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Qualitative approximation of solutions to difference equations of various types

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Abstract. In this paper we study the asymptotic behavior of solutions to difference equations of various types. We present sufficient conditions for the existence of solutions with prescribed asymptotic behavior, and establish some results concerning approximations of solutions, extending some of our previous results. Our approach allows us to control the degree of approximation. As a measure of approximation we use $\mathrm{o}(u_n)$ where u is an arbitrary fixed positive nonincreasing sequence.

Keywords: difference equation, approximative solution, prescribed asymptotic behavior, Volterra difference equation.

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1 Introduction

Let $\mathbb{N}$, $\mathbb{R}$ denote the set of positive integers and real numbers respectively. The space of all sequences $x : \mathbb{N} \to \mathbb{R}$ we denote by $\mathbb{R}^{\mathbb{N}}$. Assume $m, k \in \mathbb{N}$, $a, b : \mathbb{N} \to \mathbb{R}$. In this paper we will examine the asymptotic properties of the solutions of various specific cases of the following equations

$$\Delta^m x_n = a_n F(x)(n) + b_n, \quad F : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^{\mathbb{N}}, \tag{E}$$

$$\Delta(r_n \Delta x_n) = a_n F(x)(n) + b_n, \quad F : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^{\mathbb{N}}, \quad r : \mathbb{N} \to (0, \infty). \tag{QE}$$

In particular, we will examine the properties of solutions to equations of the form

$$\Delta^m x_n = a_n f(n, x_{\sigma_1(n)}, \dots, x_{\sigma_k(n)}) + b_n, \quad f : \mathbb{N} \times \mathbb{R}^{k+1} \to \mathbb{R}, \quad \sigma_1, \dots, \sigma_k : \mathbb{N} \to \mathbb{N},$$

$$\Delta^m x_n = a_n f(n, x_n, \Delta x_n, \Delta^2 x_n, \dots, \Delta^k x_n) + b_n, \quad f : \mathbb{N} \times \mathbb{R}^{k+2} \to \mathbb{R},$$

and discrete Volterra equations of the form

$$\Delta^m x_n = b_n + \sum_{k=1}^{n} K(n,k) f(k, x_k), \quad K : \mathbb{N} \times \mathbb{N} \to \mathbb{R}, \quad f : \mathbb{N} \times \mathbb{R} \to \mathbb{R}.$$

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By a solution of (E) we mean a sequence $x : \mathbb{N} \to \mathbb{R}$ satisfying (E) for all large n. Analogously we define a solution of (QE).

The study of asymptotic properties of solutions of differential and difference equations is of great importance. Hence many papers are devoted to this subject. For differential equations see, for example, [2, 6, 11, 13, 24, 25, 28, 29]. Asymptotic properties of solutions of ordinary difference equations were investigated in [12,30,32–34,39]. Several related results for discrete Volterra equations can be found in [3–5, 8, 10, 10, 14, 19–22] and for quasi-difference equations in [1,7,26,27,31,38].

In recent years the author presented a new theory of the study of asymptotic properties of the solutions to difference equations. This theory is based mainly on the examination of the behavior of the iterated remainder operator and on the application of asymptotic difference pairs. This approach allows us to control the degree of approximation. The properties of the iterated remainder operator are presented in [15]. Asymptotic difference pairs were introduced and used in [17]. They were also used in [18] and [21].

In this paper, in Lemma 2.1, we present a new type of asymptotic difference pair. Using Lemma 2.1 and some earlier results, we get a number of theorems about the asymptotic properties of the solutions. Let u be a positive and nonincreasing sequence. Lemma 2.1 allows us to use $\mathrm{o}(u_n)$ as a measure of approximation of solutions. Asymptotic pair technique does not work in the case of equations of type (QE). In this case, instead of Lemma 2.1, we use Lemma 2.3.

The paper is organized as follows. In Section 2, we introduce some notation and terminology. Moreover, in Lemma 2.1 and Lemma 2.3 we present the basic tools that will be used in the main part of the paper. In Section 3, we present our main results concerning the existence of solutions with prescribed asymptotic behavior. We essentially use here a fixed point theory which is frequently used in literature, see for example [1–7, 11–31, 35–37]. This section is divided into four parts devoted to various types of equations. In Section 4, we establish some results concerning approximations of solutions.

2 Preliminaries

If $x, y : \mathbb{N} \to \mathbb{R}$, then xy and $|x|$ denote the sequences defined by $xy(n) = x_n y_n$ and $|x|(n) = |x_n|$ respectively. Moreover

$$\|x\| = \sup_{n \in \mathbb{N}} |x_n|, \quad c_0 = \{z : \mathbb{N} \to \mathbb{R} : \lim_{n \to \infty} z_n = 0\}.$$

Assume $k \in \mathbb{N}$. We say that a function $f : \mathbb{N} \times \mathbb{R}^k \to \mathbb{R}$ is locally equibounded if for any $t \in \mathbb{R}^k$ there exists a neighborhood U of t in $\mathbb{R}^k$ such that f is bounded on $\mathbb{N} \times U$.

We say that a subset B of $\mathbb{R}^{\mathbb{N}}$ is bounded if there exists a constant M such that $\|a - b\| \le M$ for any $a, b \in B$. We regard any bounded subset of $\mathbb{R}^{\mathbb{N}}$ as a metric space with metric d defined by $d(a, b) = \|a - b\|$. Assume $Y \subset X \subset \mathbb{R}^{\mathbb{N}}$ and Y is bounded. We say that an operator $F : X \to \mathbb{R}^{\mathbb{N}}$, is mezocontinuous on Y if for any fixed index n the function $\varphi_n : Y \to \mathbb{R}$ defined by $\varphi_n(y) = F(y)(n)$ is uniformly continuous.

