



Ground state for Choquard equation with doubly critical growth nonlinearity

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Abstract. In this paper we consider nonlinear Choquard equation

$$-\Delta u + V(x)u = (I_\alpha * F(u))f(u) \quad \text{in } \mathbb{R}^N,$$

where $V \in C(\mathbb{R}^N)$, I_α denotes the Riesz potential, $f(t) = |t|^{p-2}t + |t|^{q-2}t$ for all $t \in \mathbb{R}$, $N \geq 5$ and $\alpha \in (0, N - 4)$. Under suitable conditions on V , we obtain that the Choquard equation with doubly critical growth nonlinearity, i.e., $p = (N + \alpha)/N$, $q = (N + \alpha)/(N - 2)$, has a nonnegative ground state solution by variational methods.

Keywords: Choquard equation, doubly critical nonlinearity, ground state solution.

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1 Introduction and main results

In this paper we consider nonlinear Choquard equation

$$-\Delta u + V(x)u = (I_\alpha * F(u))f(u) \quad \text{in } \mathbb{R}^N, \quad (1.1)$$

where $N \geq 5$, $\alpha \in (0, N - 4)$, I_α is the Riesz potential given by


$$I_\alpha(x) = \frac{\Gamma((N - \alpha)/2)}{2^\alpha \pi^{N/2} \Gamma(\alpha/2) |x|^{N-\alpha}}, \quad x \in \mathbb{R}^N \setminus \{0\},$$

Γ denotes the Gamma function, $F(t) = |t|^p/p + |t|^q/q$, $f(t) = |t|^{p-2}t + |t|^{q-2}t$ for all $t \in \mathbb{R}$, and the potential function $V \in C(\mathbb{R}^N)$ and satisfies

(V) there exist $V_0, V_\infty > 0$ such that $V_0 \leq V(x) \leq V_\infty$ for all $x \in \mathbb{R}^N$, and $\lim_{|x| \rightarrow \infty} V(x) = V_\infty$.

In the case $F(t) = |t|^p$, $f(t) = |t|^{p-2}t$ for all $t \in \mathbb{R}$, and $V = 1$, the Choquard equation (1.1) reduces to the general Choquard equation

$$-\Delta u + u = (I_\alpha * |u|^p)|u|^{p-2}u \quad \text{in } \mathbb{R}^N. \quad (1.2)$$

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When $N = 3, \alpha = 2$, and $p = 2$, the equation (1.2) has appeared in many interesting physical models and is known as the well-known Choquard–Pekar equation [6, 15], the Schrödinger–Newton equation [2, 3, 10, 18], and the stationary Hartree equation. In this case, the existence of ground states of equation (1.2) was obtained in [6, 8, 9] by variational methods.

In view of the Hardy–Littlewood–Sobolev inequality, see Lemma 2.1 below, it can be shown that the energy functional corresponding to (1.2), for every $\alpha \in (0, N)$, is well defined on $H^1(\mathbb{R}^N)$ and belongs to C^1 if

$$\frac{N + \alpha}{N} \leq p \leq \frac{N + \alpha}{N - 2},$$

where $(N + \alpha)/N$ is called the lower critical exponent and $(N + \alpha)/(N - 2)$ is called the upper critical exponent. V. Moroz and J. Van Schaftingen established the existence of ground state solutions to the Choquard equation (1.2) in [11] if p is in the subcritical range, namely $p \in ((N + \alpha)/N, (N + \alpha)/(N - 2))$, and some qualitative properties. By the Pohožaev identity [4, 5, 12], the Choquard equation (1.2) has no nontrivial ground state solution when $p \leq (N + \alpha)/N$ or $p \geq (N + \alpha)/(N - 2)$. For the more content of the equation (1.2), we refer the interested reader to the guide [14].

When V is a positive constant, $F \in C^1$ and satisfies

(F₁) there exists a positive constant C such that

$$|tF'(t)| \leq C(|t|^{(N+\alpha)/N} + |t|^{(N+\alpha)/(N-2)}), \quad t \in \mathbb{R},$$

(F₂) $\lim_{t \rightarrow \infty} F(t)/|t|^{(N+\alpha)/(N-2)} = 0$ and $\lim_{t \rightarrow 0} F(t)/|t|^{(N+\alpha)/N} = 0$,

(F₃) there exists a constant $t_0 \in \mathbb{R} \setminus \{0\}$ such that $F(t_0) \neq 0$,

Moroz and Van Schaftingen [13] proved the existence of ground state to the equation (1.1). J. Seok [17] acts against the subcriticality condition (F₂), and consider that F is doubly critical, i.e.,

$$F(t) = \frac{1}{p}|t|^p + \frac{1}{q}|t|^q, \quad p = \frac{N + \alpha}{N}, \quad q = \frac{N + \alpha}{N - 2}.$$

The functional $\int_{\mathbb{R}^N} (I_\alpha * F(u))F(u)$ contains two terms $\int_{\mathbb{R}^N} (I_\alpha * |u|^p)|u|^p$ and $\int_{\mathbb{R}^N} (I_\alpha * |u|^q)|u|^q$. For the related critical problems involving only a single critical exponent, we refer to [1, 12, 16]. However, few work concerns the case that F is doubly critical. J. Seok cleverly estimated the energy, overcome the lack of compactness, and proved that the equation (1.1) admits a nontrivial solution under appropriate assumptions on α and N in radial space $H_r^1(\mathbb{R}^N)$. Two natural questions arise. Does the solution has the least energy among nontrivial solutions of equation (1.1) in $H^1(\mathbb{R}^N)$? Furthermore, does the equation (1.1) has ground state solution in $H^1(\mathbb{R}^N)$ if V is not a constant? To the best of our knowledge, there are no results on these questions. The present paper is devoted to these aspects and answers these questions. Our main result is as follows.

Theorem 1.1. *Let $N \geq 5, \alpha \in (0, N - 4)$, the potential V satisfy the condition (V), and $f(t) = |t|^{p-2}t + |t|^{q-2}t$ for all $t \in \mathbb{R}$, where $p = (N + \alpha)/N$ and $q = (N + \alpha)/(N - 2)$. Then the equation (1.1) has a nonnegative ground state solution provided $V(x) < V_\infty$ for all $x \in \mathbb{R}^N$.*

We say that a function $u \in H^1(\mathbb{R}^N)$ is a solution to (1.1) if $J'(u) = 0$, for the definition of J , see (2.2) below. The solution u obtained in Theorem 1.1 is a ground state solution in the sense that it minimizes the corresponding energy functional J among all nontrivial solutions.

Since the appearance of the potential V breaks down the invariance under translations in \mathbb{R}^N , we cannot use the translation invariant argument directly. To overcome this challenge, we need use the comparison arguments between the minimax level of the energy functional corresponding to (1.1) and that to the limit equation

$$-\Delta u + V_\infty u = (I_\alpha * F(u))f(u) \quad \text{in } \mathbb{R}^N. \quad (1.3)$$

Thus, we first need to study the existence of ground state solution to the equation (1.3). The result is stated as follows.

