Electronic Journal of Qualitative Theory of Differential Equations
2019, No. **43**, 1–17; https://doi.org/10.14232/ejqtde.2019.1.43 www.math.u-szeged.hu/ejqtde/

On the existence and exponential stability for differential equations with multiple constant delays and nonlinearity depending on fractional substantial integrals

Milan Medveď[1] and **Michal Pospíšil**✉[1, 2]

[1]Department of Mathematical Analysis and Numerical Mathematics,
Faculty of Mathematics, Physics and Informatics, Comenius University in Bratislava,
Mlynská dolina, 842 48 Bratislava, Slovak Republic
[2]Mathematical Institute, Slovak Academy of Sciences,
Štefánikova 49, 814 73 Bratislava, Slovak Republic

Received 25 March 2019, appeared 27 June 2019
Communicated by Josef Diblík

Abstract. An existence result is proved for systems of differential equations with multiple constant delays, time-dependent coefficients and the right-hand side depending on fractional substantial integrals. Results on exponential stability for such equations are proved for linearly bounded nonlinearities and power type nonlinearities. An illustrative example is also given.

Keywords: multiple delays, fractional substantial integral, exponential stability, multi-delayed matrix exponential, logarithmic matrix norm.

2010 Mathematics Subject Classification: 34K20, 26A33.

1 Introduction

It is well known that the trivial solution of the linear fractional differential equation

$$^{C}D^{\alpha}x(t) = Ax(t), \quad x(t) \in \mathbb{R}^N,\ \alpha \in (0,1), \tag{1.1}$$

where A is a constant matrix and $^{C}D^{\alpha}x(t)$ is the Caputo fractional derivative can be asymptotically, but not exponentially stable. It is asymptotically stable if and only if $|\arg(\lambda)| > \frac{\alpha\pi}{2}$ for any eigenvalue of the matrix A (see e.g. [4,11,16]). However, for special types of fractional differential equations their solutions can be exponentially stable. In the paper [15], a sufficient condition for the exponential stability of the trivial solution of the nonlinear multi-delay fractional differential equation

$$^{C}D^{\alpha}\left(h(t)\left(\dot{x}(t) - Ax(t) - B_1x(t-\tau_1) - \cdots - B_mx(t-\tau_m)\right)\right) = f(x(t), x(t-\tau_1), \ldots, x(t-\tau_m))$$

✉Corresponding author. Email: Michal.Pospisil@fmph.uniba.sk

was proved. In the paper [3], the equation

$$\dot{x}(t) = Ax(t) + f\left(t, x(t), {}^{RL}I^{\alpha_1}x(t), \dots, {}^{RL}I^{\alpha_m}x(t)\right), \tag{1.2}$$

where ${}^{RL}I^{\alpha_1}x(t), \dots, {}^{RL}I^{\alpha_m}x(t)$ are the Riemann–Liouville integrals, was studied. An existence result and a sufficient condition for the exponential stability of the trivial solution of this equation was proved. In the paper [2], an analogous problem was solved for an equation of the form (1.2) with Caputo–Fabrizio fractional integrals instead of the Riemann–Liouville integrals.

In this paper, we study systems of differential equations with multiple constant delays, time-dependent coefficients and the right-hand side depending on fractional substantial integrals, defined below. Originally, the formula for a solution of the initial-function problem

$$\dot{x}(t) = Ax(t) + B_1x(t-\tau_1) + \cdots + B_nx(t-\tau_n) + f(t), \quad t \geq 0, \tag{1.3}$$

$$x(t) = \varphi(t), \qquad t \in [-\tau, 0] \tag{1.4}$$

where $\tau = \max_{i=1,\dots,n} \tau_i$, was stated in [14, Theorem 10] using so-called multi-delayed matrix exponential, which is an inductively built matrix polynomial of a degree depending on time. This result was later simplified in [18] using the unilateral Laplace transform to obtain a closed-form formula (see Theorem 2.1 below). We remark that the delayed matrix exponential for the equation with one constant delay was introduced in the paper [7].

In the present paper, we make use of this formula to prove existence and exponential stability results for delayed differential equation (DDE) with multiple constant delays and nonlinearity depending on fractional substantial integrals of order $\beta > 0$ with a positive parameter γ (see e.g. [4,6]),

$$I^{(\beta,\gamma)}x(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \mathrm{e}^{-\gamma(t-s)} x(s) ds.$$

In particular, we consider the Cauchy problem

$$\begin{aligned}
\dot{x}(t) &= A(t)x(t) + B_1(t)x(t-\tau_1) + \cdots + B_n(t)x(t-\tau_n) \\
&\quad + \mathcal{F}\big(t, x(t), x(t-\tau_1), \dots, x(t-\tau_n), \\
&\qquad I^{(\beta_{01},\gamma_{01})}x(t), \dots, I^{(\beta_{0m_0},\gamma_{0m_0})}x(t), \\
&\qquad I^{(\beta_{11},\gamma_{11})}x(t-\tau_1), \dots, I^{(\beta_{nm_n},\gamma_{nm_n})}x(t-\tau_n)\big), \quad t \geq 0. \\
x(t) &= \varphi(t), \quad t \in [-\tau, 0],
\end{aligned} \tag{1.5}$$

where $A, B_1, \dots, B_n$ are continuous matrix functions,

$$\mathcal{F}(t, u_0, \dots, u_n, v_{00}, \dots, v_{0m_0}, v_{11}, \dots, v_{nm_n})$$

is a continuous function of all its variables and $\varphi \in C([-\tau, 0], \mathbb{R}^N)$. This work is a continuation of [12,13], where an analogous problem was investigated without the presence of delays.

We note that in [14] and [17] the matrices $A, B_1, \dots, B_n$ were supposed to be pairwise permutable, i.e., $AB_i = B_iA$, $B_iB_j = B_jB_i$ for each $i, j = 1, \dots, n$. But our existence result, Theorem 3.1, holds without any permutability assumption. For the stability results, Theorems 4.1 and 5.1, we only assume that the matrix functions $A(t)$, $B_1(t)$, $\dots$, $B_n(t)$ are permutable at some points t_0, t_1, $\dots$, t_n, respectively.

In the whole paper, we shall denote $\|\cdot\|$ the norm of a vector and the corresponding induced matrix norm. Further, $\mathbb{N}$ and $\mathbb{N}_0$ denote the set of all positive and nonnegative integers, respectively. We also assume the property of an empty sum, $\sum_{i\in\emptyset} z(i) = 0$ for any function z.

To make our stability results more applicable, we use the logarithmic matrix norm in assumptions. Analogous results can be obtained using the largest real value of all the eigenvalues of $A(t_0)$, $\max_{\lambda_A\in\sigma(A(t_0))} \operatorname{Re}\lambda_A$, or a weighted logarithmic matrix norm [8]. However, then one has to work with the estimation

$$\|\mathrm{e}^{At}\| \le c_1 \mathrm{e}^{c_2 t} \tag{1.6}$$

with some positive constants c_1, c_2, where c_1 is not immediately known. So, the area of exponential stability can not be predetermined. By the logarithmic norm, (1.6) holds with $c_1 = 1$.

The paper is organized as follows. In the following section, we collect some known results and definitions. Section 3 is devoted to the existence result of a unique solution of the initial-function problem (1.5). Sections 4 and 5 contain results on the exponential stability of a trivial solution of a class of nonlinear DDEs with the linearly bounded nonlinearity and nonlinearity bounded by some powers of its arguments, respectively. In final Section 6, we present an example illustrating the theoretical results.

2 Preliminary results

Let us recall a result from [18, Theorem 3.3] (see also [17, Theorem 2.15] for the case with variable delays) on the representation of a solution of a DDE with multiple delays.

Theorem 2.1. *Let $n\in\mathbb{N}$, $0<\tau_1,\dots,\tau_n\in\mathbb{R}$, $\tau := \max\{\tau_1,\tau_2,\dots,\tau_n\}$, $A, B_1,\dots,B_n$ be pairwise permutable constant $N\times N$ matrices, $\varphi\in C([-\tau,0],\mathbb{R}^N)$, and $f:[0,\infty)\to\mathbb{R}^N$ be a given function. Then the solution of the Cauchy problem (1.3), (1.4) has the form*

$$x(t) = \begin{cases} \varphi(t), & -\tau\le t<0,\\ \mathcal{B}(t)\varphi(0) + \sum_{j=1}^n B_j \int_0^{\tau_j} \mathcal{B}(t-s)\varphi(s-\tau_j)ds + \int_0^t \mathcal{B}(t-s)f(s)ds, & 0\le t \end{cases}$$

where

$$\mathcal{B}(t) = \mathrm{e}^{At} \sum_{\substack{\sum_{m=1}^n k_m\tau_m\le t\\ k_1,\dots,k_n\ge 0}} \frac{(t-\sum_{m=1}^n k_m\tau_m)^{\sum_{m=1}^n k_m}}{k_1!\dots k_n!} \prod_{m=1}^n \widetilde{B}_m^{k_m}$$

for any $t\in\mathbb{R}$, and $\widetilde{B}_m = B_m \mathrm{e}^{-A\tau_m}$ for each $m=1,\dots,n$.

