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Existence and multiplicity of solutions for a Dirichlet problem involving the discrete $p(x)$-Laplacian operator

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Abstract. In the present paper, using the three critical points theorem and variational method, we study the existence and multiplicity of solutions for a Dirichlet problem involving the discrete $p(x)-$Laplacian operator.

Keywords: discrete boundary value problem, critical point theory, discrete $p(x)$-Laplacian.

MSC: 47A75, 35B38, 35P30, 34L05, 34L30.

1. INTRODUCTION

In this paper, we study the existence and multiplicity of solutions of the discrete boundary value problem

$$\begin{cases} -\Delta\left(|\Delta u(k-1)|^{p(k-1)-2}\,\Delta u(k-1)\right)+|u(k-1)|^{p(k-1)-2}\,u(k-1)=\lambda f\left(k,u\left(k\right)\right), \\ u(0)=u(T+1)=0, \end{cases} \tag{$\mathbf{P}$}$$

where $f\left(k,t\right)=|t|^{q(k)-2}\,t-|t|^{s(k)-2}\,t$ for all $t\in\mathbb{R}^{+}$ and $k\in\mathbb{Z}[1,T]$, $T\geq 2$ is a positive integer, λ is a positive constant and $\Delta u(k)=u(k+1)-u(k)$ is the forward difference operator. Here and hereafter, $\mathbb{Z}\left[a,b\right]$ denotes the discrete interval $\{a,a+1,...,b\}$ with a and b are integers such that $a<b$. Moreover, we assume that functions $p:\mathbb{Z}\left[0,T\right]\rightarrow[2,\infty)$ and $q,s:\mathbb{Z}\left[1,T\right]\rightarrow[2,\infty)$ are bounded and we denote

$$p^{-}=\min_{\mathbb{Z}[0,T]}p\leq p^{+}=\max_{\mathbb{Z}[0,T]}p,\ q^{-}=\min_{\mathbb{Z}[1,T]}q\leq q^{+}=\max_{\mathbb{Z}[1,T]}q$$

$$s^{-}=\min_{\mathbb{Z}[1,T]}s\leq s^{+}=\max_{\mathbb{Z}[1,T]}s$$

It is well known that in various fields of research, such as computer science, mechanical engineering, control systems, artificial or biological neural networks, economics and many others, the mathematical modeling of important questions leads naturally to the consideration of nonlinear difference equations. For this reason, in recent years, many authors have widely developed various methods and techniques, such as fixed points theorems or upper and lower solutions methods, to study discrete problems [7]. In order to support the above ideas we refer the reader to [2, 3, 4, 8, 10, 14, 17, 23, 24] and the reference there in.

We note that problem ($\mathbf{P}$) is the discrete variant of the variable exponent anisotropic problem

$$\begin{cases} -\sum_{i=1}^{N}\frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)-2}\frac{\partial u}{\partial x_i}\right)=\lambda\left(|u|^{q(x)-2}\,u-|u|^{s(x)-2}\,u\right),\ x\in\Omega, \\ u=0,\ x\in\partial\Omega, \end{cases} \tag{1.1}$$

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where $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ is a bounded domain with smooth boundary, $\lambda > 0$ is a real number, and $p_i(x), q(x), s(x)$ are continuous on $\overline{\Omega}$ such that $2 \leq p_i(x) < N$ and $q(x), s(x) > 1$ for any $x \in \overline{\Omega}$ and all $i \in \mathbb{Z}[1, N]$. Problems like (1.1) have been intensively studied in the last decades, since they can model various phenomena arising from the study of elastic mechanics (Zhikov [25]), electrorheological fluids (Acerbi and Mingione [1], Diening [12], Halsey [13], Růžička [22], Mihăilescu and Rădulescu [18 − 21]) and image restoration (Chen, Levine and Rao [11]).

In [16], the author studied an elliptic equation with nonstandard growth conditions and the Neumann boundary condition. He established the existence of at least three solutions by using as the main tool a variational principle due to Ricceri.

In [17], the authors studied the problem

$$\begin{cases} -\Delta\left(|\Delta u(k-1)|^{p(k-1)-2}\,\Delta u(k-1)\right) = \lambda\,|u(k)|^{q(k)-2}\,u(k), \;\; k \in \mathbb{Z}[1,T], \\ u(0) = u(T+1) = 0, \end{cases} \tag{1.2}$$

where $T \geq 2$ is a positive integer, the functions $p : \mathbb{Z}[0,T] \to [2,\infty)$ and $q : \mathbb{Z}[1,T] \to [2,\infty)$ are bounded and λ is a positive constant. By using critical point theory, they showed the existence of a continuous spectrum of eigenvalues for the problem (1.2).