Theorem 1.2. *Let $N \geq 5, \alpha \in (0, N - 4)$, $f(t) = |t|^{p-2}t + |t|^{q-2}t$ for all $t \in \mathbb{R}$, where $p = (N + \alpha)/N$ and $q = (N + \alpha)/(N - 2)$. Then the equation (1.3) has a nonnegative ground state solution.*

The proof of Theorem 1.2 relies on two ingredients: the nontrivial nature of solution to the equation (1.3) up to translation under the strict inequality

$$c < \min \left\{ \frac{1}{2} \left(1 - \frac{1}{p} \right) (pV_\infty^p S_1^p)^{1/(p-1)}, \frac{1}{2} \left(1 - \frac{1}{q} \right) (qS_2^q)^{1/(q-1)} \right\}$$

obtained by a concentration-compactness argument (Lemma 3.2) and the proof of the latter strict inequality (Lemma 3.1).

The rest of this paper is organized as follows. We give some preliminaries in Section 2. Theorems 1.2 and 1.1 are proved in Sections 3 and 4, respectively.

Throughout this paper we always use the following notations. The letters $C_i, i = 1, 2, \dots$ and C are positive constants which may change from line to line. $\mathbb{R}_+ = [0, \infty)$. $B_R(y)$ denotes the open ball centered at y with radius R in \mathbb{R}^N . For each $s \in [1, \infty)$, $L^s(\mathbb{R}^N)$ denotes the Lebesgue space with the norm $\|u\|_s = (\int_{\mathbb{R}^N} |u|^s)^{1/s}$, $u \in L^s(\mathbb{R}^N)$.

2 Preliminaries

In this section, we give some preliminaries. When V satisfies the condition (V), the following lemmas are all set up.

Let $H^1(\mathbb{R}^N)$ be the usual Sobolev space. According to the conditions of the function V , we can define an equivalent norm on $H^1(\mathbb{R}^N)$,

$$(u, v) = \int_{\mathbb{R}^N} (\nabla u \cdot \nabla v + Vuv), \quad \|u\| = (u, u)^{1/2}, \quad u, v \in H^1(\mathbb{R}^N).$$

$H^1(\mathbb{R}^N)$ is embedded continuously into $L^s(\mathbb{R}^N)$ for each $s \in [2, 2^*]$. Thus, for each $s \in [2, 2^*]$, there exists a positive constant C_s such that

$$\|u\|_s \leq C_s \|u\|, \quad u \in H^1(\mathbb{R}^N). \quad (2.1)$$

The energy functional J associated to the equation (1.1) is defined by

$$J(u) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + Vu^2) - \frac{1}{2} \int_{\mathbb{R}^N} (I_\alpha * F(u))F(u), \quad u \in H^1(\mathbb{R}^N). \quad (2.2)$$

By the Hardy–Littlewood–Sobolev inequality, see Lemma 2.1 below, we know that J is well defined on $H^1(\mathbb{R}^N)$ and belongs to C^1 , and its derivative is given by

$$\langle J'(u), v \rangle = \int_{\mathbb{R}^N} (\nabla u \cdot \nabla v + Vuv) - \int_{\mathbb{R}^N} (I_\alpha * F(u))f(u)v, \quad u, v \in H^1(\mathbb{R}^N).$$

Therefore, a weak solution of the equation (1.1) corresponds to a critical point of the energy functional J .

We consider the following constraint minimization problem

$$c := \inf_{\mathcal{N}} J, \quad (2.3)$$

where \mathcal{N} denotes the Nehari manifold

$$\begin{aligned} \mathcal{N} &= \{u \in H^1(\mathbb{R}^N) \setminus \{0\} : I(u) := \langle J'(u), u \rangle = 0\}, \\ I(u) &= \|u\|^2 - \int_{\mathbb{R}^N} \left[I_\alpha * \left(\frac{1}{p}|u|^p + \frac{1}{q}|u|^q \right) \right] (|u|^p + |u|^q), \quad u \in H^1(\mathbb{R}^N), \end{aligned} \quad (2.4)$$

and \mathcal{N} is C^1 .

To study the constraint minimization problem related with (1.1), we need to recall the following well-known Hardy–Littlewood–Sobolev inequality, see [7].

Lemma 2.1 (Hardy–Littlewood–Sobolev inequality). *Let $r, s > 1$ and $\mu \in (0, N)$ with*

$$\frac{1}{r} + \frac{\mu}{N} + \frac{1}{s} = 2.$$

Then there exists a sharp constant $C(N, \mu, r) > 0$ such that for all $u \in L^r(\mathbb{R}^N)$ and $v \in L^s(\mathbb{R}^N)$,

$$\left| \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{u(x)v(y)}{|x-y|^\mu} dx dy \right| \leq C(N, \mu, r) |u|_r |v|_s. \quad (2.5)$$

The sharp constant satisfies that

$$C(N, \mu, r) \leq \frac{N}{N-\mu} (|\mathbb{S}^{N-1}|/N)^{\mu/N} \frac{1}{rs} \left(\left(\frac{\mu/N}{1-1/r} \right)^{\mu/N} + \left(\frac{\mu/N}{1-1/s} \right)^{\mu/N} \right).$$

If $r = s = 2N/(2N - \mu)$, then

$$C(N, \mu, r) = C(N, \mu) = \pi^{\mu/2} \frac{\Gamma(N/2 - \mu/2)}{\Gamma(N - \mu/2)} \left(\frac{\Gamma(N/2)}{\Gamma(N)} \right)^{-1 + \mu/N},$$

and there is equality in (2.5) if and only if $v = Cu$ and

$$u(x) = A(\gamma^2 + |x - a|^2)^{-(2N-\mu)/2}$$

for some $A \in \mathbb{C}$, $0 \neq \gamma \in \mathbb{R}$ and $a \in \mathbb{R}^N$.

Notice that, when $\mu = N - \alpha$, by the Hardy–Littlewood–Sobolev inequality, for each $u \in H^1(\mathbb{R}^N)$, the integral

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x)|^\beta |u(y)|^\theta}{|x-y|^{N-\alpha}} dx dy$$

is well defined if

$$\beta, \theta \in \left[\frac{N+\alpha}{N}, \frac{N+\alpha}{N-2} \right].$$

Let

$$\Phi(u) = \int_{\mathbb{R}^N} (I_\alpha * |u|^\beta) |u|^\theta, \quad u \in H^1(\mathbb{R}^N),$$

where $\alpha \in (0, N)$, and $\beta, \theta \in [(N + \alpha)/N, (N + \alpha)/(N - 2)]$. By the Hardy–Littlewood–Sobolev and the Hölder inequalities, a standard analysis shows that the following properties hold.

Lemma 2.2. *If $\{u_n\} \subset H^1(\mathbb{R}^N)$ is a sequence converging weakly to u in $H^1(\mathbb{R}^N)$ as $n \rightarrow \infty$, then we have*

$$\Phi(u) \leq \liminf_{n \rightarrow \infty} \Phi(u_n). \quad (2.6)$$

$$\langle \Phi'(u_n), v \rangle \rightarrow \langle \Phi'(u), v \rangle, \quad v \in H^1(\mathbb{R}^N). \quad (2.7)$$

Proof. Assume that $\{v_n\}$ is an arbitrary subsequence of $\{u_n\}$. Since $v_n \rightarrow u$ in $L^s_{\text{loc}}(\mathbb{R}^N)$ for $s \in [1, 2^*)$, there exists a subsequence $\{w_n\}$ of $\{v_n\}$ such that $w_n \rightarrow u$ a.e. on \mathbb{R}^N .