Combining an estimation of the multi-delayed matrix exponential, [14, Lemma 13], with the representations of solutions of (1.3), (1.4) from [14] and Theorem 2.1, we obtain the following statement.

Lemma 2.2. *Let $n\in\mathbb{N}$, $0<\tau_1,\dots,\tau_n\in\mathbb{R}$, $B_1,\dots,B_n$ be pairwise permutable constant $N\times N$ matrices. If $\alpha_1,\dots,\alpha_n\in\mathbb{R}$ are such that $\|B_i\|\le\alpha_i \mathrm{e}^{\alpha_i\tau_i}$ for each $i=1,\dots,n$, then*

$$\left\| \sum_{\substack{\sum_{m=1}^n k_m\tau_m\le t\\ k_1,\dots,k_n\ge 0}} \frac{(t-\sum_{m=1}^n k_m\tau_m)^{\sum_{m=1}^n k_m}}{k_1!\dots k_n!} \prod_{m=1}^n B_m^{k_m} \right\| \le \mathrm{e}^{(\alpha_1+\dots+\alpha_n)t}$$

for any $t\in\mathbb{R}$.

We will investigate the exponential stability with respect to a ball in the sense of the next definition.

Definition 2.3. The zero solution of equation (1.3) is exponentially stable with respect to the ball $\Omega(r) := \{h \in \mathbb{R}^N \mid \|h\| \leq r\}$ if there are positive constants c_1, c_2 such that any solution x of (1.3) satisfying initial condition (1.4) with $\varphi(t) \in \Omega(r)$ for all $t \in [-\tau, 0]$ fulfills $\|x(t)\| \leq c_1 \mathrm{e}^{-c_2 t}$ for all $t \geq 0$.

Exponential stability of a trivial solution of other delay equations is understood analogously.

The logarithmic norm of a square matrix A is defined by

$$\mu(A) = \lim_{\varepsilon \to 0^+} \frac{\|I + \varepsilon A\| - 1}{\varepsilon}.$$

The properties we need are concluded in the following lemma (see e.g. [5]).

Lemma 2.4. *The logarithmic norm of a matrix A satisfies:*

1. $-\|A\| \leq -\mu(-A) \leq \operatorname{Re} \sigma(A) \leq \mu(A) \leq \|A\|$,
2. $\|\mathrm{e}^{At}\| \leq \mathrm{e}^{\mu(A)t}$ *for all* $t \geq 0$.

We shall also need the following integral inequality, which was proved in [10] for integer powers. The authors did not realize/mention that their proof works even in the more general setting with real exponents.

Lemma 2.5. *Let $2 \leq n \in \mathbb{N}$, $c \geq 0$, $f_i(t)$ for $i = 1, \dots, n$ be nonnegative continuous functions defined on $[a, b]$ and $1 = q_1 < q_2 \leq q_3 \leq \dots \leq q_n$ be real numbers. If a positive differentiable real-valued function $z(t)$ satisfies*

$$z(t) \leq c + \int_a^t \sum_{i=1}^n f_i(s) z^{q_i}(s) ds, \quad t \in [a, b]$$

and

$$1 - (q_n - 1) \int_a^b \sum_{i=2}^n c^{q_i - 1} f_i(s) \exp\left((q_n - 1) \int_a^s f_1(\sigma) d\sigma\right) ds > 0,$$

then

$$z(t) \leq \frac{c \exp\left(\int_a^t f_1(s) ds\right)}{\left(1 - (q_n - 1) \int_a^t \sum_{i=2}^n c^{q_i - 1} f_i(s) \exp\left((q_n - 1) \int_a^s f_1(\sigma) d\sigma\right) ds\right)^{\frac{1}{q_n - 1}}}.$$

Proof. The proof is exactly the same as the proof of [10, Theorem 2.6]. □

3 Existence result

Here we prove an existence and uniqueness result for a solution of the initial-function problem (1.5).

Theorem 3.1. *Let $I = [0, A] \subset \mathbb{R}$ for some $A > 0$, $G \subset \mathbb{R}^N$ be a region, $H \subset \mathbb{R}^{m_0} \times \dots \times \mathbb{R}^{m_n}$ be a region containing $0 \in \mathbb{R}^{m_0} \times \dots \times \mathbb{R}^{m_n}$, $\mathcal{F} \in C(I \times G^{n+1} \times H, \mathbb{R}^N)$ is a continuous locally Lipschitz function. Then for any $\varphi \in C([-\tau, 0], G)$ there exists $\delta > 0$ such that the initial function problem (1.5) has a unique solution $x(t)$ on the interval $I_\delta = [-\tau, \delta]$.*

Proof. Let $b_i, b_{ij} > 0$, $i = 0, \ldots, n$, $j = 1, \ldots, m_i$ be such that

$$G_{b_i} := \{x \in \mathbb{R}^N \mid \|x - \varphi(-\tau_i)\| \le b_i\} \subset G, \quad i = 0, \ldots, n$$

for $\tau_0 = 0$, and

$$V := \{(v_{01}, \ldots, v_{nm_n}) \in \mathbb{R}^{m_0} \times \cdots \times \mathbb{R}^{m_n} \mid \|v_{ij}\| \le b_{ij},\ i = 0, \ldots, n,\ j = 1, \ldots, m_i\} \subset H.$$

Let $0 < a < A$ be such that

$$\max_{\sigma \in [0, \min\{a, \tau_i\}]} \|\varphi(\sigma - \tau_i) - \varphi(-\tau_i)\| \le b_i, \quad i = 1, \ldots, n. \tag{3.1}$$

From now on, we shall assume without any loss of generality that $a \le \min_{i=1\ldots,n} \tau_i$. Note that (3.1) then implies

$$\max_{\sigma \in [0,a]} \|\varphi(\sigma - \tau_i)\| \le b_i + \|\varphi(-\tau_i)\|, \quad i = 1, \ldots, n. \tag{3.2}$$

So, we have $G_0 := [0, a] \times G_{b_0} \times \cdots \times G_{b_n} \times V \subset I \times G^{n+1} \times H$. Let us denote

$$M_0 := \max_{t \in [0,a],\, x \in G_{b_0}} \|A(t)x\|, \qquad M_A := \max_{t \in [0,a]} \|A(t)\|,$$

$$M_i := \max_{t \in [0,a],\, x \in G_{b_i}} \|B_i(t)x\|, \quad i = 1, \ldots, n,$$

$$M_{\mathcal{F}} := \max_{(t, u_0, \ldots, u_n, v_{01}, \ldots, v_{nm_n}) \in G_0} \mathcal{F}(t, u_0, \ldots, u_n, v_{01}, \ldots, v_{nm_n}).$$

Let $L_i, L_{ij} > 0$, $i = 0, \ldots, n$, $j = 1, \ldots, m_i$ be such that

$$\begin{aligned} &\|\mathcal{F}(t, u_0, \ldots, u_n, v_{01}, \ldots, v_{nm_n}) - \mathcal{F}(t, \tilde{u}_0, \ldots, \tilde{u}_n, \tilde{v}_{01}, \ldots, \tilde{v}_{nm_n})\| \\ &\quad \le \sum_{i=0}^{n} L_i \|u_i - \tilde{u}_i\| + \sum_{i=0}^{n} \sum_{j=1}^{m_i} L_{ij} \|v_{ij} - \tilde{v}_{ij}\| \end{aligned}$$

for all $(t, u_0, \ldots, u_n, v_{01}, \ldots, v_{nm_n}), (t, \tilde{u}_0, \ldots, \tilde{u}_n, \tilde{v}_{01}, \ldots, \tilde{v}_{nm_n}) \in G_0$. Finally, let

$$0 < \delta < \min\left\{a, c, \frac{b_0}{M_0 + \cdots + M_n + M_{\mathcal{F}}}, \kappa^{-1}\right\}$$

with

$$c \le \min_{\substack{i=0,\ldots,n \\ j=1,\ldots,m_i}} \left(\frac{b_{ij}\Gamma(1 + \beta_{ij})}{b_i + \|\varphi(-\tau_i)\|}\right)^{\frac{1}{\beta_{ij}}}, \quad \kappa = M_A + L_0 + \sum_{j=1}^{m_0} \frac{L_{0j} c^{\beta_{0j}}}{\Gamma(1 + \beta_{0j})}.$$