Let $m \in \mathbb{N}$. We will use the following notations

$$\mathrm{A}(m) := \left\{ a \in \mathbb{R}^{\mathbb{N}} : \sum_{n=1}^{\infty} n^{m-1} |a_n| < \infty \right\},$$

$$\mathrm{S}(m) = \left\{ a \in \mathbb{R}^{\mathbb{N}} : \text{ the series } \sum_{i_1=1}^{\infty} \sum_{i_2=i_1}^{\infty} \cdots \sum_{i_m=i_{m-1}}^{\infty} a_{i_m} \text{ is convergent} \right\}.$$

For any $a \in \mathrm{S}(m)$ we define the sequence $r^m(a)$ by

$$r^m(a)(n) = \sum_{i_1=n}^{\infty} \sum_{i_2=i_1}^{\infty} \cdots \sum_{i_m=i_{m-1}}^{\infty} a_{i_m}. \tag{2.1}$$

Then $\mathrm{S}(m)$ is a linear subspace of c_0, $r^m(a) \in c_0$ for any $a \in \mathrm{S}(m)$ and

$$r^m : \mathrm{S}(m) \to c_0$$

is a linear operator which we call the *remainder operator of order m*. If $a \in \mathrm{A}(m)$, then $a \in \mathrm{S}(m)$ and

$$r^m(a)(n) = \sum_{j=n}^{\infty} \binom{m-1+j-n}{m-1} a_j = \sum_{k=0}^{\infty} \binom{m+k-1}{m-1} a_{n+k} \tag{2.2}$$

for any $n \in \mathbb{N}$. Moreover

$$\Delta^m(r^m(a))(n) = (-1)^m a_n \tag{2.3}$$

for any $a \in \mathrm{A}(m)$ and any $n \in \mathbb{N}$. For more information about the remainder operator see [15].

We say that a pair (A, Z) of linear subspaces of $\mathbb{R}^{\mathbb{N}}$ is an asymptotic difference pair of order m or, simply, m-pair if $A \subset \Delta^m Z$, $w + z \in Z$ for any eventually zero sequence w and any $z \in Z$, and $ba \in A$ for any bounded sequence b and any $a \in \mathrm{A}$. We say that an m-pair (A, Z) is evanescent if $Z \subset c_0$.

Lemma 2.1. *Assume $m \in \mathbb{N}$, a positive sequence u is nonincreasing,*

$$A = \left\{ a \in \mathbb{R}^{\mathbb{N}} : \sum_{n=1}^{\infty} \frac{n^{m-1}|a_n|}{u_n} < \infty \right\}, \quad Z = \left\{ z \in \mathbb{R}^{\mathbb{N}} : z_n = \mathrm{o}(u_n) \right\}.$$

Then (A, Z) is an evanescent m-pair.

Proof. It is clear that $ba \in A$ for any bounded sequence b and any $a \in \mathrm{A}$. Obviously $w + z \in Z$ for any eventually zero sequence w and any $z \in Z$. Let $a \in A$. Since u is nonincreasing, we have $a \in \mathrm{A}(m)$. Define sequences w, a^+, a^- by

$$w_n = \frac{|a_n|}{u_n}, \quad a_n^+ = \max(0, a_n), \quad a_n^- = -\min(0, a_n).$$

Then $0 \le a^+ \le |a|$. Hence $a^+ \in \mathrm{A}(m)$ and using (2.2) we get

$$\begin{aligned} r^m(a^+)(n) &= \sum_{k=0}^{\infty} \binom{m+k-1}{m-1} a_{n+k}^+ \le \sum_{k=0}^{\infty} \binom{m+k-1}{m-1} |a_{n+k}| \\ &= \sum_{k=0}^{\infty} \binom{m+k-1}{m-1} u_{n+k} w_{n+k} \le \sum_{k=0}^{\infty} \binom{m+k-1}{m-1} u_n w_{n+k} = u_n r^m(w)(n). \end{aligned}$$

Therefore

$$0 \le \frac{r^m(a^+)(n)}{u_n} \le r^m(w)(n).$$

By (2.1), $r^m(w)(n) = \mathrm{o}(1)$. Hence $r^m(a^+)(n) = \mathrm{o}(u_n)$. Analogously, $r^m(a^-)(n) = \mathrm{o}(u_n)$. Thus

$$r^m(a)(n) = r^m(a^+ - a^-)(n) = r^m(a^+)(n) - r^m(a^-)(n) = \mathrm{o}(u_n).$$

Hence $r^m A \subset Z$. Now, using (2.3), we obtain

$$A = (-1)^m A = \Delta^m r^m A \subset \Delta^m Z.$$ □

Lemma 2.2. *Assume $m \in \mathbb{N}$, $a \in \mathbb{R}^{\mathbb{N}}$, $u : \mathbb{N} \to (0, \infty)$, $\Delta u \leq 0$, and*

$$\sum_{n=1}^{\infty} \frac{n^{m-1}|a_n|}{u_n} < \infty.$$

Then $a \in \mathrm{A}(m)$ and $r^m(a)(n) = \mathrm{o}(u_n)$.