By Fatou's lemma, we have $\Phi(u) \leq \liminf_{n \rightarrow \infty} \Phi(w_n)$. Thus, (2.6) holds.

Next, we will prove (2.7). Using the Hardy–Littlewood–Sobolev inequality and the symmetry property of convolution, we deduce that, for $\beta, \theta \in [(N + \alpha)/N, (N + \alpha)/(N - 2)]$,

$$\begin{aligned} & \int_{\mathbb{R}^N} |(I_\alpha * |w_n|^\beta) |w_n|^{\theta-2} w_n v - (I_\alpha * |u|^\beta) |u|^{\theta-2} u v| \\ & \leq \int_{\mathbb{R}^N} |(I_\alpha * |w_n|^\beta) (|w_n|^{\theta-2} w_n - |u|^{\theta-2} u) v| + \int_{\mathbb{R}^N} |I_\alpha * (|w_n|^\beta - |u|^\beta) |u|^{\theta-2} u v| \\ & \leq C |w_n|_{2\beta N/(N+\alpha)}^\beta (|w_n|^{\theta-2} w_n - |u|^{\theta-2} u) v|_{2N/(N+\alpha)} + \int_{\mathbb{R}^N} |(I_\alpha * |u|^{\theta-2} u v) (|w_n|^\beta - |u|^\beta)|. \end{aligned}$$

Set $2N/(N + \alpha) = r$. Since $\{|w_n|^{\theta-2} w_n - |u|^{\theta-2} u\}^r$ is bounded in $L^{\theta/(\theta-1)}(\mathbb{R}^N)$ and $|w_n|^{\theta-2} w_n \rightarrow |u|^{\theta-2} u$ a.e. on \mathbb{R}^N , it follows from $|v|^r \in L^\theta(\mathbb{R}^N)$ that $(|w_n|^{\theta-2} w_n - |u|^{\theta-2} u) v|_r \rightarrow 0$. Further, since $\{w_n\}$ is bounded in $L^{\beta r}(\mathbb{R}^N)$, we see that

$$C |w_n|_{2\beta N/(N+\alpha)}^\beta (|w_n|^{\theta-2} w_n - |u|^{\theta-2} u) v|_{2N/(N+\alpha)} \rightarrow 0. \quad (2.8)$$

Since $I_\alpha * |u|^{\theta-2} u v \in L^{r/(r-1)}(\mathbb{R}^N)$, $\{|w_n|^\beta - |u|^\beta\}$ is bounded in $L^r(\mathbb{R}^N)$ and $|w_n|^\beta \rightarrow |u|^\beta$ a.e. on \mathbb{R}^N , we see that

$$\int_{\mathbb{R}^N} |(I_\alpha * |u|^{\theta-2} u v) (|w_n|^\beta - |u|^\beta)| \rightarrow 0. \quad (2.9)$$

It follows from (2.8) and (2.9) that $\langle \Phi'(w_n), v \rangle \rightarrow \langle \Phi'(u), v \rangle$. Thus, (2.7) is true. \square

Lemma 2.3. *For each $u \in H^1(\mathbb{R}^N) \setminus \{0\}$, there exists a unique $t_u > 0$ such that $t_u u \in \mathcal{N}$. Moreover, $J(t_u u) = \max_{t \in \mathbb{R}_+} J(tu)$.*

Proof. For each $u \in H^1(\mathbb{R}^N) \setminus \{0\}$, the function $g(t) := J(tu)$ takes the form $C_1 t^2 - C_2 t^{2p} - C_3 t^{p+q} - C_4 t^{2q}$ for all $t \in \mathbb{R}_+$. By Remark 2.4 below, we see that g has a unique positive critical point t_u corresponding to its maximum, i.e., $g'(t_u) = 0$ and $g(t_u) = \max_{\mathbb{R}_+} g$. Hence, $I(t_u u) = t_u g'(t_u) = 0$ and $J(t_u u) = \max_{t \in \mathbb{R}_+} J(tu)$. \square

Remark 2.4. Let a, b, c be positive constants. By elementary calculation one obtains that the function

$$g(t) = t^2 - at^{2p} - bt^{p+q} - ct^{2q}, \quad t \in \mathbb{R}_+,$$

has a unique positive critical point t_0 with $g'(t) > 0$ for all $t \in (0, t_0)$, and $g'(t) < 0$ for all $t \in (t_0, \infty)$. Thus, g takes the maximum at $t = t_0$.

Lemma 2.5. *There exist positive constants δ and ρ such that $\|u\| \geq \delta$ and $\langle I'(u), u \rangle \leq -\rho$ for all $u \in \mathcal{N}$.*

Proof. Because of the definition of \mathcal{N} , by (2.4), the Hardy–Littlewood–Sobolev inequality and (2.1), we can derive that

$$\begin{aligned} \|u\|^2 &\leq \frac{1}{p} \int_{\mathbb{R}^N} (I_\alpha * (|u|^p + |u|^q)) (|u|^p + |u|^q) \\ &\leq C_1 \|u\|^{2p} + C_2 \|u\|^{p+q} + C_3 \|u\|^{2q}, \quad u \in \mathcal{N}. \end{aligned} \quad (2.10)$$

Since $p, q > 1$, there exists a positive constant δ such that $\|u\| \geq \delta$ for all $u \in \mathcal{N}$.

Furthermore, by (2.4) and (2.10), we have that

$$\begin{aligned} -\langle I'(u), u \rangle &= \frac{2(p-1)}{p} \int_{\mathbb{R}^N} (I_\alpha * |u|^p) |u|^p + \frac{2(q-1)}{q} \int_{\mathbb{R}^N} (I_\alpha * |u|^q) |u|^q \\ &\quad + \frac{(p+q-2)(p+q)}{pq} \int_{\mathbb{R}^N} (I_\alpha * |u|^p) |u|^q \\ &\geq \frac{2(p-1)}{p} \left[\int_{\mathbb{R}^N} (I_\alpha * |u|^p) |u|^p + \int_{\mathbb{R}^N} (I_\alpha * |u|^q) |u|^q + 2 \int_{\mathbb{R}^N} (I_\alpha * |u|^p) |u|^q \right] \\ &\geq 2(p-1) \|u\|^2 \geq 2(p-1) \delta^2. \end{aligned}$$

Set $\rho = 2(p-1)\delta^2$. The proof is completed. \square

To obtain a $(PS)_c$ sequence of the energy functional J , we show that the functional has the mountain pass geometry.

Lemma 2.6. *The functional J satisfies the mountain pass geometry, that is,*

- (i) *there exist $r, \eta > 0$ such that $J(u) \geq \eta$ for all $u \in \partial B_r = \{u \in H^1(\mathbb{R}^N) : \|u\| = r\}$, and $J(u) > 0$ for all u with $0 < \|u\| \leq r$;*
- (ii) *there exists $u_0 \in H^1(\mathbb{R}^N)$ such that $\|u_0\| > r$ and $J(u_0) < 0$.*

Proof. (i) By the Hardy–Littlewood–Sobolev inequality and (2.1), we derive that

$$J(u) \geq \frac{1}{2} \|u\|^2 - C_1 \|u\|^{2p} - C_2 \|u\|^{p+q} - C_3 \|u\|^{2q}.$$

Then (i) follows if $r > 0$ is small enough.