Consider the Banach space $C_\delta := C(I_\delta, \mathbb{R}^N)$ endowed with the maximum norm, i.e., $\|x\| = \max_{t \in I_\delta} \|x(t)\|$ for $x \in C_\delta$, and define the successive approximations $\{x_k\}_{k=0}^{\infty} \subset C_\delta$ by

$$x_0(t) = \begin{cases} \varphi(t), & t \in [-\tau, 0), \\ \varphi(0), & t \in [0, \delta], \end{cases}$$

$$x_{k+1}(t) = \begin{cases} \varphi(t), & t \in [-\tau, 0), \\ \varphi(0) + \int_0^t A(s) x_k(s) ds + \sum_{i=1}^{n} \int_0^t B_i(s) x_k(s - \tau_i) ds & \\ + \int_0^t \mathcal{F}\Big(s, x_k(s), x_k(s - \tau_1), \ldots, x_k(s - \tau_n), \ldots, & \\ \frac{1}{\Gamma(\beta_{01})} \int_0^s (s - \sigma)^{\beta_{01} - 1} e^{-\gamma_{01}(s - \sigma)} x_k(\sigma) d\sigma, \ldots, & \\ \frac{1}{\Gamma(\beta_{0m_0})} \int_0^s (s - \sigma)^{\beta_{0m_0} - 1} e^{-\gamma_{0m_0}(s - \sigma)} x_k(\sigma) d\sigma, & \\ \frac{1}{\Gamma(\beta_{11})} \int_0^s (s - \sigma)^{\beta_{11} - 1} e^{-\gamma_{11}(s - \sigma)} x_k(\sigma - \tau_1) d\sigma, \ldots, & \\ \frac{1}{\Gamma(\beta_{nm_n})} \int_0^s (s - \sigma)^{\beta_{nm_n} - 1} e^{-\gamma_{nm_n}(s - \sigma)} x_k(\sigma - \tau_n) d\sigma\Big), & t \in [0, \delta] \end{cases}$$

for $k = 0, 1, \dots$

First, we show that $x_1(t)$ is well defined. For any $s \in [0,t] \subset [0,\delta]$ we have $s \in [0,a]$,

$$\|x_0(s) - \varphi(0)\| \le \max_{\sigma\in[0,\delta]} \|x_0(\sigma) - \varphi(0)\| = \|\varphi(0) - \varphi(0)\| = 0 \le b_0,$$

i.e., $x_0(s) \in G_{b_0}$, and

$$\begin{aligned}\|x_0(s-\tau_i) - \varphi(-\tau_i)\| &\le \max_{\sigma\in[0,\delta]} \|x_0(\sigma-\tau_i) - \varphi(-\tau_i)\| \\ &\le \max_{\sigma\in[0,a]} \|\varphi(\sigma-\tau_i) - \varphi(-\tau_i)\| \le b_i\end{aligned} \tag{3.3}$$

for each $i = 1, \dots, n$ by (3.1), i.e., $x_0(s-\tau_i) \in G_{b_i}$. Next, using the estimation

$$\begin{aligned}\frac{1}{\Gamma(\beta_{ij})}\int_0^s (s-\sigma)^{\beta_{ij}-1}\mathrm{e}^{-\gamma_{ij}(s-\sigma)}d\sigma &= \frac{1}{\Gamma(\beta_{ij})}\int_0^s \sigma^{\beta_{ij}-1}\mathrm{e}^{-\gamma_{ij}\sigma}d\sigma \\ &\le \frac{1}{\Gamma(\beta_{ij})}\int_0^s \sigma^{\beta_{ij}-1}d\sigma = \frac{s^{\beta_{ij}}}{\beta_{ij}\Gamma(\beta_{ij})} = \frac{s^{\beta_{ij}}}{\Gamma(1+\beta_{ij})} \\ &\le \frac{\delta^{\beta_{ij}}}{\Gamma(1+\beta_{ij})} \le \frac{c^{\beta_{ij}}}{\Gamma(1+\beta_{ij})}\end{aligned}$$

for all $s \in [0,t] \subset [0,\delta]$ and each $i = 0, \dots, n$, $j = 1, \dots, m_i$, we derive

$$\begin{aligned}&\left\|\frac{1}{\Gamma(\beta_{0j})}\int_0^s (s-\sigma)^{\beta_{0j}-1}\mathrm{e}^{-\gamma_{0j}(s-\sigma)}x_0(\sigma)d\sigma\right\| \\ &\quad\le \max_{\sigma\in[0,\delta]}\|x_0(\sigma)\|\frac{c^{\beta_{0j}}}{\Gamma(1+\beta_{0j})} = \frac{\|\varphi(0)\|c^{\beta_{0j}}}{\Gamma(1+\beta_{0j})} \le \frac{\|\varphi(0)\|b_{0j}}{b_0 + \|\varphi(0)\|} \le b_{0j}\end{aligned}$$

for each $j = 1, \dots, m_0$, and

$$\begin{aligned}&\left\|\frac{1}{\Gamma(\beta_{ij})}\int_0^s (s-\sigma)^{\beta_{ij}-1}\mathrm{e}^{-\gamma_{ij}(s-\sigma)}x_0(\sigma-\tau_i)d\sigma\right\| \\ &\quad\le \max_{\sigma\in[0,\delta]}\|x_0(\sigma-\tau_i)\|\frac{c^{\beta_{ij}}}{\Gamma(1+\beta_{ij})} = \max_{\sigma\in[0,\delta]}\|\varphi(\sigma-\tau_i)\|\frac{c^{\beta_{ij}}}{\Gamma(1+\beta_{ij})} \\ &\quad\le \frac{(b_i + \|\varphi(-\tau_i)\|)\,c^{\beta_{ij}}}{\Gamma(1+\beta_{ij})} \le b_{ij}\end{aligned} \tag{3.4}$$

for each $i = 1, \dots, n$, $j = 1, \dots, m_i$ where we applied (3.2). Note that estimations (3.3), (3.4) are valid for x_k instead of x_0 without any respect to k, since it holds $x_k(\sigma-\tau_i) = \varphi(\sigma-\tau_i)$ for any $\sigma \in [0,\delta]$ as $0 < \delta \le a \le \min_{i=1\dots,n}\tau_i$. Therefore, the inclusion

$$\begin{aligned}\Big(&s, x_k(s), x_k(s-\tau_1), \dots, x_k(s-\tau_n), I^{(\beta_{01},\gamma_{01})}x_k(s), \dots, I^{(\beta_{0m_0},\gamma_{0m_0})}x_k(s), \\ &I^{(\beta_{11},\gamma_{11})}x_k(s-\tau_1), \dots, I^{(\beta_{nm_n},\gamma_{nm_n})}x_k(s-\tau_n)\Big) \in G_0, \quad \forall s \in [0,\delta]\end{aligned} \tag{3.5$_k$}$$

holds for $k = 0$, i.e., $(3.5)_0$ holds. That means that the argument of $\mathcal{F}$ in the definition of $x_1(t)$ is in G_0. So, $x_1(t)$ is well defined.

Now, assume $(3.5)_{k-1}$ for some $k \in \mathbb{N}$. We will show that $(3.5)_k$ follows, i.e., $x_{k+1}(t)$ is well defined on I_δ. By the above arguments, to show $(3.5)_k$ it is enough to prove $x_k(s) \in G_{b_0}$ and $\|I^{(\beta_{0j},\gamma_{0j})}x_k(s)\| \le b_{0j}$ for all $s \in [0,\delta]$ and $j = 1, \dots, m_0$. Firstly,

$$\|x_k(s) - \varphi(0)\| \le \max_{\sigma\in[0,\delta]}\|x_k(\sigma) - \varphi(0)\| \le \delta(M_0 + \cdots + M_n + M_{\mathcal{F}}) \le b_0.$$

Secondly, using the latter estimation,

$$\begin{aligned}\left\|\frac{1}{\Gamma(\beta_{0j})}\int_0^s (s-\sigma)^{\beta_{0j}-1}\mathrm{e}^{-\gamma_{0j}(s-\sigma)}x_k(\sigma)d\sigma\right\| &\leq \max_{\sigma\in[0,\delta]}\|x_k(\sigma)\|\frac{c^{\beta_{0j}}}{\Gamma(1+\beta_{0j})}\\ \leq \left(\max_{\sigma\in[0,\delta]}\|x_k(\sigma)-\varphi(0)\|+\|\varphi(0)\|\right)\frac{c^{\beta_{0j}}}{\Gamma(1+\beta_{0j})} &\leq \frac{(b_0+\|\varphi(0)\|)c^{\beta_{0j}}}{\Gamma(1+\beta_{0j})}\leq b_{0j}.\end{aligned}$$

So, we have inductively proved that all $x_k(t)$, $k\in\mathbb{N}$ are well-defined functions from C_δ.