Proof. The assertion is a consequence of the proof of Lemma 2.1. □

Lemma 2.3. *Assume $a, r, u : \mathbb{N} \to \mathbb{R}$, $r > 0$, $u > 0$, $\Delta u \leq 0$, and*

$$\sum_{k=1}^{\infty} \frac{1}{u_k r_k} \sum_{j=k}^{\infty} |a_j| < \infty.$$

Then

$$\sum_{k=n}^{\infty} \frac{1}{r_k} \sum_{j=k}^{\infty} a_j = \mathrm{o}(u_n).$$

Proof. Define sequences z, w by

$$z_n = \sum_{k=n}^{\infty} \frac{1}{u_k r_k} \sum_{j=k}^{\infty} |a_j|, \quad w_n = \sum_{k=n}^{\infty} \frac{1}{r_k} \sum_{j=k}^{\infty} a_j.$$

By assumption, $z_n = \mathrm{o}(1)$. Moreover

$$u_n^{-1}|w_n| \leq u_n^{-1} \sum_{k=n}^{\infty} \frac{1}{r_k} \sum_{j=k}^{\infty} |a_j| = \sum_{k=n}^{\infty} \frac{1}{u_n r_k} \sum_{j=k}^{\infty} |a_j|.$$

Since $\Delta u_n^{-1} \geq 0$, we get

$$u_n^{-1}|w_n| \leq \sum_{k=n}^{\infty} \frac{1}{u_k r_k} \sum_{j=k}^{\infty} |a_j| = z_n = \mathrm{o}(1).$$

Hence $|w_n| = u_n \mathrm{o}(1) = \mathrm{o}(u_n)$. Therefore $w_n = \mathrm{o}(u_n)$. □

3 Solutions with prescribed asymptotic behavior

Assume $b, u \in \mathbb{R}^{\mathbb{N}}$ and u is positive and nonincreasing. In this section we present sufficient conditions for the existence of solution x with the asymptotic behavior

$$x_n = y_n + \mathrm{o}(u_n)$$

where y is a given solution of the equation $\Delta^m y_n = b_n$ or the equation $\Delta(r_n \Delta y_n) = b_n$.

3.1 Abstract equations

Theorem 3.1. *Assume $m \in \mathbb{N}$, $a, b, u : \mathbb{N} \to \mathbb{R}$, $c \in (0, \infty)$, $F : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^{\mathbb{N}}$, y is a solution of the equation $\Delta^m y_n = b_n$,*

$$u > 0, \quad \Delta u \leq 0, \quad \sum_{n=1}^{\infty} \frac{n^{m-1}|a_n|}{u_n} < \infty, \quad U = \{x \in \mathbb{R}^{\mathbb{N}} : |x - y| \leq c\},$$

and F is bounded and mezocontinuous on U. Then there exists a solution x of the equation

$$\Delta^m x_n = a_n F(x)(n) + b_n$$

such that $x_n = y_n + \mathrm{o}(u_n)$.

Proof. The assertion is a consequence of Lemma 2.1 and [18, Corollary 4.3]. □

3.2 Functional equations

Theorem 3.2. *Assume $m, k \in \mathbb{N}$, $a, b, u : \mathbb{N} \to \mathbb{R}$, $c \in (0, \infty)$, $f : \mathbb{N} \times \mathbb{R}^k \to \mathbb{R}$, y is a solution of the equation $\Delta^m y_n = b_n$,*

$$u > 0, \quad \Delta u \leq 0, \quad \sum_{n=1}^{\infty} \frac{n^{m-1}|a_n|}{u_n} < \infty, \quad Y = \bigcup_{n \in \mathbb{N}} [y_n - c, y_n + c],$$

$$\sigma_1, \dots, \sigma_k : \mathbb{N} \to \mathbb{N}, \quad \lim_{n \to \infty} \sigma_i(n) = \infty \quad \textit{for} \quad i = 1, \dots, k,$$

and f is continuous and bounded on $\mathbb{N} \times Y^k$. Then there exists a solution x of the equation

$$\Delta^m x_n = a_n f(n, x_{\sigma_1(n)}, \dots, x_{\sigma_k(n)}) + b_n$$

such that $x_n = y_n + \mathrm{o}(u_n)$.

Proof. Define an operator $F : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^{\mathbb{N}}$ and a subset U of $\mathbb{R}^{\mathbb{N}}$ by

$$F(x)(n) = f(n, x_{\sigma_1(n)}, \dots, x_{\sigma_k(n)}), \quad U = \{x \in \mathbb{R}^{\mathbb{N}} : |x - y| \leq c\}.$$

Then F is bounded on U. By [18, Example 3.4] F is mezocontinuous on U. Using Theorem 3.1 we obtain the result. □

Corollary 3.3. *Assume $m, k \in \mathbb{N}$, $a, b, u : \mathbb{N} \to \mathbb{R}$, $f : \mathbb{N} \times \mathbb{R}^k \to \mathbb{R}$,*

$$u > 0, \quad \Delta u \leq 0, \quad \sum_{n=1}^{\infty} \frac{n^{m-1}|a_n|}{u_n} < \infty,$$

$$\sigma_1, \dots, \sigma_k : \mathbb{N} \to \mathbb{N}, \quad \lim_{n \to \infty} \sigma_i(n) = \infty \quad \textit{for} \quad i = 1, \dots, k,$$

and f is continuous and locally equibounded. Then for any bounded solution y of the equation $\Delta^m y_n = b_n$, there exists a solution x of the equation

$$\Delta^m x_n = a_n f(n, x_{\sigma_1(n)}, \dots, x_{\sigma_k(n)}) + b_n$$

such that $x_n = y_n + \mathrm{o}(u_n)$.