In [15], the authors dealt with the following problem which is a generalization of (1.2)

$$\begin{cases} -\Delta\left(a(k-1, \Delta u(k-1))\right) = f(k), \;\; k \in \mathbb{Z}[1,T], \\ u(0) = u(T+1) = 0, \end{cases} \tag{1.3}$$

where $T \geq 2$ is a positive integer, $\Delta u(k) = u(k+1) - u(k)$ is the forward difference operator and $a(k, \xi) = |\xi|^{p(k)-2}\,\xi$ such that $k \in \mathbb{Z}[0,T]$ and $\xi \in \mathbb{R}$. By using critical point theory, the authors proved the existence and uniqueness of weak solutions for the problem (1.3).

For the case $p(x) = p = const$, the authors investigated the following problem

$$\begin{cases} -\Delta\left(\phi_p\left(\Delta u(k-1)\right)\right) = \lambda f\left(k, u(k)\right), \;\; k \in \mathbb{Z}[1,T], \\ u(0) = u(T+1) = 0, \end{cases} \tag{1.4}$$

where $T \geq 2$ is a positive integer, $[1,T]$ is the discrete interval $\{1, ..., T\}$, λ is a positive real parameter, $\phi_p(s) = |s|^{p-2}\,s$, $1 < p < \infty$ and $f : [1,T] \times \mathbb{R} \to \mathbb{R}$ is a continuous function (see [9]). By using critical point theory, they showed the existence of multiple solutions for the problem (1.4).

This paper is organized as follows. In Section 2, we present some necessary preliminary results. In Section 3, using three critical points theorem and the variational method we show the existence and multiplicity of solutions of problem (**P**).

2. PRELIMINARIES

Let W be the function space

$$W = \{u : \mathbb{Z}[0, 1+T] \to \mathbb{R};\ \text{such that}\ \ u(0) = u(T+1) = 0\},$$

Then, W is a T-dimensional Hilbert space with the inner product

$$\langle u, v\rangle = \sum_{k=1}^{T+1} \Delta u(k-1)\Delta v(k-1), \;\; \forall u, v \in W.$$

(see [3]). The associated norm is defined by

$$\|u\| = \left(\sum_{k=1}^{T+1} |\Delta u(k-1)|^2 \right)^{1/2}.$$

On the other hand, it is useful to introduce other norms on W, namely

$$|u|_m = \left(\sum_{k=1}^{T} |u(k)|^m \right)^{1/m}, \ \forall u \in W \text{ and } m \geq 2.$$

In [10] it is verified that

$$T^{(2-m)/2m} |u|_2 \leq |u|_m \leq T^{1/m} |u|_2 , \ \forall u \in W \text{ and } m \geq 2. \tag{2.1}$$

Remark 2.1. There exists $j \in [1, T]$ such that $u(j) = \max_{k \in [1,T]} |u(k)|$ for every $u \in W$. Therefore, since $u(0) = u(T+1) = 0$, a straightforward computation gives

$$u(j) \leq \frac{1}{2} \sum_{k=1}^{T+1} |u(k+1) - u(k)| ,$$

and by using the discrete Hölder inequality, we have that

$$\max_{k \in [1,T]} |u(k)| \leq \frac{\sqrt{T+1}}{2} \|u\| .$$

Lemma 2.1.[17]
(a) There exist two positive constants C_1 and C_2 such that

$$\sum_{k=1}^{T+1} |\Delta u(k-1)|^{p(k-1)} \geq C_1 \|u\|^{p^-} - C_2, \ \forall u \in W \textit{ with } \|u\| > 1.$$

(b) There exists a positive constant C_3 such that

$$\sum_{k=1}^{T+1} |\Delta u(k-1)|^{p(k-1)} \geq C_3 \|u\|^{p^+} , \ \forall u \in W \textit{ with } \|u\| < 1.$$

(c) For any $m \geq 2$ there exists a positive constant c_m such that

$$\sum_{k=1}^{T} |u(k)|^m \leq c_m \sum_{k=1}^{T+1} |\Delta u(k-1)|^m , \ \forall u \in W.$$