In the next step, we show that $x_k(t)$ converges uniformly on I_δ to a solution of (1.5) as $k\to\infty$. Using the identity $x_k(s-\tau_i)-x_{k-1}(s-\tau_i)=0$ for all $s\in[0,\delta]$ and $k\in\mathbb{N}$, we can estimate

$$\begin{aligned}\|x_{k+1}-x_k\| &= \max_{t\in[0,\delta]}\|x_{k+1}(t)-x_k(t)\|\\ &\leq \max_{t\in[0,\delta]}\Bigg[M_A\int_0^t\|x_k(s)-x_{k-1}(s)\|ds+\int_0^t\Bigg(L_0\|x_k(s)-x_{k-1}(s)\|\\ &\qquad+\sum_{j=1}^{m_0}\frac{L_{0j}}{\Gamma(\beta_{0j})}\int_0^s(s-\sigma)^{\beta_{0j}-1}\mathrm{e}^{-\gamma_{0j}(s-\sigma)}\|x_k(\sigma)-x_{k-1}(\sigma)\|d\sigma\Bigg)ds\Bigg]\\ &\leq \delta\|x_k-x_{k-1}\|\left(M_A+L_0+\sum_{j=1}^{m_0}\frac{L_{0j}c^{\beta_{0j}}}{\Gamma(1+\beta_{0j})}\right)=\delta\kappa\|x_k-x_{k-1}\|\end{aligned}$$

for each $k\in\mathbb{N}$. Therefore,

$$\|x_{k+1}-x_k\|\leq(\delta\kappa)^k\|x_1-x_0\|,\quad k\in\mathbb{N}_0.$$

Consequently,

$$\sum_{i=1}^{k}\|x_i(t)-x_{i-1}(t)\|\leq\|x_1-x_0\|\sum_{i=0}^{k-1}(\delta\kappa)^i,\quad\forall t\in[0,\delta],\,k\in\mathbb{N}.$$

Hence, $\sum_{i=0}^{\infty}(\delta\kappa)^i<\infty$ implies the uniform convergence of the series $\sum_{i=1}^{\infty}(x_i(t)-x_{i-1}(t))$ on I_δ. So, using $x_k=x_0+\sum_{i=1}^{k}(x_i-x_{i-1})$ for each $k\in\mathbb{N}$, we see that the sequence $\{x_k(t)\}_{k=0}^{\infty}$ converges uniformly on I_δ to the continuous function $x=x_0+\sum_{i=1}^{\infty}(x_i-x_{i-1})\in C_\delta$, which is a unique solution of (1.5). □

4 Exponential stability for linearly bounded right-hand side

In this section, we prove a sufficient condition for the exponential stability of a trivial solution of the DDE with variable coefficients, multiple delays and nonlinearity depending on fractional substantial integrals,

$$\begin{aligned}\dot{x}(t) &= A(t)x(t)+B_1(t)x(t-\tau_1)+\cdots+B_n(t)x(t-\tau_n)\\ &\quad+F(t,x(t),x(t-\tau_1),\ldots,x(t-\tau_n))\\ &\quad+f\Big(t,I^{(\beta_{01},\gamma_{01})}x(t),\ldots,I^{(\beta_{0m_0},\gamma_{0m_0})}x(t),\\ &\qquad I^{(\beta_{11},\gamma_{11})}x(t-\tau_1),\ldots,I^{(\beta_{nm_n},\gamma_{nm_n})}x(t-\tau_n)\Big),\quad t\geq 0.\end{aligned}\tag{4.1}$$

For better clarity, we conclude the main assumptions here:

(H1) there are positive numbers r_i and $q_i, \Theta_i, t_i \geq 0$, $i = 0, \dots, n$ such that

$$\|A(t) - A(t_0)\| \leq q_0 e^{-r_0|t-t_0|}|t-t_0|^{\Theta_0},$$
$$\|B_i(t) - B_i(t_i)\| \leq q_i e^{-r_i|t-t_i|}|t-t_i|^{\Theta_i}, \quad i = 1, \dots, n$$

for all $t \geq 0$;

(H2) there are constants $\alpha_1, \dots, \alpha_n$ such that

$$\|B_i(t_i) e^{-A(t_0)\tau_i}\| \leq \alpha_i e^{\alpha_i \tau_i}$$

for each $i = 1, \dots, n$;

(H3) it holds $\gamma_{ij} > \rho = -\mu(A(t_0)) - \alpha > 0$ for each $i = 0, \dots, n$, $j = 1, \dots, m_0$, where $\alpha = \alpha_1 + \cdots + \alpha_n$ and $\mu(A(t_0))$ is the logarithmic norm of the constant matrix $A(t_0)$;

(H4) for a constant $0 < r \leq \infty$ there are positive constants ϑ_i and $\delta_i \geq 0$ for $i = 0, \dots, n$ such that

$$\|F(t, u_0, \dots, u_n)\| \leq \sum_{i=0}^{n} \delta_i e^{-\vartheta_i t}\|u_i\|$$

for all $t \geq 0$ and $u_i \in \Omega(r)$, $i = 0, \dots, n$;

(H5) there are $m_i \in \mathbb{N}$ positive constants μ_{ij} and $\eta_{ij} \geq 0$ for $i = 0, \dots, n$, $j = 1, \dots, m_i$ such that

$$\|f(t, v_{01}, \dots, v_{0m_0}, v_{11}, \dots, v_{nm_n})\| \leq \sum_{i=0}^{n} \sum_{j=1}^{m_i} \eta_{ij} e^{-\mu_{ij} t}\|v_{ij}\|$$

for all $t \geq 0$ and $v_{ij} \in \Omega(r)$, $i = 0, \dots, n$, $j = 1, \dots, m_i$.

Without conditions (H4), (H5), equation (4.1) could not have an exponentially stable trivial solution (see e.g. [3,9]).

Theorem 4.1. *Let $n \in \mathbb{N}$, $0 < \tau_1, \dots, \tau_n \in \mathbb{R}$, $\tau := \max\{\tau_1, \tau_2, \dots, \tau_n\}$, $A, B_1, \dots, B_n$ be $N \times N$-matrix valued functions, and suppose that the assumptions (H1)–(H5) are satisfied. If $A(t_0)$, $B_1(t_1)$, $\dots$, $B_n(t_n)$ are pairwise permutable, then the trivial solution of equation (4.1) is exponentially stable with respect to the ball $\Omega(\lambda)$ with*

$$\lambda = \frac{r \min\{1, \underline{\gamma}\}}{e^K \left(1 + \sum_{j=1}^{n} \|B_j(t_j)\| \frac{e^{\rho\tau_j} - 1}{\rho}\right)} \tag{4.2}$$

where $\underline{\gamma} = \min_{\substack{i=0,\dots,n \\ j=1,\dots,m_i}} \gamma_{ij}^{\beta_{ij}}$,

$$\begin{aligned} K = {} & \frac{2q_0\Gamma(\Theta_0 + 1)}{r_0^{\Theta_0+1}} + \frac{\delta_0}{\vartheta_0} + \sum_{j=1}^{m_0} \frac{\eta_{0j}}{\mu_{0j}(\gamma_{0j} - \rho)^{\beta_{0j}}} \\ & + \sum_{i=1}^{n} e^{\rho\tau_i} \left(\frac{2q_i\Gamma(\Theta_i + 1)}{r_i^{\Theta_i+1}} + \frac{\delta_i}{\vartheta_i} + \sum_{j=1}^{m_i} \frac{\eta_{ij}}{\mu_{ij}(\gamma_{ij} - \rho)^{\beta_{ij}}} \right). \end{aligned} \tag{4.3}$$

Proof. For simplicity in notation, we shall write $F(t)$ and $f(t)$ omitting most of their arguments. Let x be a solution of equation (4.1) on the interval $[0, T)$, $0 < T < \infty$ with the initial function $\varphi \in C([-\tau, 0], \mathbb{R}^N)$ satisfying

$$\|\varphi\| = \max_{t\in[-\tau,0]} \|\varphi(t)\| \le \lambda.$$

Let us rewrite equation (4.1) as follows:

$$\begin{aligned}\dot{x}(t) = A(t_0)x(t) + B_1(t_1)x(t-\tau_1) + \cdots + B_n(t_n)x(t-\tau_n) + (A(t) - A(t_0))x(t)\\ + (B_1(t) - B_1(t_1))x(t-\tau_1) + \cdots + (B_n(t) - B_n(t_n))x(t-\tau_n) + F(t) + f(t), \quad t \ge 0.\end{aligned}$$

By Theorem 2.1, x has the form

$$\begin{aligned}x(t) &= \mathcal{B}(t)\varphi(0) + \sum_{j=1}^{n} B_j(t_j)\int_0^{\tau_j} \mathcal{B}(t-s)\varphi(s-\tau_j)ds\\ &\quad + \int_0^t \mathcal{B}(t-s)\big((A(s) - A(t_0))x(s) + (B_1(s) - B_1(t_1))x(s-\tau_1)\\ &\quad + \cdots + (B_n(s) - B_n(t_n))x(s-\tau_n)\big)ds + \int_0^t \mathcal{B}(t-s)(F(s) + f(s))ds\end{aligned}$$

for $t \in [0, T]$, where

$$\mathcal{B}(t) = \mathrm{e}^{A(t_0)t} \sum_{\substack{\sum_{m=1}^n k_m\tau_m \le t\\ k_1,\dots,k_n\ge 0}} \frac{(t - \sum_{m=1}^n k_m\tau_m)^{\sum_{m=1}^n k_m}}{k_1!\dots k_n!} \prod_{m=1}^{n} \widetilde{B}_m^{k_m}$$

and $\widetilde{B}_m = B_m(t_m)\mathrm{e}^{-A(t_0)\tau_m}$ for each $m = 1, \dots, n$.