Proof. Assume y is a bounded solution of the equation $\Delta^m y_n = b_n$, $c > 0$, and

$$Y = \bigcup_{n\in\mathbb{N}} [y_n - c, y_n + c].$$

Then Y^k is a bounded subset of $\mathbb{R}^k$. For any $t \in Y^k$ there exist a neighborhood U_t of t and a positive constant M_t such that $|f(n,u)| \le M_t$ for any $(n,u) \in \mathbb{N} \times U_t$. Choose $t_1, \dots, t_p \in Y^k$ such that

$$Y^k \subset U_{t_1} \cup U_{t_2} \cup \cdots \cup U_{t_p}.$$

If $M = \max(M_{t_1}, \dots, M_{t_p})$, then $|f(n,u)| \le M$ for any $(n,u) \in \mathbb{N} \times Y^k$. Now, using Theorem 3.2 we obtain the result. □

Corollary 3.4. *Assume $m, k \in \mathbb{N}$, $a, b, u : \mathbb{N} \to \mathbb{R}$, $f : \mathbb{N} \times \mathbb{R}^k \to \mathbb{R}$,*

$$u > 0, \quad \Delta u \le 0, \quad \sum_{n=1}^{\infty} \frac{n^{m-1}|a_n|}{u_n} < \infty,$$

$$\sigma_1, \dots, \sigma_k : \mathbb{N} \to \mathbb{N}, \quad \lim_{n\to\infty} \sigma_i(n) = \infty \quad \text{for} \quad i = 1, \dots, k,$$

and f is continuous and bounded. Then for any solution y of the equation $\Delta^m y_n = b_n$, there exists a solution x of the equation

$$\Delta^m x_n = a_n f(n, x_{\sigma_1(n)}, \dots, x_{\sigma_k(n)}) + b_n$$

such that $x_n = y_n + \mathrm{o}(u_n)$.

Proof. The assertion is an immediate consequence of Theorem 3.2. □

Theorem 3.5. *Assume $m, k \in \mathbb{N}$, $a, b, u : \mathbb{N} \to \mathbb{R}$, $c \in (0, \infty)$, $f : \mathbb{N} \times \mathbb{R}^{k+1} \to \mathbb{R}$, y is a solution of the equation $\Delta^m y_n = b_n$,*

$$u > 0, \quad \Delta u \le 0, \quad \sum_{n=1}^{\infty} \frac{n^{m-1}|a_n|}{u_n} < \infty,$$

and f is continuous and bounded on the set

$$Y = \bigcup_{n\in\mathbb{N}} \{n\} \times [y_n - c, y_n + c] \times [\Delta y_n - c, \Delta y_n + c] \times \cdots \times [\Delta^k y_n - c, \Delta^k y_n + c].$$

Then there exists a solution x of the equation

$$\Delta^m x_n = a_n f(n, x_n, \Delta x_n, \Delta^2 x_n, \dots, \Delta^k x_n) + b_n$$

such that $x_n = y_n + \mathrm{o}(u_n)$.

Proof. Define an operator $F : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^{\mathbb{N}}$ and a subset U of $\mathbb{R}^{\mathbb{N}}$ by

$$F(x)(n) = f(n, x_n, \Delta x_n, \Delta^2 x_n, \dots, \Delta^k x_n), \quad U = \{x \in \mathbb{R}^{\mathbb{N}} : |x - y| \le 2^{-k} c\}.$$

Assume $x \in U$, $n \in \mathbb{N}$, and $j \in \{1, \dots, k\}$. Then

$$|\Delta x_n - \Delta y_n| \le |x_{n+1} - y_{n+1}| + |x_n - y_n| \le (2^{-k} + 2^{-k})c \le c,$$

$$|\Delta^2 x_n - \Delta^2 y_n| \le 2^2 2^{-k} c \le c, \ \dots, \ |\Delta^j x_n - \Delta^j y_n| \le 2^j 2^{-k} c \le c.$$

Hence $(n, x_n, \Delta x_n, \Delta^2 x_n, \dots, \Delta^k x_n) \in Y$. Therefore F is bounded on U. By [18, Example 3.5] F is mezocontinuous on U. Using Theorem 3.1 we obtain the result. □

Corollary 3.6. *Assume $m, k \in \mathbb{N}$, $a, b, u : \mathbb{N} \to \mathbb{R}$, $f : \mathbb{N} \times \mathbb{R}^{k+1} \to \mathbb{R}$,*

$$u > 0, \quad \Delta u \le 0, \quad \sum_{n=1}^{\infty} \frac{n^{m-1}|a_n|}{u_n} < \infty,$$

and f is continuous and locally equibounded. Then for any bounded solution y of the equation $\Delta^m y_n = b_n$, there exists a solution x of the equation

$$\Delta^m x_n = a_n f(n, x_n, \Delta x_n, \Delta^2 x_n, \dots, \Delta^k x_n) + b_n$$

such that $x_n = y_n + \mathrm{o}(u_n)$.