(ii) For any given $u \in H^1(\mathbb{R}^N) \setminus \{0\}$, the function $g(t) := J(tu)$ take the form $C_1 t^2 - C_2 t^{2p} - C_3 t^{p+q} - C_4 t^{2q}$ for all $t \in \mathbb{R}_+$. Since $g(0) = 0$ and $\lim_{t \rightarrow \infty} g(t) = -\infty$, there exists $t_0 > 0$ large enough such that (ii) holds for $u_0 = t_0 u$. \square

We define

$$c_1 = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} J(\gamma(t)),$$

where $\Gamma = \{\gamma \in C([0,1], H^1(\mathbb{R}^N)) : \gamma(0) = 0, J(\gamma(1)) < 0\}$. Then it follows from Lemma 2.6 (i) that $c_1 > 0$. Furthermore, we can show that the minimax value c_1 also can be characterized by $c = c_1 = c_2$, where c is defined in (2.3), and

$$c_2 = \inf_{H^1(\mathbb{R}^N) \setminus \{0\}} \max_{t \in \mathbb{R}_+} J(tu).$$

According to Lemma 2.6 and [19, Theorem 2.8, p.41], there is a $(PS)_c$ sequence $\{u_n\} \subset H^1(\mathbb{R}^N)$, that is,

$$J'(u_n) \rightarrow 0, \quad J(u_n) \rightarrow c.$$

Lemma 2.7. Let $\{u_n\} \subset H^1(\mathbb{R}^N)$ be a $(PS)_c$ sequence of J . Then $\{u_n\}$ is bounded in $H^1(\mathbb{R}^N)$.

Proof. Since $\{u_n\} \subset H^1(\mathbb{R}^N)$ is a $(PS)_c$ sequence, we have that

$$c + o(1) + o(1)\|u_n\| = J(u_n) - \frac{1}{2p}I(u_n) \geq \frac{1}{2} \left(1 - \frac{1}{p}\right) \|u_n\|^2.$$

Because of $p > 1$, the above inequality induce that $\{u_n\}$ is bounded. \square

Lemma 2.8. Let $\{u_n\} \subset H^1(\mathbb{R}^N)$ be a $(PS)_c$ sequence of J and $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^N)$. If $u \neq 0$, then the equation (1.1) has a nonnegative ground state solution of the equation (1.1).

Proof. Since $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^N)$, it follows from (2.7) that $J'(u) = 0$. Because $u \neq 0$, we know that $u \in \mathcal{N}$ is a nonzero critical point of J . Using (2.6), we obtain

$$\begin{aligned} c &\leq J(u) - \frac{1}{2}I(u) \\ &= \frac{p-1}{2p^2} \int_{\mathbb{R}^N} (I_\alpha * |u|^p)|u|^p + \frac{q-1}{2q^2} \int_{\mathbb{R}^N} (I_\alpha * |u|^q)|u|^q + \frac{p+q-2}{2pq} \int_{\mathbb{R}^N} (I_\alpha * |u|^p)|u|^q \\ &\leq \liminf_{n \rightarrow \infty} \left[\frac{p-1}{2p^2} \int_{\mathbb{R}^N} (I_\alpha * |u_n|^p)|u_n|^p + \frac{q-1}{2q^2} \int_{\mathbb{R}^N} (I_\alpha * |u_n|^q)|u_n|^q \right. \\ &\quad \left. + \frac{p+q-2}{2pq} \int_{\mathbb{R}^N} (I_\alpha * |u_n|^p)|u_n|^q \right] \\ &= \liminf_{n \rightarrow \infty} \left[J(u_n) - \frac{1}{2}I(u_n) \right] = c, \end{aligned}$$

which implied that $J(u) = c$.

Consider $w = |u|$. An easy computation shows that $w \in \mathcal{N}$ and $J(w) = J(u) = c$. It follows from the Lagrange multiplier theorem that $J'(w) = \lambda I'(w)$ for some $\lambda \in \mathbb{R}$. Hence, $\lambda \langle I'(w), w \rangle = \langle J'(w), w \rangle = 0$. By Lemma 2.5, we know that $\langle I'(w), w \rangle \leq -\rho$. Thus, $\lambda = 0$, which implies that w is a nonnegative solution of the equation (1.1). Since $J(w) = c$, it is a nonnegative ground state solution to the equation (1.1). \square

3 Proof of Theorem 1.2

Before giving the proof of Theorem 1.1, we need give the proof of Theorem 1.2. In this section, $V = V_\infty$.

The following two inequalities are special cases of the Hardy–Littlewood–Sobolev inequality. The first one is

$$S_1 \left(\int_{\mathbb{R}^N} (I_\alpha * |u|^{(N+\alpha)/N}) |u|^{(N+\alpha)/N} \right)^{N/(N+\alpha)} \leq \int_{\mathbb{R}^N} u^2, \quad u \in H^1(\mathbb{R}^N). \quad (3.1)$$

The second one is

$$S_2 \left(\int_{\mathbb{R}^N} (I_\alpha * |v|^{(N+\alpha)/(N-2)}) |v|^{(N+\alpha)/(N-2)} \right)^{(N-2)/(N+\alpha)} \leq \int_{\mathbb{R}^N} |\nabla v|^2, \quad v \in H^1(\mathbb{R}^N). \quad (3.2)$$

By Lemma 2.1 and [1, Lemma 2.2], we see that the best constants S_1 and S_2 are achieved if and only if $u_\lambda(x) = C_1 \lambda^{N/2} / (\lambda^2 + |x|^2)^{N/2}$ for all $x \in \mathbb{R}$, and $v_\lambda(x) = C_2 \lambda^{(N-2)/2} / (\lambda^2 + |x|^2)^{(N-2)/2}$ for all $x \in \mathbb{R}$, respectively. The next lemma comes from [17], for reader's convenience, we give a detailed proof.

Lemma 3.1. *There exists $u_0 \in H^1(\mathbb{R}^N) \setminus \{0\}$ such that*

$$\sup_{t \in \mathbb{R}_+} J(tu_0) < \min \left\{ \frac{1}{2} \left(1 - \frac{1}{p}\right) (pV_\infty^p S_1^p)^{1/(p-1)}, \frac{1}{2} \left(1 - \frac{1}{q}\right) (qS_2^q)^{1/(q-1)} \right\}.$$