For now, let us assume that $r = \infty$. The case $r < \infty$ is postponed to the end of the proof.

Using the assumptions and Lemmas 2.2, 2.4, we obtain

$$\|\mathcal{B}(t)\| \le \|\mathrm{e}^{A(t_0)t}\|\mathrm{e}^{\alpha t} \le \mathrm{e}^{(\mu(A(t_0))+\alpha)t} = \mathrm{e}^{-\rho t}$$

for any $t \ge 0$. Hence

$$\begin{aligned}\mathrm{e}^{\rho t}\|x(t)\| &\le \|\varphi(0)\| + \sum_{j=1}^{n} \|B_j(t_j)\| \int_0^{\tau_j} \mathrm{e}^{\rho s}\|\varphi(s-\tau_j)\|ds\\ &\quad + \int_0^t \mathrm{e}^{\rho s}\big(\|A(s) - A(t_0)\|\|x(s)\| + \|B_1(s) - B_1(t_1)\|\|x(s-\tau_1)\|\\ &\quad + \cdots + \|B_n(s) - B_n(t_n)\|\|x(s-\tau_n)\|\big)ds + \int_0^t \mathrm{e}^{\rho s}(\|F(s)\| + \|f(s)\|)ds.\end{aligned}$$

Note that

$$\|I^{(\beta,\gamma)}h(t)\| \le \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1}\mathrm{e}^{-\gamma(t-s)}\|h(s)\|ds. \tag{4.4}$$

Therefore, denoting $u(t) := \mathrm{e}^{\rho t}\|x(t)\|$,

$$C := \|\varphi\| \left(1 + \sum_{j=1}^{n} \|B_j(t_j)\| \frac{\mathrm{e}^{\rho\tau_j} - 1}{\rho}\right) \tag{4.5}$$

and using assumptions (H1), (H4), (H5), we obtain

$$
\begin{aligned}
u(t) \le C &+ \int_0^t \Big(q_0 e^{-r_0|s-t_0|} |s-t_0|^{\Theta_0} u(s) \\
&+ \sum_{i=1}^{n} q_i e^{-r_i|s-t_i|} |s-t_i|^{\Theta_i} e^{\rho\tau_i} u(s-\tau_i) \Big) ds \\
&+ \int_0^t \Big(\delta_0 e^{-\vartheta_0 s} u(s) + \sum_{i=1}^{n} \delta_i e^{-\vartheta_i s} e^{\rho\tau_i} u(s-\tau_i) \Big) ds \\
&+ \int_0^t e^{\rho s} \Big(\sum_{j=1}^{m_0} \frac{\eta_{0j}}{\Gamma(\beta_{0j})} e^{-\mu_{0j}s} \int_0^s (s-\sigma)^{\beta_{0j}-1} e^{-\gamma_{0j}(s-\sigma)} e^{-\rho\sigma} u(\sigma) d\sigma \\
&\qquad + \sum_{i=1}^{n} \sum_{j=1}^{m_i} \frac{\eta_{ij}}{\Gamma(\beta_{ij})} e^{-\mu_{ij}s} \int_0^s (s-\sigma)^{\beta_{ij}-1} e^{-\gamma_{ij}(s-\sigma)} e^{-\rho(\sigma-\tau_i)} u(\sigma-\tau_i) d\sigma \Big) ds.
\end{aligned}
$$

Let us denote $\Psi(t)$ the right-hand side of the latter inequality. Clearly, it is a nondecreasing function satisfying $\Psi(0) = C$. To estimate the delayed terms, we use the inequality

$$
\begin{aligned}
u(s-\tau_i) &\le \max_{\sigma\in[0,s]} u(\sigma-\tau_i) \le \max_{\sigma\in[-\tau,s]} u(\sigma) \\
&= \max\Big\{ \max_{\sigma\in[-\tau,0]} u(\sigma), \max_{\sigma\in[0,s]} u(\sigma) \Big\} \\
&\le \max\Big\{ \max_{\sigma\in[-\tau,0]} e^{\rho\sigma}\|\varphi(\sigma)\|, \max_{\sigma\in[0,s]} \Psi(\sigma) \Big\} \le \max\{C, \Psi(s)\} = \Psi(s)
\end{aligned}
$$

for any $s \in [0,t]$ and each $i = 1, \dots, n$. So we obtain

$$
\begin{aligned}
& e^{\rho s} \int_0^s (s-\sigma)^{\beta_{0j}-1} e^{-\gamma_{0j}(s-\sigma)} e^{-\rho\sigma} u(\sigma) d\sigma \\
&\quad \le \Psi(s) \int_0^s \sigma^{\beta_{0j}-1} e^{-(\gamma_{0j}-\rho)\sigma} d\sigma \le \Psi(s) \int_0^\infty \sigma^{\beta_{0j}-1} e^{-(\gamma_{0j}-\rho)\sigma} d\sigma \\
&\quad = \frac{\Psi(s)\Gamma(\beta_{0j})}{(\gamma_{0j}-\rho)^{\beta_{0j}}}
\end{aligned} \tag{4.6}
$$

for all $s \in [0,t]$ and each $j = 1, \dots, m_0$. Analogously,

$$
e^{\rho s} \int_0^s (s-\sigma)^{\beta_{ij}-1} e^{-\gamma_{ij}(s-\sigma)} e^{-\rho(\sigma-\tau_i)} u(\sigma-\tau_i) d\sigma \le \frac{\Psi(s) e^{\rho\tau_i} \Gamma(\beta_{ij})}{(\gamma_{ij}-\rho)^{\beta_{ij}}} \tag{4.7}
$$

for all $s \in [0,t]$ and each $i = 1, \dots, n$, $j = 1, \dots, m_i$. Therefore, we arrive at

$$
\Psi(t) \le C + \int_0^t b(s)\Psi(s) ds, \quad t \in [0,T] \tag{4.8}
$$

where

$$
\begin{aligned}
b(s) &= q_0 e^{-r_0|s-t_0|} |s-t_0|^{\Theta_0} + \delta_0 e^{-\vartheta_0 s} + \sum_{j=1}^{m_0} \frac{\eta_{0j} e^{-\mu_{0j}s}}{(\gamma_{0j}-\rho)^{\beta_{0j}}} \\
&\quad + \sum_{i=1}^{n} e^{\rho\tau_i} \Big(q_i e^{-r_i|s-t_i|} |s-t_i|^{\Theta_i} + \delta_i e^{-\vartheta_i s} + \sum_{j=1}^{m_i} \frac{\eta_{ij} e^{-\mu_{ij}s}}{(\gamma_{ij}-\rho)^{\beta_{ij}}} \Big).
\end{aligned} \tag{4.9}
$$

Note that

$$\begin{aligned}\int_0^t e^{-r_i|s-t_i|}|s-t_i|^{\Theta_i}ds &\le \int_0^\infty e^{-r_i|s-t_i|}|s-t_i|^{\Theta_i}ds \\ &= \int_{-t_i}^0 e^{-r_i|s|}|s|^{\Theta_i}ds + \int_0^\infty e^{-r_is}s^{\Theta_i}ds \\ &\le 2\int_0^\infty e^{-r_is}s^{\Theta_i}ds = \frac{2\Gamma(\Theta_i+1)}{r_i^{\Theta_i+1}}\end{aligned}$$

for each $i = 0, \dots, n$. So, it holds

$$\int_0^t b(s)ds \le \int_0^\infty b(s)ds \le K.$$

Applying the Gronwall's inequality to (4.8) then gives

$$\Psi(t) \le C\exp\left\{\int_0^t b(s)ds\right\} \le Ce^K < \infty$$

for any $t \ge 0$. That means

$$\|x(t)\| = e^{-\rho t}u(t) \le e^{-\rho t}\Psi(t) \le Ce^K e^{-\rho t} \quad \forall t \in [0, T). \tag{4.10}$$

Since the right-hand side is independent of T, the estimation holds for any $t \ge 0$.