Theorem 2.1.[6] *Let X be a separable and reflexive real Banach space; $\Phi : X \to \mathbb{R}$ a continuously Gâteaux differentiable and sequentially weakly lower semi-continuous functional whose Gâteaux derivative admits a continuous inverse on X^*;$\Psi : X \to \mathbb{R}$ a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact. Assume that,*
(i) $\lim_{\|u\| \to \infty} J_\lambda(u) = \lim_{\|u\| \to \infty} (\Phi(u) + \lambda \Psi(u)) = \infty$ *for all* $\lambda > 0$,
(ii) There are $r \in \mathbb{R}$ and u_0, $u_1 \in X$ such that $\Phi(u_0) < r < \Phi(u_1)$,
(iii) $\inf_{u \in \Phi^{-1}((-\infty, r])} \Psi(u) > \frac{(\Phi(u_1) - r)\Psi(u_0) + (r - \Phi(u_0))\Psi(u_1)}{\Phi(u_1) - \Psi(u_1)}$.

Then there exist an open interval $\Lambda \subset (0,\infty)$ and a positive real number ρ such that for each $\lambda \in \Lambda$ the equation

$$\Phi'(u) + \lambda\Psi'(u) = 0$$

has at least three solutions in X whose norms are lees than ρ.

3. AUXILIARY RESULTS

For any $\lambda > 0$ the energy functional corresponding to problem (**P**) is defined as $J_\lambda : W \to \mathbb{R}$,

$$\begin{aligned} J_\lambda(u) &= \sum_{k=1}^{T+1}\left(\frac{1}{p(k-1)}\left|\Delta u(k-1)\right|^{p(k-1)} + \frac{1}{p(k-1)}\left|u(k-1)\right|^{p(k-1)}\right) \\ &\quad -\lambda\left(\sum_{k=1}^{T}\frac{1}{q(k)}\left|u(k)\right|^{q(k)} - \sum_{k=1}^{T}\frac{1}{s(k)}\left|u(k)\right|^{s(k)}\right) \end{aligned}$$

Let

$$\Phi(u) = \sum_{k=1}^{T+1}\left(\frac{1}{p(k-1)}\left|\Delta u(k-1)\right|^{p(k-1)} + \frac{1}{p(k-1)}\left|u(k-1)\right|^{p(k-1)}\right)$$

and

$$\Psi(u) = -\sum_{k=1}^{T}\frac{1}{q(k)}\left|u(k)\right|^{q(k)} + \sum_{k=1}^{T}\frac{1}{s(k)}\left|u(k)\right|^{s(k)}.$$

From the standard arguments $J_\lambda \in C^1(W,\mathbb{R})$ and its derivative is given by

$$\begin{aligned} \left\langle J'_\lambda(u), v\right\rangle &= \sum_{k=1}^{T+1}\left|\Delta u(k-1)\right|^{p(k-1)-2}\Delta u(k-1)\Delta v(k-1) + \left|u(k-1)\right|^{p(k-1)-2}u(k-1)v(k-1) \\ &-\lambda\left[\sum_{k=1}^{T}\left|u(k)\right|^{q(k)-2}u(k)v(k) - \sum_{k=1}^{T}\left|u(k)\right|^{s(k)-2}u(k)v(k)\right], \end{aligned}$$

for any $u, v \in W$.

Definition 3.1. We say that $u \in W$ is a weak solution of (**P**) if

$$\begin{aligned} &\sum_{k=1}^{T+1}\left(\left|\Delta u(k-1)\right|^{p(k-1)-2}\Delta u(k-1)\Delta\varphi(k-1) + \left|u(k-1)\right|^{p(k-1)-2}u(k-1)\varphi(k-1)\right) \\ &= \lambda\left(\sum_{k=1}^{T}\left|u(k)\right|^{q(k)-2}u(k)\varphi(k) - \sum_{k=1}^{T}\left|u(k)\right|^{s(k)-2}u(k)\varphi(k)\right), \end{aligned}$$

where $\varphi \in W$. Hence, the critical points of functional J_λ are the weak solutions for problem (**P**).

