P_n(x, y; \lambda, \varepsilon)$, $\dot{y} = x + \varepsilon Q_n(x, y; \lambda, \varepsilon)$. Notice that, in this case equation (2.3) does not hold. As example, straightforward calculations with the general quadratic system

$$\dot{x} = -y + \varepsilon \sum_{i+j=2} a_{ij}(\varepsilon) x^i y^j, \qquad \dot{y} = x + \varepsilon \sum_{i+j=2} b_{ij}(\varepsilon) x^i y^j,$$

having analytic coefficients a_{ij} and b_{ij} at $\varepsilon = 0$ produce the following averaged functions:

$$\begin{aligned} f_1(z;\lambda) &\equiv 0, \\ f_2(z;\lambda) &= z^3\, \xi_{20}(\lambda), \\ f_3(z;\lambda) &= z^3\, (\xi_{30}(\lambda) + z\xi_{31}(\lambda)), \\ f_4(z;\lambda) &= z^3\, (\xi_{40}(\lambda) + z\xi_{41}(\lambda) + z^2\xi_{42}(\lambda)), \end{aligned}$$

with $\xi_{ij} \in \mathbb{R}[\lambda]$. Here $\lambda \in \mathbb{R}^{18}$ denotes the parameter vector whose components are the values $a_{ij}(0)$, $b_{ij}(0)$ and its derivatives $a_{ij}^{(k)}(0)$ and $b_{ij}^{(k)}(0)$ of order $k \in \{1, 2\}$. Moreover, ξ_{20} divides $\xi_{31}(\lambda)$. Hence, in order to obtain a limit cycle bifurcation from some periodic orbit $x^2 + y^2 = z_0^2$ of the harmonic oscillator, the parameters $\lambda = \lambda^*$ must satisfy $f_2(z;\lambda^*) = f_3(z;\lambda^*) \equiv 0$, in which case $\xi_{41}(\lambda^*) = 0$ and λ^* can be chosen such that the equation $f_4(z;\lambda^*) = 0$ has exactly one solution $z = z_0 > 0$.

Acknowledgements

The authors are partially supported by MINECO grant number MTM2017-84383-P and AGAUR grant number 2017SGR-1276.

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