Proof. Let us define two functions

$$u_\lambda(x) := C_1 \frac{\lambda^{N/2}}{(\lambda^2 + |x|^2)^{N/2}}, \quad v_\lambda(x) := C_2 \frac{\lambda^{(N-2)/2}}{(\lambda^2 + |x|^2)^{(N-2)/2}}, \quad x \in \mathbb{R}^N, \lambda > 0,$$

which are the extremal functions of inequalities (3.1) and (3.2), respectively. Since $N \geq 5$, $u_\lambda, v_\lambda \in H^1(\mathbb{R}^N)$. By computing, we have that for each $\lambda > 0$

$$u_\lambda(x) = \lambda^{-N/2} u_1(x/\lambda), \quad v_\lambda(x) = \lambda^{-(N-2)/2} v_1(x/\lambda), \quad x \in \mathbb{R}^N,$$

$$|u_\lambda|_2 = |u_1|_2, \quad |v_\lambda|_2 = \lambda |v_1|_2, \quad (3.3)$$

$$|\nabla u_\lambda|_2 = \lambda^{-1} |\nabla u_1|_2, \quad |\nabla v_\lambda|_2 = |\nabla v_1|_2, \quad (3.4)$$

$$\int_{\mathbb{R}^N} (I_\alpha * |u_\lambda|^p) |u_\lambda|^p = \int_{\mathbb{R}^N} (I_\alpha * |u_1|^p) |u_1|^p, \quad (3.5)$$

$$\int_{\mathbb{R}^N} (I_\alpha * |u_\lambda|^q) |u_\lambda|^q = \lambda^{-q} \int_{\mathbb{R}^N} (I_\alpha * |u_1|^q) |u_1|^q, \quad (3.6)$$

$$\int_{\mathbb{R}^N} (I_\alpha * |u_\lambda|^q) |u_\lambda|^q = \lambda^{-2q} \int_{\mathbb{R}^N} (I_\alpha * |u_1|^q) |u_1|^q, \quad (3.7)$$

$$\int_{\mathbb{R}^N} (I_\alpha * |v_\lambda|^p) |v_\lambda|^p = \lambda^{2p} \int_{\mathbb{R}^N} (I_\alpha * |v_1|^p) |v_1|^p,$$

$$\int_{\mathbb{R}^N} (I_\alpha * |v_\lambda|^q) |v_\lambda|^q = \lambda^p \int_{\mathbb{R}^N} (I_\alpha * |v_1|^q) |v_1|^q,$$

$$\int_{\mathbb{R}^N} (I_\alpha * |v_\lambda|^q) |v_\lambda|^q = \int_{\mathbb{R}^N} (I_\alpha * |v_1|^q) |v_1|^q.$$

The constants C_1 and C_2 are chosen to satisfy

$$\int_{\mathbb{R}^N} u_1^2 = \int_{\mathbb{R}^N} (I_\alpha * |u_1|^p) |u_1|^p, \quad \int_{\mathbb{R}^N} |\nabla v_1|^2 = \int_{\mathbb{R}^N} (I_\alpha * |v_1|^q) |v_1|^q. \quad (3.8)$$

Let $s_\lambda > 0$ and $t_\lambda > 0$ satisfy

$$J(s_\lambda u_\lambda) = \max_{s \in \mathbb{R}_+} J(su_\lambda), \quad J(t_\lambda v_\lambda) = \max_{t \in \mathbb{R}_+} J(tv_\lambda). \quad (3.9)$$

Then there exist $\bar{s}_\lambda > s_\lambda$ and $\bar{t}_\lambda > t_\lambda$ such that $J(\bar{s}_\lambda u_\lambda) < 0$ and $J(\bar{t}_\lambda v_\lambda) < 0$. Thus, by defining $\gamma_1(t) = t\bar{s}_\lambda u_\lambda$ and $\gamma_2(t) = t\bar{t}_\lambda v_\lambda$ for all $t \in [0, 1]$, we see that

$$c \leq \min \left\{ \max_{t \in [0,1]} J(\gamma_1(t)), \max_{t \in [0,1]} J(\gamma_2(t)) \right\} = \min \{J(s_\lambda u_\lambda), J(t_\lambda v_\lambda)\}.$$

It follows from (3.9), (3.3)–(3.7) that

$$\begin{aligned}
0 &= \frac{d}{dt} [J(tu_\lambda)] \Big|_{t=s_\lambda} \\
&= s_\lambda \int_{\mathbb{R}^N} |\nabla u_\lambda|^2 + s_\lambda \int_{\mathbb{R}^N} V_\infty u_\lambda^2 - \frac{s_\lambda^{2p-1}}{p} \int_{\mathbb{R}^N} (I_\alpha * |u_\lambda|^p) |u_\lambda|^p \\
&\quad - \frac{(p+q)s_\lambda^{p+q-1}}{pq} \int_{\mathbb{R}^N} (I_\alpha * |u_\lambda|^p) |u_\lambda|^q - \frac{s_\lambda^{2q-1}}{q} \int_{\mathbb{R}^N} (I_\alpha * |u_\lambda|^q) |u_\lambda|^q \\
&= \frac{s_\lambda}{\lambda^2} \int_{\mathbb{R}^N} |\nabla u_1|^2 + s_\lambda \int_{\mathbb{R}^N} V_\infty u_1^2 - \frac{s_\lambda^{2p-1}}{p} \int_{\mathbb{R}^N} (I_\alpha * |u_1|^p) |u_1|^p \\
&\quad - \frac{(p+q)s_\lambda^{p+q-1}}{pq\lambda^q} \int_{\mathbb{R}^N} (I_\alpha * |u_1|^p) |u_1|^q - \frac{s_\lambda^{2q-1}}{q\lambda^{2q}} \int_{\mathbb{R}^N} (I_\alpha * |u_1|^q) |u_1|^q,
\end{aligned} \tag{3.10}$$

which implies that

$$0 \leq \frac{1}{\lambda^2} \int_{\mathbb{R}^N} |\nabla u_1|^2 + \int_{\mathbb{R}^N} V_\infty u_1^2 - \frac{s_\lambda^{2p-2}}{p} \int_{\mathbb{R}^N} (I_\alpha * |u_1|^p) |u_1|^p. \tag{3.11}$$

Let $s_\infty = \limsup_{\lambda \rightarrow \infty} s_\lambda$. Suppose that $s_\infty = \infty$. Then we get a contradiction by (3.11). Thus $s_\infty < \infty$. Taking again $\lambda \rightarrow \infty$ in (3.10), we obtain

$$s_\infty \int_{\mathbb{R}^N} V_\infty u_1^2 = \frac{s_\infty^{2p-1}}{p} \int_{\mathbb{R}^N} (I_\alpha * |u_1|^p) |u_1|^p,$$

which from (3.8) implies $s_\infty = (pV_\infty)^{1/(2p-2)}$. Furthermore, we can prove that $\lim_{\lambda \rightarrow \infty} s_\lambda = (pV_\infty)^{1/(2p-2)}$. Hence,

$$\begin{aligned}
J(s_\lambda u_\lambda) &= \frac{s_\lambda^2}{2} \int_{\mathbb{R}^N} |\nabla u_\lambda|^2 + \frac{s_\lambda^2}{2} \int_{\mathbb{R}^N} V_\infty u_\lambda^2 - \frac{s_\lambda^{2p}}{2p^2} \int_{\mathbb{R}^N} (I_\alpha * |u_\lambda|^p) |u_\lambda|^p \\
&\quad - \frac{s_\lambda^{p+q}}{pq} \int_{\mathbb{R}^N} (I_\alpha * |u_\lambda|^p) |u_\lambda|^q - \frac{s_\lambda^{2q}}{2q^2} \int_{\mathbb{R}^N} (I_\alpha * |u_\lambda|^q) |u_\lambda|^q \\
&= \frac{s_\lambda^2}{2\lambda^2} \int_{\mathbb{R}^N} |\nabla u_1|^2 + \frac{s_\lambda^2}{2} \int_{\mathbb{R}^N} V_\infty u_1^2 - \frac{s_\lambda^{2p}}{2p^2} \int_{\mathbb{R}^N} (I_\alpha * |u_1|^p) |u_1|^p \\
&\quad - \frac{s_\lambda^{p+q}}{pq\lambda^q} \int_{\mathbb{R}^N} (I_\alpha * |u_1|^p) |u_1|^q - \frac{s_\lambda^{2q}}{2q^2\lambda^{2q}} \int_{\mathbb{R}^N} (I_\alpha * |u_1|^q) |u_1|^q \\
&\leq \frac{1}{2} \left(V_\infty s_\lambda^2 - \frac{s_\lambda^{2p}}{p^2} \right) \int_{\mathbb{R}^N} u_1^2 - \frac{1}{\lambda^q} \left[\frac{s_\lambda^{p+q}}{pq} \int_{\mathbb{R}^N} (I_\alpha * |u_1|^p) |u_1|^q - \frac{s_\lambda^2}{2\lambda^{2-q}} \int_{\mathbb{R}^N} |\nabla u_1|^2 \right].
\end{aligned}$$