Theorem 3.1. *Suppose that the functions $p : \mathbb{Z}[0,T] \to [2,\infty)$ and $q, s : \mathbb{Z}[1,T] \to [2,\infty)$ are bounded and $s^+ < q^- \le q^+ < p^-$. Then there exist an open interval $\Lambda \subset (0,\infty)$ and a constant $\rho > 0$ such that for any $\lambda \in \Lambda$, problem (P) has at least three weak solutions whose norms are less than ρ.*

Proof of Theorem 3.1.

(i) First, we point out that

$$|u(k)|^{q(k)} \le |u(k)|^{q^-} + |u(k)|^{q^+}, \ \forall k \in \mathbb{Z}[1,T] \text{ and } u \in W.$$

Thus, we have

$$\sum_{k=1}^{T} |u(k)|^{q(k)} \le \sum_{k=1}^{T} |u(k)|^{q^+} + \sum_{k=1}^{T} |u(k)|^{q^-}.$$

Using the above inequality, we obtain

$$\begin{aligned} J_\lambda(u) &= \sum_{k=1}^{T+1} \left(\frac{1}{p(k-1)} |\Delta u(k-1)|^{p(k-1)} + \frac{1}{p(k-1)} |u(k-1)|^{p(k-1)} \right) \\ &\quad -\lambda \left(\sum_{k=1}^{T} \frac{1}{q(k)} |u(k)|^{q(k)} - \sum_{k=1}^{T} \frac{1}{s(k)} |u(k)|^{s(k)} \right) \end{aligned}$$

$$\ge \frac{1}{p^+} \sum_{k=1}^{T+1} \left(|\Delta u(k-1)|^{p(k-1)} + |u(k-1)|^{p(k-1)} \right) - \frac{\lambda}{q^-} \left(|u|_{q^-}^{q^-} + |u|_{q^+}^{q^+} \right). \tag{3.2}$$

Now, consider the case $\|u\| > 1$ for $u \in W$. Then, using Lemma 2.1 (c) and relation (2.1), we get that

$$\begin{aligned} |u|_{q^-}^{q^-} + |u|_{q^+}^{q^+} &\le c_{q^-} \sum_{k=1}^{T+1} |\Delta u(k-1)|^{q^-} + c_{q^+} \sum_{k=1}^{T+1} |\Delta u(k-1)|^{q^+} \\ &\le T c_{q^-} \|u\|^{q^-} + T c_{q^+} \|u\|^{q^+}. \end{aligned} \tag{3.3}$$

For the case $\|u\| > 1$ for $u \in W$, using Lemma 2.1 (a), the relation (2.1) and (3.3), the inequality (3.2) imply that

$$J_\lambda(u) \ge \frac{C_1}{p^+} \|u\|^{p^-} - \frac{C_2}{p^+} - \frac{\lambda}{q^-} \left(T c_{q^-} \|u\|^{q^-} + T c_{q^+} \|u\|^{q^+} \right).$$

Since $p^- > q^+$, then $\lim\limits_{\|u\| \to \infty} (\Phi(u) + \lambda \Psi(u)) = \infty$, and hence, Theorem 2.1 (i) is verified.

In the following, we show that Theorem 2.1 (ii) and Theorem 2.1 (iii) hold.

Define $H : W \times [0,\infty) \to \mathbb{R}$ by

$$H(k,t) = \frac{1}{q(k)} |t|^{q(k)} - \frac{1}{s(k)} |t|^{s(k)}, \ \forall k \in \mathbb{Z}[1,T] \text{ and } t \in [0,\infty).$$

It is clear that $H(k,t)$ and $H_t(k,t)$ are continuous for all $k \in \mathbb{Z}[1,T]$ and satisfies the following equality

$$\begin{aligned} H_t(k,t) &= |t|^{q(k)-1} - |t|^{s(k)-1} \\ &= |t|^{s(k)-1} \left(|t|^{q(k)-s(k)} - 1 \right). \end{aligned}$$

Thus, $H_t(k,t) \geq 0$ for all $t \geq 1$ and all $k \in \mathbb{Z}[1,T]$ and $H_t(k,t) \leq 0$ for all $t \leq 1$ and all $k \in \mathbb{Z}[1,T]$. It follows that $H(k,t)$ is increasing for $t \in (1,\infty)$, and decreasing for $t \in (0,1)$. Moreover,

$$\lim_{t\to\infty} H(k,t) = \infty, \text{ for all } k \in \mathbb{Z}[1,T]$$

Using the above information, we get that there exists $\delta > 1$ such that

$$H(k,t) \geq 0 = H(k,0) \geq H(k,\tau), \ k \in \mathbb{Z}[1,T], \ t > \delta \text{ and } \tau \in (0,1). \tag{3.4}$$