The condition (4.2) on λ enables to apply estimations of $\|F(t)\|$ and $\|f(t)\|$ during the proof. If condition (4.2) holds, from (4.10), one can see that $\|x(t)\| \le r$ for all $t \in [0, T)$. Clearly, it is true also for $t \in [-\tau, 0]$. Next, from (4.4) and (4.10), we get

$$\|I^{(\beta,\gamma)}x(t)\| \le \frac{Ce^K}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}e^{-\gamma(t-s)}ds \le \frac{Ce^K}{\gamma^\beta}. \tag{4.11}$$

The same holds with $x(t-\tau_i)$ for any $i = 1, \dots, n$ instead of $x(t)$. So again, we can apply the estimation of $\|f(t)\|$ due to (4.2).

Finally, if $r < \infty$, the statement follows from the previous case using the Urysohn's lemma [1, Lemma 10.2]. □

We would like to emphasize that in the above theorem, the commutativity of matrix functions A, B_1, ..., B_n at general t is not required.

5 Exponential stability for power nonlinearities on right-hand side

Here we investigate the case of more general functions F and f on the right-hand side of equation (4.1). In particular, we consider the modified assumptions:

(H4') for a constant $0 < r \le \infty$ there are $\vartheta_i > 0$, $\delta_i, \tilde{\delta}_i, \tilde{\vartheta}_i \ge 0$ and $\omega_i > 1$ for $i = 0, \dots, n$ such that

$$\|F(t, u_0, \dots, u_n)\| \le \sum_{i=0}^n \left(\delta_i e^{-\vartheta_i t}\|u_i\| + \tilde{\delta}_i e^{-\tilde{\vartheta}_i t}\|u_i\|^{\omega_i}\right)$$

for all $t \ge 0$ and $u_i \in \Omega(r)$, $i = 0, \dots, n$;

(H5′) there are $m_i \in \mathbb{N}$, $\mu_{ij} > 0$, $\eta_{ij}, \tilde{\eta}_{ij}, \tilde{\mu}_{ij} \geq 0$ and $\omega_{ij} > 1$ for $i = 0, \dots, n$, $j = 1, \dots, m_i$ such that

$$\|f(t, v_{01}, \dots, v_{0m_0}, v_{11}, \dots, v_{nm_n})\| \leq \sum_{i=0}^{n} \sum_{j=1}^{m_i} \left(\eta_{ij} e^{-\mu_{ij} t} \|v_{ij}\| + \tilde{\eta}_{ij} e^{-\tilde{\mu}_{ij} t} \|v_{ij}\|^{\omega_{ij}}\right)$$

for all $t \geq 0$ and $v_{ij} \in \Omega(r)$, $i = 0, \dots, n$, $j = 1, \dots, m_i$.

We will assume that at least one of $\tilde{\delta}_i$, $\tilde{\eta}_{ij}$, $i = 0, \dots, n$, $j = 1, \dots, m_i$ is nonzero, so that this is not the case of Theorem 4.1.

Theorem 5.1. *Let $n \in \mathbb{N}$, $0 < \tau_1, \dots, \tau_n \in \mathbb{R}$, $\tau := \max\{\tau_1, \tau_2, \dots, \tau_n\}$, $A, B_1, \dots, B_n$ be $N \times N$-matrix valued functions, and suppose that the assumptions (H1), (H2), (H3), (H4′) and (H5′) are satisfied. If $A(t_0)$, $B_1(t_1)$, …, $B_n(t_n)$ are pairwise permutable, then the trivial solution of equation (4.1) is exponentially stable with respect to the ball $\Omega(\lambda)$ with*

$$\lambda < \min\left\{\lambda_1, \lambda_2, r \min\{1, \underline{\gamma}\}\right\} \tag{5.1}$$

where

$$\lambda_i = \frac{C_i}{1 + \sum_{j=1}^{n} \|B_j(t_j)\| \frac{e^{\rho\tau_j} - 1}{\rho}}, \quad i = 1, 2,$$

C_1 is the root of the equation

$$\sum_{i=0}^{n} \left(C_1^{\omega_i - 1} K_i + \sum_{j=1}^{m_i} C_1^{\omega_{ij} - 1} K_{ij} \right) = \frac{1}{(\omega - 1) e^{(\omega - 1)K}} \tag{5.2}$$

and C_2 is the smallest positive root of the equation

$$\frac{C_2 e^K}{\left[1 - (\omega - 1) e^{(\omega-1)K} \left(\sum_{i=0}^{n} \left(C_2^{\omega_i - 1} K_i + \sum_{j=1}^{m_i} C_2^{\omega_{ij} - 1} K_{ij}\right)\right)\right]^{\frac{1}{\omega - 1}}} = r \min\{1, \underline{\gamma}\}, \tag{5.3}$$

where K is defined by (4.3), $\omega = \max_{\substack{i=0,\dots,n \\ j=1,\dots,m_i}} \{\omega_i, \omega_{ij}\} > 1$, $\underline{\gamma} = \min_{\substack{i=0,\dots,n \\ j=1,\dots,m_i}} \gamma_{ij}^{\beta_{ij}}$,

$$K_0 = \frac{\tilde{\delta}_0}{\tilde{\vartheta}_0 + \rho(\omega_0 - 1)}, \qquad K_i = \frac{\tilde{\delta}_i e^{\rho\omega_i\tau_i}}{\tilde{\vartheta}_i + \rho(\omega_i - 1)}, \quad i = 1, \dots, n,$$

$$K_{0j} = \frac{\tilde{\eta}_{0j}}{(\gamma_{0j} - \rho)^{\beta_{0j}\omega_{0j}} (\tilde{\mu}_{0j} + \rho(\omega_{0j} - 1))}, \quad j = 1, \dots, m_0,$$

$$K_{ij} = \frac{\tilde{\eta}_{ij} e^{\rho\omega_{ij}\tau_i}}{(\gamma_{ij} - \rho)^{\beta_{ij}\omega_{ij}} (\tilde{\mu}_{ij} + \rho(\omega_{ij} - 1))}, \quad i = 1, \dots, n, \ j = 1, \dots, m_i,$$

Proof. First, we assume that $r = \infty$. Following the proof of Theorem 4.1, we arrive at

$$\begin{aligned} u(t) &= e^{\rho t}\|x(t)\| \\ &\leq C + \int_0^t \left(q_0 e^{-r_0|s - t_0|} |s - t_0|^{\Theta_0} u(s) + \sum_{i=1}^{n} q_i e^{-r_i|s - t_i|} |s - t_i|^{\Theta_i} e^{\rho\tau_i} u(s - \tau_i) \right) ds \\ &\quad + \int_0^t \Bigg(\delta_0 e^{-\vartheta_0 s} u(s) + \tilde{\delta}_0 e^{-\tilde{\vartheta}_0 s} e^{-\rho(\omega_0 - 1)s} u^{\omega_0}(s) \\ &\qquad + \sum_{i=1}^{n} \left(\delta_i e^{-\vartheta_i s} e^{\rho\tau_i} u(s - \tau_i) + \tilde{\delta}_i e^{-\tilde{\vartheta}_i s} e^{-\rho(\omega_i - 1)s + \rho\omega_i\tau_i} u^{\omega_i}(s - \tau_i) \right) \Bigg) ds \end{aligned}$$

$$+\int_0^t \mathrm{e}^{\rho s}\Bigg(\sum_{j=1}^{m_0}\bigg(\frac{\eta_{0j}}{\Gamma(\beta_{0j})}\mathrm{e}^{-\mu_{0j}s}\int_0^s (s-\sigma)^{\beta_{0j}-1}\mathrm{e}^{-\gamma_{0j}(s-\sigma)}\mathrm{e}^{-\rho\sigma}u(\sigma)d\sigma$$

$$+\frac{\tilde{\eta}_{0j}}{\Gamma(\beta_{0j})^{\omega_{0j}}}\mathrm{e}^{-\tilde{\mu}_{0j}s}\left(\int_0^s (s-\sigma)^{\beta_{0j}-1}\mathrm{e}^{-\gamma_{0j}(s-\sigma)}\mathrm{e}^{-\rho\sigma}u(\sigma)d\sigma\right)^{\omega_{0j}}\bigg)$$