Note that the function $f(s) := V_\infty p^2 s^2 - s^{2p}$, $s \in \mathbb{R}_+$, attains its maximum at $s = s_\infty$. This shows that

$$\frac{1}{2} \left(V_\infty s_\lambda^2 - \frac{s_\lambda^{2p}}{p^2} \right) \int_{\mathbb{R}^N} u_1^2 \leq \frac{1}{2} \left(1 - \frac{1}{p} \right) (pV_\infty^p S_1^p)^{1/(p-1)}.$$

It follows from $4 + \alpha < N$ that $q < 2$ and

$$\lim_{\lambda \rightarrow \infty} \left[\frac{s_\lambda^{p+q}}{pq} \int_{\mathbb{R}^N} (I_\alpha * |u_1|^p) |u_1|^q - \frac{s_\lambda^2}{2\lambda^{2-q}} \int_{\mathbb{R}^N} |\nabla u_1|^2 \right] = \frac{s_\infty^{p+q}}{pq} \int_{\mathbb{R}^N} (I_\alpha * |u_1|^p) |u_1|^q > 0.$$

Thus,

$$J(s_\lambda u_\lambda) < \frac{1}{2} \left(1 - \frac{1}{p}\right) (pV_\infty^p S_1^p)^{1/(p-1)}$$

for sufficiently large $\lambda > 0$.

Similarly we have

$$\begin{aligned} 0 &= \frac{d}{dt} [J(tv_\lambda)] \Big|_{t=t_\lambda} \\ &= t_\lambda \int_{\mathbb{R}^N} |\nabla v_\lambda|^2 + t_\lambda \int_{\mathbb{R}^N} V_\infty v_\lambda^2 - \frac{t_\lambda^{2p-1}}{p} \int_{\mathbb{R}^N} (I_\alpha * |v_\lambda|^p) |v_\lambda|^p \\ &\quad - \frac{(p+q)t_\lambda^{p+q-1}}{pq} \int_{\mathbb{R}^N} (I_\alpha * |v_\lambda|^p) |v_\lambda|^q - \frac{t_\lambda^{2q-1}}{q} \int_{\mathbb{R}^N} (I_\alpha * |v_\lambda|^q) |v_\lambda|^q \\ &= t_\lambda \int_{\mathbb{R}^N} |\nabla v_1|^2 + t_\lambda \lambda^2 \int_{\mathbb{R}^N} V_\infty v_1^2 - \frac{t_\lambda^{2p-1} \lambda^{2p}}{p} \int_{\mathbb{R}^N} (I_\alpha * |v_1|^p) |v_1|^p \\ &\quad - \frac{(p+q)t_\lambda^{p+q-1} \lambda^p}{pq} \int_{\mathbb{R}^N} (I_\alpha * |v_1|^p) |v_1|^q - \frac{t_\lambda^{2q-1}}{q} \int_{\mathbb{R}^N} (I_\alpha * |v_1|^q) |v_1|^q, \end{aligned} \quad (3.12)$$

which implies that

$$0 \leq \int_{\mathbb{R}^N} |\nabla v_1|^2 + \lambda^2 \int_{\mathbb{R}^N} V_\infty v_1^2 - \frac{t_\lambda^{2q-2}}{q} \int_{\mathbb{R}^N} (I_\alpha * |v_1|^q) |v_1|^q. \quad (3.13)$$

Let $t_0 := \limsup_{\lambda \rightarrow 0} t_\lambda$. Then we can get that $t_0 < \infty$ by (3.13). Taking again $\lambda \rightarrow 0$ in (3.12), we get

$$t_0 \int_{\mathbb{R}^N} |\nabla v_1|^2 = \frac{t_0^{2q-1}}{q} \int_{\mathbb{R}^N} (I_\alpha * |v_1|^q) |v_1|^q,$$

which implies $t_0 = q^{1/(2q-2)}$. Furthermore, we can prove that $\lim_{\lambda \rightarrow 0} t_\lambda = q^{1/(2q-2)}$. Thus,

$$\begin{aligned} J(t_\lambda v_\lambda) &= \frac{t_\lambda^2}{2} \int_{\mathbb{R}^N} |\nabla v_\lambda|^2 + \frac{t_\lambda^2}{2} \int_{\mathbb{R}^N} V_\infty v_\lambda^2 - \frac{t_\lambda^{2p}}{2p^2} \int_{\mathbb{R}^N} (I_\alpha * |v_\lambda|^p) |v_\lambda|^p \\ &\quad - \frac{t_\lambda^{p+q}}{pq} \int_{\mathbb{R}^N} (I_\alpha * |v_\lambda|^p) |v_\lambda|^q - \frac{t_\lambda^{2q}}{2q^2} \int_{\mathbb{R}^N} (I_\alpha * |v_\lambda|^q) |v_\lambda|^q \\ &= \frac{t_\lambda^2}{2} \int_{\mathbb{R}^N} |\nabla v_1|^2 + \frac{t_\lambda^2 \lambda^2}{2} \int_{\mathbb{R}^N} V_\infty v_1^2 - \frac{t_\lambda^{2p} \lambda^{2p}}{2p^2} \int_{\mathbb{R}^N} (I_\alpha * |v_1|^p) |v_1|^p \\ &\quad - \frac{t_\lambda^{p+q} \lambda^p}{pq} \int_{\mathbb{R}^N} (I_\alpha * |v_1|^p) |v_1|^q - \frac{t_\lambda^{2q}}{2q^2} \int_{\mathbb{R}^N} (I_\alpha * |v_1|^q) |v_1|^q \\ &\leq \frac{1}{2} \left(t_\lambda^2 - \frac{t_\lambda^{2q}}{q^2} \right) \int_{\mathbb{R}^N} |\nabla v_1|^2 - \lambda^p \left[\frac{t_\lambda^{p+q}}{pq} \int_{\mathbb{R}^N} (I_\alpha * |v_1|^p) |v_1|^q - \frac{t_\lambda^2 \lambda^{2-p}}{2} \int_{\mathbb{R}^N} V_\infty v_1^2 \right]. \end{aligned}$$