Let a, b be two real numbers such that $0 < a < \min\left\{1, \frac{\sqrt{T+1}}{2}\right\}$ and $b > \delta$. It is clear that b satisfies $b^{p^-}(T+1) > 1$. Then relation (3.4) implies

$$\sum_{k=1}^{T} \sup_{0\leq t\leq a} H(k,t) \leq 0 < \frac{1}{b^{p^-}}\left(\frac{2a}{\sqrt{T+1}}\right)^{p^+} \sum_{k=1}^{T} H(k,b).$$

We consider $u_0, u_1 \in W$, $u_0(k) = 0$, $u_1(k) = b$ for any $k \in \mathbb{Z}[1,T]$. We also define $r = \frac{1}{p^+}\left(\frac{2a}{\sqrt{T+1}}\right)^{p^+}$. Clearly, $r \in (0,1)$. A simple computation implies

$$\Phi(u_0) = \Psi(u_0) = 0,$$

$$\begin{aligned}
\Phi(u_1) &= \sum_{k=1}^{T+1}\left(\frac{1}{p(k-1)}|\Delta u_1(k-1)|^{p(k-1)} + \frac{1}{p(k-1)}|u_1(k-1)|^{p(k-1)}\right) \\
&\geq \frac{1}{p^+}b^{p^-}(T+1) > \frac{1}{p^+}\left(\frac{2a}{\sqrt{T+1}}\right)^{p^+} = r,
\end{aligned}$$

and

$$\Psi(u_1) = -\sum_{k=1}^{T} H(k,b).$$

Thus, we obtain

$$\Phi(u_0) < r < \Phi(u_1).$$

Hence, Theorem 2.1 (ii) is verified.

On the other hand, we have

$$\begin{aligned}
-\frac{(\Phi(u_1) - r)\Psi(u_0) + (r - \Phi(u_0))\Psi(u_1)}{\Phi(u_1) - \Phi(u_0)} &= -r\frac{\Psi(u_1)}{\Phi(u_1)} \\
&= r\frac{\sum_{k=1}^{T} H(k,b)}{\sum_{k=1}^{T+1}\frac{1}{p(k-1)}b^{p(k-1)}} > 0.
\end{aligned}$$

Now, we investigate the case $\Phi(u) \leq r < 1$ for $u \in W$. Then, using Lemma 2.1 (b) and for $C_3 = T^{\frac{p^+-2}{2}}$, we have

$$\begin{aligned}
\frac{1}{p^+}\|u\|^{p^+} &\leq \frac{C_3}{p^+}\|u\|^{p^+} \leq \sum_{k=1}^{T+1}\frac{1}{p(k-1)}|\Delta u(k-1)|^{p(k-1)} \\
&\leq r < 1.
\end{aligned}$$

Thus, using Remark 2.1, for any $u \in W$ with $\Phi(u) \le r$ we obtain that

$$|u(k)| \le \frac{\sqrt{T+1}}{2}\|u\| \le \frac{\sqrt{T+1}}{2}\left(p^{+}r\right)^{\frac{1}{p^{+}}} = a,\ \forall k \in \mathbb{Z}[1,T]$$

where $r < \frac{1}{p^{+}}$.

The above inequality means that

$$-\inf_{u\in\Phi^{-1}((-\infty,r])}\Psi(u) = \sup_{u\in\Phi^{-1}((-\infty,r])} -\Psi(u) \le \sum_{k=1}^{T}\sup_{0\le t\le a} H(k,t) \le 0.$$

Therefore, we obtain

$$-\inf_{u\in\Phi^{-1}((-\infty,r])}\Psi(u) < r\frac{\sum_{k=1}^{T} H(k,b)}{\sum_{k=1}^{T+1}\frac{1}{p(k-1)}b^{p(k-1)}},$$

or

$$\inf_{u\in\Phi^{-1}((-\infty,r])}\Psi(u) > \frac{(\Phi(u_1)-r)\Psi(u_0)+(r-\Phi(u_0))\Psi(u_1)}{\Phi(u_1)-\Phi(u_0)}.$$

Theorem 2.1 (iii) is verified.