Proof. Assume y is a bounded solution of the equation $\Delta^m y_n = b_n$, $c > 0$, and

$$Y_k = \bigcup_{n \in \mathbb{N}} [y_n - c, y_n + c] \times [\Delta y_n - c, \Delta y_n + c] \times \cdots \times [\Delta^k y_n - c, \Delta^k y_n + c].$$

As in the proof of Corollary 3.3 one can show that f is bounded on $\mathbb{N} \times Y_k$. Now, using Theorem 3.5 we obtain the result. □

Corollary 3.7. *Assume $m, k \in \mathbb{N}$, $a, b, u : \mathbb{N} \to \mathbb{R}$, $f : \mathbb{N} \times \mathbb{R}^{k+1} \to \mathbb{R}$,*

$$u > 0, \quad \Delta u \le 0, \quad \sum_{n=1}^{\infty} \frac{n^{m-1}|a_n|}{u_n} < \infty,$$

and f is continuous and bounded. Then for any solution y of the equation $\Delta^m y_n = b_n$, there exists a solution x of the equation

$$\Delta^m x_n = a_n f(n, x_n, \Delta x_n, \Delta^2 x_n, \dots, \Delta^k x_n) + b_n$$

such that $x_n = y_n + \mathrm{o}(u_n)$.

Proof. The assertion is an immediate consequence of Theorem 3.5. □

3.3 Discrete Volterra equations

Theorem 3.8. *Assume $m \in \mathbb{N}$, $a, b, u : \mathbb{N} \to \mathbb{R}$, $K : \mathbb{N} \times \mathbb{N} \to \mathbb{R}$, $f : \mathbb{N} \times \mathbb{R} \to \mathbb{R}$,*

$$u > 0, \quad \Delta u \le 0, \quad \sigma : \mathbb{N} \to \mathbb{N}, \quad \lim_{n \to \infty} \sigma(n) = \infty, \quad \sum_{n=1}^{\infty} \frac{n^{m-1}}{u_n} \sum_{k=1}^{n} |K(n,k)| < \infty,$$

y is a solution of the equation $\Delta^m y_n = b_n$, and there exists a uniform neighborhood U of the set $y(\mathbb{N})$ such that the restriction $f|\mathbb{N} \times U$ is continuous and bounded. Then there exists a solution x of the equation

$$\Delta^m x_n = b_n + \sum_{k=1}^{n} K(n,k) f(k, x_{\sigma(k)})$$

such that $x_n = y_n + \mathrm{o}(u_n)$.

Proof. The assertion is a consequence of Lemma 2.1 and [21, Theorem 3.1]. □

3.4 Quasi-difference equations

Asymptotic pair technique does not work in the case of equations of type (QE). Therefore, in this subsection we will use Lemma 2.3. Moreover, we will need the following two lemmas.

Lemma 3.9 ([23, Lemma 5]). *If $\sum_{k=1}^{\infty}\frac{1}{r_k}\sum_{i=k}^{\infty}|u_i| < \infty$, then*

$$\sum_{k=1}^{\infty}|u_k|\sum_{i=1}^{k}\frac{1}{r_i} < \infty \quad \textit{and} \quad \sum_{k=n}^{\infty}\frac{1}{r_k}\sum_{i=k}^{\infty}|u_i| \le \sum_{k=n}^{\infty}|u_k|\sum_{i=1}^{k}\frac{1}{r_i}$$

for any $n \in \mathbb{N}$.

Lemma 3.10 ([16, Lemma 4.7]). *Assume $y, \rho : \mathbb{N} \to \mathbb{R}$, and $\lim_{n\to\infty}\rho_n = 0$. In the set $X = \{x \in \mathbb{R}^{\mathbb{N}} : |x - y| \le |\rho|\}$ we define a metric by the formula*

$$d(x, z) = \|x - z\|. \tag{3.1}$$

Then any continuous map $H : X \to X$ has a fixed point.

Theorem 3.11. *Assume $a, b, r, u : \mathbb{N} \to \mathbb{R}$, $r > 0$, $u > 0$, $\Delta u \le 0$, y is a solution of the equation $\Delta(r_n\Delta y_n) = b_n$,*

$$\sum_{k=1}^{\infty}\frac{1}{u_k r_k}\sum_{j=k}^{\infty}|a_j| < \infty, \quad q \in \mathbb{N}, \quad \alpha \in (0, \infty), \quad U = \bigcup_{n=q}^{\infty}[y_n - \alpha, y_n + \alpha],$$

and $f : \mathbb{R} \to \mathbb{R}$ is continuous and bounded on U. Then there exists a solution x of the equation

$$\Delta(r_n\Delta x_n) = a_n f(x_{\sigma(n)}) + b_n$$

such that $x_n = y_n + \mathrm{o}(u_n)$.

Proof. In the proof we use the methods analogous to the methods from previous papers [22] and [23]. For $n \in \mathbb{N}$ and $x \in \mathbb{R}^{\mathbb{N}}$ let

$$F(x)(n) = a_n f(x_{\sigma(n)}). \tag{3.2}$$

There exists $L > 0$, such that

$$|f(t)| \le L \tag{3.3}$$

for any $t \in U$. Since $\Delta u \le 0$, we have

$$\sum_{k=1}^{\infty}\frac{1}{r_k}\sum_{j=k}^{\infty}|a_j| < \infty. \tag{3.4}$$

Let

$$Y = \{x \in \mathbb{R}^{\mathbb{N}} : |x - y| \le \alpha\}, \quad \rho \in \mathbb{R}^{\mathbb{N}}, \quad \rho_n = L\sum_{k=n}^{\infty}\frac{1}{r_k}\sum_{j=k}^{\infty}|a_j|.$$