Note that the function $g(t) := q^2 t^2 - t^{2q}$, $t \in \mathbb{R}_+$, attains its maximum at $t = t_0$. Hence,

$$\frac{1}{2} \left(t_\lambda^2 - \frac{t_\lambda^{2q}}{q^2} \right) \int_{\mathbb{R}^N} |\nabla v_1|^2 \leq \frac{1}{2} \left(1 - \frac{1}{q}\right) (qS_2^q)^{1/(q-1)}.$$

It follows from $p < 2$ that

$$\lim_{\lambda \rightarrow 0} \left[\frac{t_\lambda^{p+q}}{pq} \int_{\mathbb{R}^N} (I_\alpha * |v_1|^p) |v_1|^q - \frac{t_\lambda^2 \lambda^{2-p}}{2} \int_{\mathbb{R}^N} V_\infty v_1^2 \right] = \frac{t_0^{p+q}}{pq} \int_{\mathbb{R}^N} (I_\alpha * |v_1|^p) |v_1|^q > 0.$$

Thus,

$$J(t_\lambda v_\lambda) < \frac{1}{2} \left(1 - \frac{1}{q}\right) (qS_2^q)^{1/(q-1)}$$

for sufficiently small λ . \square

The next lemma establishes an important information involving the $(PS)_c$ sequence which will be crucial later on.

Lemma 3.2. *Assume that $\{u_n\} \subset H^1(\mathbb{R}^N)$ is a $(PS)_c$ sequence of J . Then there exists $\sigma > 0$ and a sequence $\{y_n\} \subset \mathbb{R}^N$ such that*

$$\limsup_{n \rightarrow \infty} \int_{B_1(y_n)} u_n^2 \geq \sigma.$$

Proof. Assuming the contrary that

$$\sup_{y \in \mathbb{R}^N} \int_{B_1(y)} u_n^2 \rightarrow 0,$$

it follows that

$$u_n \rightarrow 0 \quad \text{in } L^s(\mathbb{R}^N), \quad s \in (2, 2^*).$$

Choose t and τ close to $2N/(N + \alpha)$ satisfying

$$t < \frac{2N}{N + \alpha} < \tau, \quad \frac{1}{t} + \frac{N - \alpha}{N} + \frac{1}{\tau} = 2.$$

By the Hardy–Littlewood–Sobolev inequality, we know that for all n ,

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_n(x)|^p |u_n(y)|^q}{|x - y|^{N - \alpha}} dx dy \leq C |u_n|_{p\tau}^p |u_n|_{qt}^q,$$

from which it follows that

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_n(x)|^p |u_n(y)|^q}{|x - y|^{N - \alpha}} dx dy \rightarrow 0, \quad n \rightarrow \infty.$$

Since $\{u_n\}$ is a $(PS)_c$ sequence, we get that

$$c = \frac{1}{2} \|u_n\|^2 - \frac{1}{2p^2} \int_{\mathbb{R}^N} (I_\alpha * |u_n|^p) |u_n|^p - \frac{1}{2q^2} \int_{\mathbb{R}^N} (I_\alpha * |u_n|^q) |u_n|^q + o(1), \quad (3.14)$$

$$\|u_n\|^2 = \frac{1}{p} \int_{\mathbb{R}^N} (I_\alpha * |u_n|^p) |u_n|^p + \frac{1}{q} \int_{\mathbb{R}^N} (I_\alpha * |u_n|^q) |u_n|^q + o(1). \quad (3.15)$$

If $u_n \rightarrow 0$ in $H^1(\mathbb{R}^N)$ as $n \rightarrow \infty$, it follows from (3.14) that $c = 0$, which is a contradiction. Thus,

$$\limsup_{n \rightarrow \infty} \|u_n\| > 0. \quad (3.16)$$

By virtue of (3.1), (3.2) and (3.15), we obtain that

$$\begin{aligned} V_\infty S_1 \left(\int_{\mathbb{R}^N} (I_\alpha * |u_n|^p) |u_n|^p \right)^{1/p} + S_2 \left(\int_{\mathbb{R}^N} (I_\alpha * |u_n|^q) |u_n|^q \right)^{1/q} \\ \leq \frac{1}{p} \int_{\mathbb{R}^N} (I_\alpha * |u_n|^p) |u_n|^p + \frac{1}{q} \int_{\mathbb{R}^N} (I_\alpha * |u_n|^q) |u_n|^q + o(1). \end{aligned} \quad (3.17)$$

Let

$$a := \limsup_{n \rightarrow \infty} \int_{\mathbb{R}^N} (I_\alpha * |u_n|^p) |u_n|^p, \quad b := \limsup_{n \rightarrow \infty} \int_{\mathbb{R}^N} (I_\alpha * |u_n|^q) |u_n|^q,$$

both of which are finite since $\{u_n\}$ is bounded. Passing to a limit in (3.17), we have that

$$V_\infty S_1 a^{1/p} + S_2 b^{1/q} \leq \frac{a}{p} + \frac{b}{q}. \quad (3.18)$$

Moreover, by (3.15) and (3.16), we obtain $a + b > 0$. If $a < (pV_\infty S_1)^{p/(p-1)}$ and $b < (qS_2)^{q/(q-1)}$, then it follows from (3.18) that

$$0 \leq \frac{a}{p} + \frac{b}{q} - V_\infty S_1 a^{1/p} - S_2 b^{1/q} = a^{1/p} \left(\frac{1}{p} a^{(p-1)/p} - V_\infty S_1 \right) + b^{1/q} \left(\frac{1}{q} b^{(q-1)/q} - S_2 \right) < 0.$$

This is a contradiction. Thus, $a \geq (pV_\infty S_1)^{p/(p-1)}$ or $b \geq (qS_2)^{q/(q-1)}$. By (3.14) and (3.15) again, we have

$$c = \frac{1}{2p} \left(1 - \frac{1}{p} \right) \int_{\mathbb{R}^N} (I_\alpha * |u_n|^p) |u_n|^p + \frac{1}{2q} \left(1 - \frac{1}{q} \right) \int_{\mathbb{R}^N} (I_\alpha * |u_n|^q) |u_n|^q + o(1).$$

It follows that either $c \geq 2^{-1}(1 - 1/p)(pV_\infty^p S_1^p)^{1/(p-1)}$ or $c \geq 2^{-1}(1 - 1/q)(qS_2^q)^{1/(q-1)}$, which contradicts to fact stated in Lemma 3.1. \square

Proof of Theorem 1.2. By Lemma 3.1, there exists a $(PS)_c$ sequence $\{u_n\} \subset H^1(\mathbb{R}^N)$ with

$$0 < c < \min \left\{ \frac{1}{2} \left(1 - \frac{1}{p} \right) (pV_\infty^p S_1^p)^{1/(p-1)}, \frac{1}{2} \left(1 - \frac{1}{q} \right) (qS_2^q)^{1/(q-1)} \right\}.$$

By Lemma 3.2, there exist $\sigma > 0$ and a sequence $\{y_n\} \subset \mathbb{R}^N$ such that

$$\limsup_{n \rightarrow \infty} \int_{B_1(y_n)} u_n^2 \geq \sigma.$$

Since J and J' are both invariant by translation, it follows that $\{v_n\} \subset H^1(\mathbb{R}^N)$ is still a $(PS)_c$ sequence and

$$\limsup_{n \rightarrow \infty} \int_{B_1(0)} v_n^2 \geq \sigma, \quad (3.19)$$

where $v_n(\cdot) = u_n(\cdot + y_n)$ for all n . It follows from Lemma 2.7 that $\{v_n\}$ is bounded. We may assume that $v_n \rightharpoonup v$ in $H^1(\mathbb{R}^N)$. Thus, $v \neq 0$ by (3.19) and Lemma 2.8 implies the desired results. The proof is completed. \square

4 Proof of Theorem 1.1

In this section, the potential function $V < V_\infty$ for all $x \in \mathbb{R}^N$. When $V = V_\infty$, we denote J, \mathcal{N}, c, I and $\|\cdot\|$ by $J_\infty, \mathcal{N}_\infty, c_\infty, I_\infty$ and $\|\cdot\|_{V_\infty}$, respectively. Firstly, we present a key estimate for c .

