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Lyapunov regularity and triangularization for unbounded sequences

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Abstract. The notion of Lyapunov regularity for a dynamics with discrete time defined by a *bounded* sequence of matrices can be characterized in many ways, highlighting different aspects of this important property introduced by Lyapunov. In strong contrast to the case of bounded sequences, not all these properties are equivalent to regularity for *unbounded* sequences. We first show that certain properties remain equivalent for unbounded sequences of matrices. We then show that unlike for bounded sequences and, more generally, tempered sequences, certain properties related to the existence of limits for the Lyapunov exponents on the diagonal are no longer equivalent to regularity for unbounded sequences.

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1 Introduction

1.1 Main theme

In this paper we consider the notion of *Lyapunov regularity* for a dynamics with discrete time defined by a sequence of matrices that may be unbounded. More precisely, we consider a sequence of invertible $q \times q$ matrices $(A_n)_{n\in\mathbb{N}}$ with real entries and the associated dynamics

$$x_{n+1} = A_n x_n, \quad \text{for } n \in \mathbb{N}, \tag{1.1}$$

on $\mathbb{R}^q$. Let

$$\mathcal{A}_n = \begin{cases} A_{n-1} \cdots A_1 & \text{if } n > 1, \\ \mathrm{Id} & \text{if } n = 1. \end{cases} \tag{1.2}$$

Assuming that the Lyapunov exponent

$$\lambda(v) = \limsup_{n\to\infty} \frac{1}{n} \log\|\mathcal{A}_n v\|$$

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is finite for all nonzero vectors $v \in \mathbb{R}^q$, the sequence $(A_n)_{n\in\mathbb{N}}$ is said to be *Lyapunov regular* if

$$\liminf_{n\to\infty} \frac{1}{n} \log|\det \mathcal{A}_n| = \sum_{i=1}^{q} \lambda(v_i) \tag{1.3}$$

for some basis $v_1, \ldots, v_q$ for $\mathbb{R}^q$. We emphasize that the sequence need not be bounded or even tempered. We recall that a sequence $(A_n)_{n\in\mathbb{N}}$ is said to be *tempered* if

$$\limsup_{n\to\infty} \frac{1}{n} \log \|A_n\| \le 0, \tag{1.4}$$

where as usual

$$\|A_n\| = \sup_{v\in\mathbb{R}^q\setminus\{0\}} \frac{\|A_n v\|}{\|v\|}.$$

Our main aim is to show that whereas various characterizations of Lyapunov regularity for bounded sequences extend to unbounded sequences, various others related to the triangularization of the sequence do not. We recall that to make a *triangularization* of a sequence of $q \times q$ matrices $(A_n)_{n\in\mathbb{N}}$ corresponds to find a sequence of invertible $q \times q$ matrices $(V_n)_{n\in\mathbb{N}}$ satisfying

$$\lim_{n\to\infty} \frac{1}{n} \log\|V_n\| = \lim_{n\to\infty} \frac{1}{n} \log\|V_n^{-1}\| = 0 \tag{1.5}$$

such that the matrices

$$B_n = V_{n+1}^{-1} A_n V_n$$

are upper-triangular for each $n \in \mathbb{N}$. Any sequence $(V_n)_{n\in\mathbb{N}}$ satisfying (1.5) is called a *Lyapunov coordinate change* (see Section 3 for some of its properties). In the latter case of the triangularization of a sequence of matrices, we provide a gradation of successively weaker properties that are all equivalent for bounded sequences, by providing explicit examples of sequences of matrices for which each two of these successively weaker properties are not both satisfied (thus showing that the properties are not equivalent). This recommends caution when using Lyapunov regularity in the study of the stability of a nonlinear dynamics obtained from perturbing a linear dynamics defined by an unbounded sequence since not all the usual characterizations of regularity remain equivalent for unbounded sequences.

1.2 Lyapunov regularity

Before proceeding, we describe briefly why the theory of Lyapunov regularity plays an important role in the stability theory of differential equations and dynamical systems (we refer the reader to [5] for a detailed description). It is easy to verify (for example using the variation of parameters formula, for continuous time, or a corresponding formula for discrete time) that the *uniform* exponential stability of a linear dynamics as in (1.1) persists under sufficiently small nonlinear perturbations, that is, perturbations of the form

$$x_{n+1} = A_n x_n + f_n(x_n)$$

with the maps f_n sufficiently small in some appropriate sense. In general this is no longer true when the exponential stability is not uniform, that is, when the time that it takes for the iteration of the dynamics to reach a given neighborhood of zero with exponential decay depends on the initial time. The notion of Lyapunov regularity was introduced by Lyapunov [12] and then studied by many others (see for example the books [1,5,9,11] and the references therein) as a means to give quantitative conditions, also involving the Lyapunov exponents, under which the *nonuniform* exponential stability of a linear dynamics persists under sufficiently small perturbations. This amounts to introduce certain regularity coefficients such that when they are sufficiently small the exponential stability persists. For example, the *Lyapunov regularity coefficient* of a sequence of $q \times q$ matrices $A = (A_n)_{n\in\mathbb{N}}$ is the number

$$\sigma(A) = \min \sum_{i=1}^{q} \lambda(v_i) - \liminf_{n\to\infty} \frac{1}{n} \log|\det \mathcal{A}_n|,$$

where the minimum is taken over all bases $v_1, \ldots, v_q$ for $\mathbb{R}^q$. One can show that the sequence A is Lyapunov regular if and only if $\sigma(A) = 0$ (see [4] for a detailed exposition of the theory).