$$+\sum_{i=1}^{n}\sum_{j=1}^{m_i}\bigg(\frac{\eta_{ij}}{\Gamma(\beta_{ij})}\mathrm{e}^{-\mu_{ij}s}\int_0^s (s-\sigma)^{\beta_{ij}-1}\mathrm{e}^{-\gamma_{ij}(s-\sigma)}\mathrm{e}^{-\rho(\sigma-\tau_i)}u(\sigma-\tau_i)d\sigma$$

$$+\frac{\tilde{\eta}_{ij}}{\Gamma(\beta_{ij})^{\omega_{ij}}}\mathrm{e}^{-\tilde{\mu}_{ij}s}\left(\int_0^s (s-\sigma)^{\beta_{ij}-1}\mathrm{e}^{-\gamma_{ij}(s-\sigma)}\mathrm{e}^{-\rho(\sigma-\tau_i)}u(\sigma-\tau_i)d\sigma\right)^{\omega_{ij}}\bigg)\Bigg)ds$$

where C is given by (4.5) and $\varphi \in C([-\tau,0],\mathbb{R}^N)$ is such that $\|\varphi\| \le \lambda$. Let us denote $\Psi(t)$ the right-hand side of the above inequality. Then $\Psi(t)$ is a nondecreasing function satisfying $\Psi(0)=C$. Analogously to (4.6) and (4.7), we derive

$$\mathrm{e}^{\rho s}\left(\int_0^s (s-\sigma)^{\beta_{0j}-1}\mathrm{e}^{-\gamma_{0j}(s-\sigma)}\mathrm{e}^{-\rho\sigma}u(\sigma)d\sigma\right)^{\omega_{0j}} \le \frac{\Psi^{\omega_{0j}}(s)\mathrm{e}^{\rho(1-\omega_{0j})s}\Gamma(\beta_{0j})^{\omega_{0j}}}{(\gamma_{0j}-\rho)^{\beta_{0j}\omega_{0j}}}$$

for all $s \in [0,t]$ and each $j = 1,\dots,m_0$, and

$$\mathrm{e}^{\rho s}\left(\int_0^s (s-\sigma)^{\beta_{ij}-1}\mathrm{e}^{-\gamma_{ij}(s-\sigma)}\mathrm{e}^{-\rho(\sigma-\tau_i)}u(\sigma-\tau_i)d\sigma\right)^{\omega_{ij}} \le \frac{\Psi^{\omega_{ij}}(s)\mathrm{e}^{\rho(1-\omega_{ij})s+\rho\omega_{ij}\tau_i}\Gamma(\beta_{ij})^{\omega_{ij}}}{(\gamma_{ij}-\rho)^{\beta_{ij}\omega_{ij}}}$$

for all $s \in [0,t]$ and each $i = 1,\dots,n$, $j = 1,\dots,m_i$. Therefore, we have

$$\Psi(t) \le C + \int_0^t b(s)\Psi(s)ds + \sum_{i=0}^{n}\int_0^t b_i(s)\Psi^{\omega_i}(s)ds + \sum_{i=1}^{n}\sum_{j=1}^{m_i}\int_0^t b_{ij}(s)\Psi^{\omega_{ij}}(s)ds \tag{5.4}$$

where $b(s)$ is given by (4.9),

$$\begin{aligned}
b_0(s) &= \tilde{\delta}_0\mathrm{e}^{-(\tilde{\vartheta}_0+\rho(\omega_0-1))s}, && \\
b_i(s) &= \tilde{\delta}_i\mathrm{e}^{\rho\omega_i\tau_i-(\tilde{\vartheta}_i+\rho(\omega_i-1))s}, && i=1,\dots,n,\\
b_{0j} &= \frac{\tilde{\eta}_{0j}\mathrm{e}^{-(\tilde{\mu}_{0j}+\rho(\omega_{0j}-1))s}}{(\gamma_{0j}-\rho)^{\beta_{0j}\omega_{0j}}}, && j=1,\dots,m_0,\\
b_{ij} &= \frac{\tilde{\eta}_{ij}\mathrm{e}^{\rho\omega_{ij}\tau_i-(\tilde{\mu}_{ij}+\rho(\omega_{ij}-1))s}}{(\gamma_{ij}-\rho)^{\beta_{ij}\omega_{ij}}}, && i=1,\dots,n,\ j=1,\dots,m_i.
\end{aligned}$$

Note that $K_i = \int_0^\infty b_i(s)ds$ and $K_{ij} = \int_0^\infty b_{ij}(s)ds$ for each $i = 0,\dots,n$, $j = 1,\dots,m_i$. Now, from assumption (5.1) on λ, we have $C < C_1$. Since the left-hand side of (5.2) is increasing in C_1, it follows

$$(\omega-1)\mathrm{e}^{(\omega-1)K}\sum_{i=0}^{n}\left(C^{\omega_i-1}K_i + \sum_{j=1}^{m_i}C^{\omega_{ij}-1}K_{ij}\right) < 1,$$

and Lemma 2.5 can be applied on inequality (5.4) to obtain

$$\Psi(t) \le C \exp\left(\int_0^t b(s)ds\right)$$
$$\times\left(1-(\omega-1)\int_0^t \sum_{i=0}^{n}\left(C^{\omega_i-1}b_i(s)+\sum_{j=1}^{m_i}C^{\omega_{ij}-1}b_{ij}(s)\right)\exp\left((\omega-1)\int_0^s b(\sigma)d\sigma\right)ds\right)^{\frac{1}{1-\omega}}$$
$$\le \frac{C\mathrm{e}^K}{\left[1-(\omega-1)\mathrm{e}^{(\omega-1)K}\left(\sum_{i=0}^{n}\left(C^{\omega_i-1}K_i+\sum_{j=1}^{m_i}C^{\omega_{ij}-1}K_{ij}\right)\right)\right]^{\frac{1}{\omega-1}}} =: \widetilde{K}$$

for all $t \in [0,T)$. Since the $\widetilde{K}$ is independent of T, $\Psi(t) \le \widetilde{K}$ for all $t \ge 0$. Hence $\|x(t)\| \le \widetilde{K}\mathrm{e}^{-\rho t}$.

Again, by (5.1), one can see that $\lambda < \lambda_2$, i.e., $C < C_2$. Let us denote $g(C_2)$ the left-hand side of (5.3). Clearly, it is a continuous function satisfying $g(0)=0$ and $g(C_2)=r\min\{1,\underline{\gamma}\}$. Moreover, we know that $g(\xi)\in[0,r\min\{1,\underline{\gamma}\})$ for $\xi\in[0,C_2)$. Thus

$$\widetilde{K} = g(C) < r\min\{1,\underline{\gamma}\} \le r.$$

From (5.1), also $\|\varphi\| < r$. So, the estimation of $\|F(t)\|$ could be applied. Similarly to (4.11), we have

$$\|I^{(\beta_{0j},\gamma_{0j})}x(t)\| \le \frac{\widetilde{K}}{\gamma_{0j}^{\beta_{0j}}} < \frac{r\underline{\gamma}}{\gamma_{0j}^{\beta_{0j}}} \le r, \quad t \ge 0$$

for each $j = 1,\dots,m_0$, and

$$\|I^{(\beta_{ij},\gamma_{ij})}x(t-\tau_i)\| \le \begin{cases} \frac{\|\varphi\|}{\gamma_{ij}^{\beta_{ij}}} \le \frac{r\min\{1,\underline{\gamma}\}}{\gamma_{ij}^{\beta_{ij}}} \le r, & t\in[0,\tau_i], \\ \frac{\widetilde{K}}{\gamma_{ij}^{\beta_{ij}}} < \frac{r\underline{\gamma}}{\gamma_{ij}^{\beta_{ij}}} \le r, & t > \tau_i \end{cases}$$

for each $i = 1,\dots,n$, $j = 1,\dots,m_i$. So, also the estimation of $\|f(t)\|$ was allowed.

Finally, the case $r < \infty$ can be proved using Urysohn's lemma as in Theorem 4.1. □

If equation (5.3) does not have a positive root, we set $C_2 = \infty$.