If $x \in Y$, then $x_n \in U$ for large n. Hence the sequence $(f(x_n))$ is bounded for any $x \in Y$. By Lemma 2.3, $\rho_n = \mathrm{o}(u_n)$. Hence there exists an index p such that $\rho_n \le \alpha$ and $\sigma(n) \ge q$ for $n \ge p$. Let

$$X = \{x \in \mathbb{R}^{\mathbb{N}} : |x - y| \le \rho \text{ and } x_n = y_n \text{ for } n < p\},$$

$$H : Y \to \mathbb{R}^{\mathbb{N}}, \qquad H(x)(n) = \begin{cases} y_n & \text{for } n < p \\ y_n + \sum_{k=n}^{\infty} \frac{1}{r_k} \sum_{j=k}^{\infty} F(x)(j) & \text{for } n \geq p. \end{cases}$$

Note that $X \subset Y$. If $x \in X$, then for $n \geq p$ we have

$$|H(x)(n) - y_n| = \left| \sum_{k=n}^{\infty} \frac{1}{r_k} \sum_{j=k}^{\infty} F(x)(j) \right| \leq \sum_{k=n}^{\infty} \frac{1}{r_k} \sum_{j=k}^{\infty} |F(x)(j)| \leq \rho_n.$$

Therefore $HX \subset X$. Let $x \in X$, and $\varepsilon > 0$. Using (3.4) and Lemma 3.9 we get

$$\sum_{k=1}^{\infty} |a_k| \sum_{i=1}^{k} \frac{1}{r_i} < \infty.$$

Choose an index $m \geq p$ and a positive constant γ such that

$$L \sum_{k=m}^{\infty} |a_k| \sum_{i=1}^{k} \frac{1}{r_i} < \varepsilon \quad \text{and} \quad \gamma \sum_{k=1}^{m} |a_k| \sum_{i=1}^{k} \frac{1}{r_i} < \varepsilon. \tag{3.5}$$

Let

$$C = \bigcup_{n=1}^{m} [y_n - \alpha, y_n + \alpha].$$

Since C is a compact subset of $\mathbb{R}$, f is uniformly continuous on C. Choose a positive δ such that if $t_1, t_2 \in C$ and $|t_2 - t_1| < \delta$, then

$$|f(t_2) - f(t_1)| < \gamma. \tag{3.6}$$

Choose $z \in X$ such that $\|x - z\| < \delta$. Then

$$\begin{aligned} \|Hx - Hz\| &= \sup_{n \geq p} \left| \sum_{k=n}^{\infty} \frac{1}{r_k} \sum_{j=k}^{\infty} (F(x)(j) - F(z)(j)) \right| \\ &\leq \sum_{k=p}^{\infty} \frac{1}{r_k} \sum_{j=k}^{\infty} |F(x)(j) - F(z)(j)| \leq \sum_{k=p}^{\infty} \frac{1}{r_k} \sum_{j=k}^{\infty} |a_j||f(x_{\sigma(j)}) - f(z_{\sigma(j)})|. \end{aligned}$$

Using Lemma 3.9, (3.6), (3.3), and (3.5) we obtain

$$\begin{aligned} \|Hx - Hz\| &\leq \sum_{k=p}^{\infty} |a_k||f(x_{\sigma(k)}) - f(z_{\sigma(k)})| \sum_{i=1}^{k} \frac{1}{r_i} \\ &\leq \gamma \sum_{k=1}^{m} |a_k| \sum_{i=1}^{k} \frac{1}{r_i} + 2L \sum_{k=m}^{\infty} |a_k| \sum_{i=1}^{k} \frac{1}{r_i} < 3\varepsilon. \end{aligned}$$

Hence the map $H : X \to X$ is continuous with respect to the metric defined by (3.1). By Lemma 3.10 there exists a point $x \in X$ such that $x = Hx$. Then for $n \geq p$ we have

$$x_n = y_n + \sum_{k=n}^{\infty} \frac{1}{r_k} \sum_{j=k}^{\infty} F(x)(j).$$

Hence, for $n \geq p$ we get

$$\begin{aligned} \Delta(r_n \Delta x_n) &= \Delta(r_n \Delta y_n) + \Delta \left(r_n \Delta \left(\sum_{k=n}^{\infty} \frac{1}{r_k} \sum_{j=k}^{\infty} F(x)(j) \right) \right) \\ &= b_n - \Delta \left(\sum_{j=n}^{\infty} F(x)(j) \right) = F(x)(n) + b_n = a_n f(x_{\sigma(n)}) + b_n \end{aligned}$$

for large n. Since $x \in X$ and $\rho_n = \mathrm{o}(u_n)$, we get $x_n = y_n + \mathrm{o}(u_n)$. $\square$

Corollary 3.12. *Assume $a, b, r, u : \mathbb{N} \to \mathbb{R}$, $r > 0$, $u > 0$, $\Delta u \le 0$, y is a solution of the equation $\Delta(r_n \Delta y_n) = 0$,*

$$\sum_{k=1}^{\infty} \frac{1}{u_k r_k} \sum_{j=k}^{\infty} (|a_j| + |b_j|) < \infty, \quad q \in \mathbb{N}, \quad \alpha \in (0, \infty), \quad U = \bigcup_{n=q}^{\infty} [y_n - \alpha, y_n + \alpha],$$

and $f : \mathbb{R} \to \mathbb{R}$ is continuous and bounded on U. Then there exists a solution x of the equation

$$\Delta(r_n \Delta x_n) = a_n f(x_{\sigma(n)}) + b_n$$

such that $x_n = y_n + \mathrm{o}(u_n)$.