A major breakthrough in the theory of Lyapunov regularity occurred when Oseledets [13] showed that in the context of ergodic theory any regularity coefficient vanishes almost everywhere (more precisely, it vanishes for almost all trajectories of a measure-preserving flow under a certain integrability assumption). This eventually led to an exponential development of the area, initially with seminal work of Pesin [14,15]. We refer the reader to the book [6] for a sufficiently detailed description of the theory, nowadays referred to as nonuniform hyperbolicity theory or Pesin theory. The first nontrivial consequence of the persistence of nonuniform exponential stability can be considered the construction of stable and unstable invariant manifolds by Pesin in [14]. It turns out that the notion of nonuniform hyperbolicity can be deduced from the existence of nonzero Lyapunov exponents using the regularity coefficient to show that the nonuniformity can be made arbitrarily small along almost all trajectories (since the regularity coefficient vanishes almost everywhere). From this point of view, Lyapunov regularity can be considered a principal technical device in the study of nonuniform hyperbolicity. This specific topic is not pursued in our paper and so we refrain from introducing the notions and results explicitly, referring instead the reader to the former references.

In our paper, Lyapunov regularity is the main topic from beginning to end. In particular, we consider various properties that are equivalent to Lyapunov regularity for bounded sequences and we establish their equivalence for arbitrary sequences (see Theorem 3.3). For example, we show that for a sequence of invertible $q \times q$ matrices $(A_n)_{n\in\mathbb{N}}$ whose Lyapunov exponent takes only finite values on $\mathbb{R}^q \setminus \{0\}$, the following properties are equivalent:

1. $(A_n)_{n\in\mathbb{N}}$ is Lyapunov regular;

2. there exist a Lyapunov coordinate change $(V_n)_{n\in\mathbb{N}}$ (see (1.5)) and a diagonal $q \times q$ matrix D such that
$$V_{n+1}^{-1} A_n V_n = D \quad \text{for all } n \in \mathbb{N};$$

3. there exists a basis $v_1, \ldots, v_q$ for $\mathbb{R}^q$ such that the limit
$$\lim_{n\to\infty} \frac{1}{n} \log\|\mathcal{A}_n v_i\|$$
exists for $i = 1, \ldots, q$ and
$$\lim_{n\to\infty} \frac{1}{n} \log \angle\big(\mathcal{A}_n v_j, \operatorname{span}\{\mathcal{A}_n v_{j+1}, \ldots, \mathcal{A}_n v_q\}\big) = 0 \tag{1.6}$$
for $j = 1, \ldots, q-1$.

We recall that the angle $\angle(v, E)$ between a vector $v \in \mathbb{R}^q$ and a subspace $E \subset \mathbb{R}^q$ is defined by

$$\angle(v, E) = \inf\{\angle(v, w) : w \in E\} \in [0, \pi/2].$$

Property 2 says that the sequence $(A_n)_{n\in\mathbb{N}}$ can be transformed into a constant diagonal sequence via a Lyapunov coordinate change. Property 3 says that the values $\lambda(v_i)$ of the Lyapunov exponent are limits (which in fact implies that $\lambda(v)$ is a limit for any v), while (1.6) implies that any two sequences $\mathcal{A}_n v_i$ and $\mathcal{A}_n v_j$ with $i \neq j$ approach at most with subexponential speed when $n \to \infty$. To a certain extent, the proofs of the equivalence between these and other properties are obtained by modifying existing arguments for bounded sequences, although we give a clean streamlined argument. At the end of Section 3 we provide a detailed list of references for the existing proofs of the relations between various properties that are equivalent to Lyapunov regularity for bounded sequences (either for discrete or continuous time).

1.3 Triangular reduction

In the second part of the paper we discuss how the reduction of a sequence of matrices to a sequence of upper-triangular matrices via a Lyapunov coordinate change relates to Lyapunov regularity. It turns out that unlike in the case of bounded sequences and, more generally, tempered sequences, some of these properties are no longer equivalent.