Lemma 4.1. *One has that $c < c_\infty$.*

Proof. By Theorem 1.2, there exists a function $u \in \mathcal{N}_\infty$ such that $J_\infty(u) = c_\infty$ and $J'_\infty(u) = 0$. Since $V < V_\infty$ for all $x \in \mathbb{R}^N$, we have that

$$I(u) = I_\infty(u) + \int_{\mathbb{R}^N} (V - V_\infty)u^2 < 0.$$

Thus, according to Remark 2.4, there exists a positive number $t_u < 1$ such that $t_u u \in \mathcal{N}$. Hence,

$$c_\infty = J_\infty(u) > J_\infty(t_u u) = J(t_u u) + \frac{1}{2} \int_{\mathbb{R}^N} (V_\infty - V)|t_u u|^2 > c.$$

The proof is completed. \square

Now, we ready to prove our main result Theorem 1.1.

Proof of Theorem 1.1. Let $\{u_n\} \subset H^1(\mathbb{R}^N)$ be a $(PS)_c$ sequence. It follows from $c > 0$ that there exists $\delta > 0$ such that $\|u_n\| \geq \delta$ for sufficiently large n . Using Lemma 2.7, we can assume that $u_n \rightharpoonup u$. To prove Theorem 1.1, by Lemma 2.8, we only need to show that $u \neq 0$. Suppose, by contradiction, that $u = 0$. Then $u_n \rightarrow 0$ in $L^s_{\text{loc}}(\mathbb{R}^N)$ for all $s \in [1, 2^*)$. Since

$$\begin{aligned} \int_{\mathbb{R}^N} (V_\infty - V)u_n^2 &= \int_{B_R} (V_\infty - V)u_n^2 + \int_{B_R^c} (V_\infty - V)u_n^2 \\ &\leq V_\infty \int_{B_R} u_n^2 + \varepsilon(R)|u_n|_2^2, \end{aligned}$$

where $\varepsilon(R) = \sup_{B_R^c} (V_\infty - V) \rightarrow 0$ as $R \rightarrow \infty$, we obtain that $\int_{\mathbb{R}^N} (V_\infty - V)u_n^2 \rightarrow 0$ as $n \rightarrow \infty$. Consequently,

$$I_\infty(u_n) = I(u_n) + \int_{\mathbb{R}^N} (V_\infty - V)u_n^2 = o(1). \quad (4.1)$$

For each n , according to Lemma 2.3, there exists $t_n > 0$ such that $t_n u_n \in \mathcal{N}_\infty$. It follows from Lemma 2.5 that there exists $\delta_\infty > 0$ such that $\|t_n u_n\|_{V_\infty} \geq \delta_\infty$ for all n . Thus, $\liminf_{n \rightarrow \infty} t_n > 0$ because $\|u_n\|_{V_\infty}$ is upper-bounded. Now we prove that $\{t_n\}$ is bounded. Otherwise, we suppose that $\limsup_{n \rightarrow \infty} t_n = \infty$. Without loss of generality, we may assume that $t_n \rightarrow \infty$ as $n \rightarrow \infty$. Since $\{t_n u_n\} \subset \mathcal{N}_\infty$, we have that for sufficiently large n ,

$$\begin{aligned} t_n^2 \|u_n\|_{V_\infty}^2 &= \frac{t_n^{2p}}{p} \int_{\mathbb{R}^N} (I_\alpha * |u_n|^p)|u_n|^p + \frac{(p+q)t_n^{p+q}}{pq} \int_{\mathbb{R}^N} (I_\alpha * |u_n|^p)|u_n|^q + \frac{t_n^{2q}}{q} \int_{\mathbb{R}^N} (I_\alpha * |u_n|^q)|u_n|^q \\ &> t_n^{2p} \left(\frac{1}{p} \int_{\mathbb{R}^N} (I_\alpha * |u_n|^p)|u_n|^p + \frac{p+q}{pq} \int_{\mathbb{R}^N} (I_\alpha * |u_n|^p)|u_n|^q + \frac{1}{q} \int_{\mathbb{R}^N} (I_\alpha * |u_n|^q)|u_n|^q \right) \\ &= t_n^{2p} (\|u_n\|^2 + o(1)) \geq t_n^{2p} (\delta^2 + o(1)), \end{aligned}$$

which deduce a contradiction. Since $\{t_n u_n\} \subset \mathcal{N}_\infty$, it follows from (4.1) that

$$\begin{aligned} \frac{1}{p} (t_n^{2(p-1)} - 1) \int_{\mathbb{R}^N} (I_\alpha * |u_n|^p)|u_n|^p + \frac{1}{q} (t_n^{2(q-1)} - 1) \int_{\mathbb{R}^N} (I_\alpha * |u_n|^q)|u_n|^q \\ + \frac{p+q}{pq} (t_n^{p+q-2} - 1) \int_{\mathbb{R}^N} (I_\alpha * |u_n|^p)|u_n|^q = o(1). \end{aligned}$$

Since $\|u_n\| \geq \delta$ for sufficiently large n , we can assume $\lim_{n \rightarrow \infty} t_n = 1$. Thus, we have that

$$\begin{aligned}
 c + o(1) &= J(u_n) - \frac{1}{2}I(u_n) \\
 &= \frac{p-1}{2p^2} \int_{\mathbb{R}^N} (I_\alpha * |u_n|^p) |u_n|^p + \frac{q-1}{2q^2} \int_{\mathbb{R}^N} (I_\alpha * |u_n|^q) |u_n|^q \\
 &\quad + \frac{p+q-2}{2pq} \int_{\mathbb{R}^N} (I_\alpha * |u_n|^p) |u_n|^q \\
 &= \frac{p-1}{2p^2} \int_{\mathbb{R}^N} (I_\alpha * |t_n u_n|^p) |t_n u_n|^p + \frac{q-1}{2q^2} \int_{\mathbb{R}^N} (I_\alpha * |t_n u_n|^q) |t_n u_n|^q \\
 &\quad + \frac{p+q-2}{2pq} \int_{\mathbb{R}^N} (I_\alpha * |t_n u_n|^p) |t_n u_n|^q + o(1) \\
 &= J_\infty(t_n u_n) - \frac{1}{2}I_\infty(t_n u_n) + o(1) \\
 &= J_\infty(t_n u_n) + o(1) \geq c_\infty + o(1),
 \end{aligned}$$

which contradict to Lemma 4.1. Thus, $u \neq 0$. The proof is completed. \square

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