Proof. Define sequences w, y' by

$$w_n = \sum_{k=n}^{\infty} \frac{1}{r_k} \sum_{j=k}^{\infty} b_j, \quad y'_n = y_n + w_n.$$

Choose a number $\alpha' \in (0, \alpha)$ and let $\beta = \alpha - \alpha'$. By Lemma 2.3, $w_n = \mathrm{o}(u_n)$. Hence there exists an index $q' \ge q$ such that $|w_n| \le \beta$ for any $n \ge q'$. Let

$$U' = \bigcup_{n=q'}^{\infty} [y'_n - \alpha', y'_n + \alpha'].$$

If $t \in U'$ and $n \ge q'$, then

$$|t - y_n| = |t - y'_n + y'_n - y_n| \le |t - y'_n| + |y'_n - y_n| \le \alpha' + |w_n| \le \alpha' + \beta = \alpha.$$

Hence $U' \subset U$. Therefore f is continuous and bounded on U'. Moreover it is easy to see that $\Delta(r_n \Delta w_n) = b_n$. Thus

$$\Delta(r_n \Delta y'_n) = \Delta(r_n \Delta y_n) + \Delta(r_n \Delta w_n) = b_n.$$

By Theorem 3.11 there exists a solution x of the equation

$$\Delta(r_n \Delta x_n) = a_n f(x_{\sigma(n)}) + b_n$$

such that $x_n = y'_n + \mathrm{o}(u_n)$. Then

$$x_n = y_n + w_n + \mathrm{o}(u_n) = y_n + \mathrm{o}(u_n).$$ □

Remark 3.13. It is easy to see that if $r : \mathbb{N} \to (0, \infty)$, then a sequence y is a solution of the equation $\Delta(r_n \Delta y_n) = 0$ if and only if there exist real constants c_1, c_2 such that

$$y_n = c_1 \sum_{j=1}^{n-1} \frac{1}{r_j} + c_2$$

for any n.

Corollary 3.14. *Assume $a, b, r, u : \mathbb{N} \to \mathbb{R}$, $r > 0$, $u > 0$, $\Delta u \le 0$,*

$$\sum_{k=1}^{\infty} \frac{1}{u_k r_k} \sum_{j=k}^{\infty} |a_j| < \infty,$$

and $f : \mathbb{R} \to \mathbb{R}$ is continuous. Then for any bounded solution y of the equation $\Delta(r_n \Delta y_n) = b_n$ there exists a solution x of the equation

$$\Delta(r_n \Delta x_n) = a_n f(x_{\sigma(n)}) + b_n$$

such that $x_n = y_n + \mathrm{o}(u_n)$.

Proof. The assertion is an easy consequence of Theorem 3.11. □

4 Asymptotic behavior of solutions

In this section we establish some results concerning approximations of solutions. The results relating to equations of type (E) are based on Lemma 4.1. In the case of equations of type (QE), we use Lemma 4.5.

Lemma 4.1 ([17, Lemma 3.7]). *Assume $m \in \mathbb{N}$, (A, Z) is an m-pair, $a \in A$, $b, x : \mathbb{N} \to \mathbb{R}$, and $\Delta^m x_n = \mathrm{O}(a_n) + b_n$. Then there exist a solution y of the equation $\Delta^m y_n = b_n$ and a sequence $z \in Z$ such that $x_n = y_n + z_n$.*

Using Lemma 2.1 and Lemma 4.1 we obtain the following three theorems.

Theorem 4.2. *Assume $m \in \mathbb{N}$, $a, b : \mathbb{N} \to \mathbb{R}$, $r, u : \mathbb{N} \to (0, \infty)$, $\Delta u \le 0$,*

$$\sum_{n=1}^{\infty} \frac{n^{m-1}|a_n|}{u_n} < \infty, \quad F : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^{\mathbb{N}},$$

and x is a solution of the equation

$$\Delta^m x_n = a_n F(x)(n) + b_n$$

such that the sequence $F(x)$ is bounded. Then there exists a solution y of the equation $\Delta^m y_n = b_n$ such that $x_n = y_n + \mathrm{o}(u_n)$.

Theorem 4.3. *Assume $m, k \in \mathbb{N}$, $a, b : \mathbb{N} \to \mathbb{R}$, $r, u : \mathbb{N} \to (0, \infty)$, $\Delta u \le 0$,*

$$\sum_{n=1}^{\infty} \frac{n^{m-1}|a_n|}{u_n} < \infty, \quad f : \mathbb{N} \times \mathbb{R}^{k+1} \to \mathbb{R}, \quad \sigma_1, \dots, \sigma_k : \mathbb{N} \to \mathbb{N},$$

and x is a solution of the equation

$$\Delta^m x_n = a_n f(n, x_{\sigma_1(n)}, \dots, x_{\sigma_k(n)}) + b_n$$

such that the sequence $f(n, x_{\sigma_1(n)}, \dots, x_{\sigma_k(n)})$ is bounded. Then there exists a solution y of the equation $\Delta^m y_n = b_n$ such that $x_n = y_n + \mathrm{o}(u_n)$.