Consequently, all the assumptions of Theorem 2.1 are satisfied. Therefore, we conclude that there exists an open interval $\Lambda \subset (0,\infty)$ and constant $\rho > 0$ such that for any $\lambda \in \Lambda$, the equation

$$\Phi'(u) + \lambda\Psi'(u) = 0$$

has at least three solutions in W whose norms are less than ρ. The proof of Theorem 3.1 is complete.

Example 3.1. For problem (**P**) if we choose $T = 2$ and p, q, s as follows

$$\begin{aligned} p(k-1) &= e^{k-1}+6, \\ q(k) &= k+4, \\ s(k) &= \ln k + 4, \end{aligned}$$

then we have

$$\begin{aligned} p^{-} &= 7,\ p^{+} = e+6, \\ q^{-} &= 5,\ q^{+} = 6, \\ s^{-} &= 4,\ s^{+} = \ln 2 + 4. \end{aligned}$$

Therefore the functions p, q and s satisfy the conditions in Theorem 3.1.

Remark 3.1. Applying ([5], Theorem 2.1) in the proof of Theorem 3.1, an upper bound of the interval of parameters λ for which (**P**) has at least three weak solutions is obtained. To be precise, in the conclusion of Theorem 3.1 one has

$$\Lambda \subseteq \left(0, h\frac{\sum_{k=1}^{T+1}\frac{b^{p(k-1)}}{p(k-1)}}{\sum_{k=1}^{T} H(k,b)}\right)$$

for each $h > 1$ and b as in the proof of Theorem 3.1 (namely, $b > \delta$ is such that $H(k,b) > 0$). ■

In the sequel, we investigate the solutions of the problem (**P**) under the following conditions,

$(g_1) : g \in C\left(\mathbb{R}^2, \mathbb{R}\right)$,
(g_2) :there exists $\eta > 0$ such that $tg(k,t) \leq 0$ for $|t| \geq \eta$ for all $k \in \mathbb{Z}[1,T]$.

Lemma 3.1. *Assume that (g_1) and (g_2) hold. Then,*

$$\sum_{k=1}^{T} G(k,u(k)) \leq B,$$

where $G(k,u(k)) = \int_0^{u(k)} g(k,t)dt$ and $B > 0$ is a constant.

Proof. From (g_1) and (g_2), for $u \in W$ we have

$$\sum_{k=1}^{T} G(k,u(k)) = \sum_{k=1}^{T} \int_0^{u(k)} g(k,t)dt \leq \sum_{k=1}^{T} \int_{-\eta}^{\eta} |g(k,t)|dt = B$$

The proof is complete.

Theorem 3.2. *Suppose that the functions $p : \mathbb{Z}[0,T] \to [2,\infty)$ and $q,s : \mathbb{Z}[1,T] \to [2,\infty)$ are bounded, $s^- > q^+$ and $\lambda \in (0,\infty)$. Then there exists at least one nontrivial weak solution $u \in W$ to problem (**P**) such that $J_\lambda(u) = \inf_{v\in W} J_\lambda(v)$.*

Proof. It is obvious that $f(k,t) = |t|^{q(k)-2} t - |t|^{s(k)-2} t \in C\left(\mathbb{R}^2, \mathbb{R}\right)$, for every $k \in \mathbb{Z}[1,T]$. Since $s^- > q^+$, we get

$$tf(k,t) = |t|^{q(k)} - |t|^{s(k)} \leq 0 \text{ for } |t| \geq 1,\ k \in \mathbb{Z}[1,T].$$

Hence, the condition (g_2) holds for $f(k,t) = |t|^{q(k)-2} t - |t|^{s(k)-2} t$. For $u \in W$ with $\|u\| > 1$ by Lemma 3.1, we get

$$\begin{aligned} J_\lambda(u) &= \sum_{k=1}^{T+1} \left(\frac{1}{p(k-1)} |\triangle u(k-1)|^{p(k-1)} + \frac{1}{p(k-1)} |u(k-1)|^{p(k-1)} \right) - \lambda \sum_{k=1}^{T} F(k,u(k)) \\ &\geq \frac{C_1}{p^+} \|u\|^{p^-} - \lambda B \to \infty \text{ as } \|u\| \to \infty. \end{aligned}$$

Since J_λ is coercive, continuous (therefore, weakly lower semi-continuous since W is finite dimensional) and Gâteaux differentiable on W, using the relation between critical points of J_λ and problem (**P**), we deduce Theorem 3.2. holds. ■

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