6 Illustrative example

Consider the following system of DDEs with one delay

$$\begin{aligned} \dot{x}(t) &= -x(t) + 3t\mathrm{e}^{-t}y(t) + \frac{x(t-1)}{2} \\ \dot{y}(t) &= -y(t) + \left(I^{(\frac{1}{2},\frac{1}{2})}x(t-1)\right)^2 \end{aligned} \tag{6.1}$$

for $t \ge 0$. In this case

$$A(t) = \begin{pmatrix} -1 & 3t\mathrm{e}^{-t} \\ 0 & -1 \end{pmatrix}, \quad B(t) = B = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 0 \end{pmatrix},$$

$n = 1$, $\tau = 1$. Clearly, $A(0)B = BA(0) = \left(\begin{smallmatrix} -1/2 & 0 \\ 0 & 0 \end{smallmatrix}\right)$. Considering the norms $\|v\| = \|v\|_1 = |v_1| + |v_2|$ for $v = (v_1, v_2) \in \mathbb{R}^2$ and $\|D\| = \|D\|_1 = \max_{j=1,2}(|d_{1j}| + |d_{2j}|)$ for $D = \left(\begin{smallmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{smallmatrix}\right)$, we get the logarithmic norm

$$\mu(D) = \mu_1(D) = \max_{j=1,2} \left(d_{jj} + \sum_{\substack{i=1 \\ i \neq j}}^{2} |d_{ij}| \right).$$

Moreover, $\|A(t) - A(0)\| = 3te^{-t}$, $\|B(t) - B(0)\| = 0$ for any $t \geq 0$. In the notation of Section 5, we have $t_1 = 0$, $q_0 = 3$, $r_0 = 1$, $\Theta_0 = 1$, $\beta_{11} = \gamma_{11} = \frac{1}{2}$ and set $q_1 = 0$, $r_1 = 1$, $\Theta_1 = 0$, $\beta_{01} = \gamma_{01} = 1$. Since $F(t, u_0, u_1) = 0$, we can take $\delta_i = \tilde{\delta}_i = 0$, $\vartheta_i = \tilde{\vartheta}_i = 1$ and $\omega_i = 2$ for $i = 1, 2$. Next, $f(t, v_{01}, v_{11}) = (0, (v_{11})_1^2)$ where $(v_{11})_1$ is the first coordinate of v_{11}. Hence, $\|f(t, v_{01}, v_{11})\| = (v_{11})_1^2 \leq \|v_{11}\|^2$ and we take $\eta_{i1} = 0$, $\mu_{i1} = 1$, $\tilde{\mu}_{i1} = 0$, $\omega_{i1} = 2$ for $i = 1, 2$ and $\tilde{\eta}_{01} = 0$, $\tilde{\eta}_{11} = 1$, $r = \infty$. Furthermore, $\|Be^{-A(0)}\| = e\|B\| = \frac{e}{2} \leq \alpha_1 e^{\alpha_1}$ if $\alpha_1 \geq 0.68508$. Taking $\alpha = \alpha_1 = 0.686$, condition (H3) has the form $\gamma_{11} = \frac{1}{2} > \rho = -\mu(A(0)) - \alpha = 1 - 0.686 = 0.314 > 0$.

We want to apply Theorem 5.1. So, we calculate the constants, $K_0 = K_1 = K_{01} = 0$, $K_{11} \doteq 32.084$ and $K = 6$. Consequently, C_1 given by (5.2) is $C_1 \doteq 7.726 \cdot 10^{-5}$ and $\lambda_1 \doteq 4.867 \cdot 10^{-5}$. Since the left-hand side of (5.3) is bounded and the right-hand side is ∞, we set $C_2 = \infty$. From Theorem 5.1 we obtain the following result.

Proposition 6.1. *The trivial solution of* (6.1) *is exponentially stable with respect to the ball* $\Omega(4.867 \cdot 10^{-5})$*, i.e., any solution of* (6.1) *satisfying* $(x(t), y(t)) = (\varphi_1(t), \varphi_2(t))$ *for* $t \in [-1, 0]$ *tends exponentially to zero provided that* $\|\varphi\| = \max_{t \in [-\tau, 0]}(|\varphi_1(t)| + |\varphi_2(t)|) \leq 4.867 \cdot 10^{-5}$.

Acknowledgements

The authors thank the anonymous reviewer for careful reading of the manuscript. M. Medveď was supported by the Grant VEGA 1/0078/17 and by the Slovak Research and Development Agency under the contract No. APVV-14-0378. M. Pospíšil was supported by the Grants VEGA 1/0078/17, VEGA-SAV 2/0153/16 and by the Slovak Research and Development Agency under the contract No. APVV-14-0378.

References

[1] G. E. Bredon, *Topology and geometry*, Graduate Texts in Mathematics, Springer-Verlag, New York, 1993. https://doi.org/10.1007/978-1-4757-6848-0; MR1224675; Zbl 0791.55001

[2] E. Brestovanská, M. Medveď, Exponential stability of solutions of a second order system of integrodifferential equations with the Caputo–Fabrizio fractional derivatives, *Progr. Fract. Differ. Appl.* **2**(2016), No. 3, 187–192. https://doi.org/10.18576/pfda/020303

[3] E. Brestovanská, M. Medveď, Exponential stability of solutions of nonlinear fractionally perturbed ordinary differential equations, *Electron. J. Differ. Equ.* **2017**, No. 280, 1–17. MR3747998

[4] M. Chen, W. Deng, Discretized fractional substantial calculus, *ESAIM Math. Model. Numer. Anal.* **49**(2015), No. 2, 373–394. https://doi.org/10.1051/m2an/2014037; MR3342210; Zbl 1314.26007

[5] W. A. Coppel, *Stability and asymptotic behavior of differential equations*, D. C. Heath, Boston, USA, 1965. MR0190463; Zbl 0154.09301

[6] R. Friedrich, F. Jenko, A. Baule, S. Eule, Anomalous diffusion of inertial, weakly damped particles, *Phys. Rev. Lett.* **96**(2006), 230601. https://doi.org/10.1103/PhysRevLett.96.230601

[7] D. Y. Khusainov, G. V. Shuklin, Linear autonomous time-delay system with permutation matrices solving, *Stud. Univ. Žilina Math. Ser.* **17**(2003), 101–108. MR2064983; Zbl 1064.34042

[8] G.-D. Hu, M. Liu, The weighted logarithmic matrix norm and bounds of the matrix exponential, *Linear Alg. Appl.* **390**(2004), 145–154. https://doi.org/10.1016/j.laa.2004.04.015; MR2083412; Zbl 1060.15024

[9] C. P. F. Li, R. Zhang, A survey on the stability of fractional differential equations, *Eur. Phys. J., Special Topics* **193**(2011), 27–47. https://doi.org/10.1140/epjst/e2011-01379-1

[10] Y. Louartassi, E. El Mazoudi, N. Elalami, A new generalization of lemma Gronwall–Bellman, *Appl. Math. Sci. (Ruse)* **6**(2012), No. 13, 621–628. MR2880885; Zbl 1248.26034

[11] D. Matignon, Stability result on fractional differential equations with applications to control processing, in: *IMACS-SMC Proceedings*, Lille, France, 1996, pp. 963–968.

[12] M. Medveď, E. Brestovanská, New conditions for the exponential stability of fractionally perturbed ODEs, *Electron. J. Qual. Theory Differ. Equ.* **2018**, No. 84, 1–14. https://doi.org/10.14232/ejqtde.2018.1.84; MR3863878; Zbl 07011131

[13] M. Medveď, E. Brestovanská, Corrigendum to the paper: New conditions for the exponential stability of fractionally perturbed ODEs, EJQTDE, No. 84 (2018), 1–14, *Electron. J. Qual. Theory Differ. Equ.* **2018**, No. 102, 1–2. https://doi.org/10.14232/ejqtde.2018.1.102; MR3896826

[14] M. Medveď, M. Pospíšil, Sufficient conditions for the asymptotic stability of nonlinear multidelay differential equations with linear parts defined by pairwise permutable matrices, *Nonlinear Anal.* **75**(2012), 3348–3363. https://doi.org/10.1016/j.na.2011.12.031; MR2891173; Zbl 1244.34096

[15] M. Medveď, M. Pospíšil, L. Škripková, On exponential stability of nonlinear fractional multidelay integro-differential equations defined by pairwise permutable matrices, *Appl. Math. Comput.* **227**(2014), 456–468. https://doi.org/10.1016/j.amc.2013.11.012; MR3146332; Zbl 1364.34115

[16] I. Petráš, *Fractional-order nonlinear systems, modeling, analysis and simulations*, Higher-Education Press, Springer, 2011. https://doi.org/10.1007/978-3-642-18101-6

[17] M. Pospíšil, Representation and stability of solutions of systems of functional differential equations with multiple delays, *Electron. J. Qual. Theory Differ. Equ.* **2012**, No. 54, 1–30. https://doi.org/10.14232/ejqtde.2012.1.54; MR2959044; Zbl 1340.34271

[18] M. Pospíšil, F. Jaroš, On the representation of solutions of delayed differential equations via Laplace transform, *Electron. J. Qual. Theory Differ. Equ.* **2016**, No. 117, 1–13. https://doi.org/10.14232/ejqtde.2016.1.117; MR3582910; Zbl 1399.34184