Theorem 4.4. *Assume $m \in \mathbb{N}$, $a, b : \mathbb{N} \to \mathbb{R}$, $r, u : \mathbb{N} \to (0, \infty)$, $\Delta u \le 0$,*

$$K : \mathbb{N} \times \mathbb{N} \to \mathbb{R}, \quad \sum_{n=1}^{\infty} \frac{n^{m-1}}{u_n} \sum_{k=1}^{n} |K(n,k)| < \infty, \quad f : \mathbb{N} \times \mathbb{R} \to \mathbb{R}, \quad \sigma : \mathbb{N} \to \mathbb{N},$$

and x is a solution of the equation

$$\Delta^m x_n = b_n + \sum_{k=1}^{n} K(n,k) f(k, x_{\sigma(k)})$$

such that the sequence $f(n, x_{\sigma(n)})$ is bounded. Then there exists a solution y of the equation $\Delta^m y_n = b_n$ such that $x_n = y_n + \mathrm{o}(u_n)$.

Lemma 4.5. *Assume $b, x : \mathbb{N} \to \mathbb{R}$, $r, u : \mathbb{N} \to (0, \infty)$, $\Delta u \le 0$,*

$$\sum_{k=1}^{\infty} \frac{1}{u_k r_k} \sum_{j=k}^{\infty} |a_j| < \infty, \quad \text{and} \quad \Delta(r_n \Delta x_n) = \mathrm{O}(a_n) + b_n$$

Then there exists a solution y of the equation $\Delta(r_n \Delta y_n) = b_n$ such that $x_n = y_n + \mathrm{o}(u_n)$.

Proof. Define a sequence w by $w_n = \Delta(r_n \Delta x_n) - b_n$. Then $w_n = \mathrm{O}(a_n)$. Hence

$$\sum_{k=1}^{\infty} \frac{1}{u_k r_k} \sum_{j=k}^{\infty} |w_j| < \infty.$$

Define a sequence z by

$$z_n = \sum_{k=n}^{\infty} \frac{1}{r_k} \sum_{j=k}^{\infty} w_j.$$

By Lemma 2.3, $z_n = \mathrm{o}(u_n)$. Let $y = x - z$. Then

$$\begin{aligned}
\Delta(r_n \Delta y_n) &= \Delta(r_n \Delta x_n) - \Delta\left(r_n \sum_{k=n}^{\infty} \frac{1}{r_k} \sum_{j=k}^{\infty} w_j\right) \\
&= \Delta(r_n \Delta x_n) + \Delta\left(\sum_{j=n}^{\infty} w_j\right) = \Delta(r_n \Delta x_n) - w_n = b_n.
\end{aligned}$$

□

Using Lemma 4.5 we obtain the following three theorems.

Theorem 4.6. *Assume $a, b : \mathbb{N} \to \mathbb{R}$, $r, u : \mathbb{N} \to (0, \infty)$, $\Delta u \le 0$, $F : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^{\mathbb{N}}$,*

$$\sum_{k=1}^{\infty} \frac{1}{u_k r_k} \sum_{j=k}^{\infty} |a_j| < \infty,$$

and x is a solution of the equation $\Delta(r_n \Delta x_n) = a_n F(x)(n) + b_n$ such that the sequence $F(x)$ is bounded. Then there exists a solution y of the equation $\Delta(r_n \Delta y_n) = b_n$, such that $x_n = y_n + \mathrm{o}(u_n)$.

Theorem 4.7. *Assume $k \in \mathbb{N}$, $a, b : \mathbb{N} \to \mathbb{R}$, $r, u : \mathbb{N} \to (0, \infty)$, $\Delta u \le 0$,*

$$\sum_{k=1}^{\infty} \frac{1}{u_k r_k} \sum_{j=k}^{\infty} |a_j| < \infty, \quad f : \mathbb{N} \times \mathbb{R}^{k+1} \to \mathbb{R}, \quad \sigma_1, \ldots, \sigma_k : \mathbb{N} \to \mathbb{N},$$

and x is a solution of the equation

$$\Delta(r_n \Delta x_n) = a_n f(n, x_{\sigma_1(n)}, \ldots, x_{\sigma_k(n)}) + b_n$$

such that the sequence $f(n, x_{\sigma_1(n)}, \ldots, x_{\sigma_k(n)})$ is bounded. Then there exists a solution y of the equation $\Delta(r_n \Delta y_n) = b_n$ such that $x_n = y_n + \mathrm{o}(u_n)$.

Theorem 4.8. *Assume $m \in \mathbb{N}$, $a, b : \mathbb{N} \to \mathbb{R}$, $r, u : \mathbb{N} \to (0, \infty)$, $\Delta u \le 0$,*

$$K : \mathbb{N} \times \mathbb{N} \to \mathbb{R}, \quad \sum_{k=1}^{\infty} \frac{1}{u_k r_k} \sum_{j=k}^{\infty} \sum_{i=1}^{j} |K(j, i)| < \infty, \quad f : \mathbb{N} \times \mathbb{R} \to \mathbb{R},$$

$\sigma : \mathbb{N} \to \mathbb{N}$, and x is a solution of the equation

$$\Delta(r_n \Delta x_n) = b_n + \sum_{k=1}^{n} K(n, k) f(k, x_k)$$

such that the sequence $f(n, x_k)$ is bounded. Then there exists a solution y of the equation $\Delta^m y_n = b_n$ such that $x_n = y_n + \mathrm{o}(u_n)$.

Remark 4.9. Theorems 4.2–4.8 do not guarantee the existence of the described solutions. In many concrete cases the existence of such solutions can be obtained. Some of such cases are presented in Section 3.

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