We first describe the type of problems in which we are interested. Let $(A_n)_{n\in\mathbb{N}}$ be a *tempered* sequence of $q \times q$ upper-triangular matrices (see (1.4)). Denoting the entries of A_n by $a_{ij}(n)$, it follows for example from Theorem 1.3.12 in [6] that if the limits

$$c_i := \lim_{n\to\infty} \frac{1}{n} \log \prod_{l=1}^{n-1} |a_{ii}(l)| \tag{1.7}$$

exist and are finite for $i = 1, \ldots, q$, then the sequence is Lyapunov regular (in which case the numbers $c_1, \ldots, c_q$ are the values of the Lyapunov exponent on $\mathbb{R}^q \setminus \{0\}$, counted with their multiplicities but possibly not ordered). On the other hand, we show in Theorem 4.1 that the existence and finiteness of the limits in (1.7) is a necessary condition for Lyapunov regularity, even if the sequence is not tempered (see (1.9) for an example of a nontempered sequence of upper-triangular matrices illustrating that the condition is not sufficient). In fact, Theorem 4.1 considers also the more general case when the sequence of matrices $(A_n)_{n\in\mathbb{N}}$ is transformed into a sequence of upper-triangular matrices via a Lyapunov coordinate change.

In strong contrast, the fact that a *nontempered* sequence $(A_n)_{n\in\mathbb{N}}$ can be reduced via a Lyapunov coordinate change to a sequence of upper-triangular matrices $B_n = (b_{ij}(n))_{1\le i\le j\le q}$ such that the limits

$$d_i := \lim_{n\to\infty} \frac{1}{n} \log \prod_{l=1}^{n-1} |b_{ii}(l)| \tag{1.8}$$

exist and are finite for $i = 1, \ldots, q$, is not sufficient for the Lyapunov regularity of the sequence $(A_n)_{n\in\mathbb{N}}$. For example, take

$$A_n = \begin{pmatrix} 1 & 2^{n-1} \\ 0 & 1 \end{pmatrix} \tag{1.9}$$

for $n \geq 1$. Then, by (1.2), we have

$$\mathcal{A}_n = \begin{pmatrix} 1 & 2^{n-1} - 1 \\ 0 & 1 \end{pmatrix} \quad \text{for } n > 1.$$

Clearly, the limits in (1.7) exist for this sequence. Moreover, the values of the associated Lyapunov exponent are $\lambda'_1 = 0$ and $\lambda'_2 = \log 2$. On the other hand, since $\det \mathcal{A}_n = 1$, we have

$$0 = \lim_{n\to\infty} \frac{1}{n} \log|\det \mathcal{A}_n| \neq \min \sum_{i=1}^{2} \lambda(v_i) = \lambda'_1 + \lambda'_2 = \log 2,$$

where the minimum is taken over all bases v_1, v_2 for $\mathbb{R}^2$, and so the sequence $(A_n)_{n\in\mathbb{N}}$ is not Lyapunov regular (see (1.3)).

In fact we provide even more detailed information on the relation between the Lyapunov regularity of a sequence of matrices and its reduction to a sequence of upper-triangular matrices via a Lyapunov coordinate change. Namely, consider the following classes of matrices:

1. let $\mathcal{S}_1$ be the set of all sequences of invertible $q \times q$ matrices that are Lyapunov regular;
2. let $\mathcal{S}_3$ be the set of all sequences of invertible $q \times q$ matrices $(A_n)_{n\in\mathbb{N}}$ such that after a reduction to a sequence of upper-triangular matrices via a Lyapunov coordinate change the limits in (1.8) exist and are finite for $i = 1, \ldots, q$;
3. let $\mathcal{S}_2$ be the set of all sequences of invertible $q \times q$ matrices $(A_n)_{n\in\mathbb{N}} \in \mathcal{S}_3$ such that, up to a permutation, the vector $(d_1, \ldots, d_q)$ given by (1.8) is the same for any Lyapunov coordinate change.

We show in Theorem 4.2 that

$$\mathcal{S}_1 \subset \mathcal{S}_2 \subset \mathcal{S}_3 \subset \mathcal{L}, \tag{1.10}$$

where $\mathcal{L}$ is the set of all sequences of invertible $q \times q$ matrices whose Lyapunov exponent takes only finite values on $\mathbb{R}^q \setminus \{0\}$. We also show that these inclusions are proper, by giving explicit examples. On the other hand, for tempered sequences of matrices the first two inclusions in (1.10) become equalities. More precisely, if $\mathcal{T}$ is the set of all tempered sequences of $q \times q$ matrices, then

$$\mathcal{S}_1 \cap \mathcal{T} = \mathcal{S}_2 \cap \mathcal{T} = \mathcal{S}_3 \cap \mathcal{T}. \tag{1.11}$$

Indeed, for example by Theorem 1.3.12 in [6], if $(B_n)_{n\in\mathbb{N}}$ is a tempered sequence of upper-triangular matrices and the limits d_i in (1.8) exist and are finite for $i = 1, \ldots, q$, then the sequence is Lyapunov regular. Hence, by Proposition 3.1 below, for tempered sequences we have $\mathcal{S}_3 \cap \mathcal{T} \subset \mathcal{S}_1 \cap \mathcal{T}$ and so it follows from (1.10) that property (1.11) holds for tempered sequences.

Our arguments are inspired by work of Barabanov and Konyukh in [3] where they established earlier corresponding results for continuous time. To the possible extent we follow their approach.

2 Gramians and volumes

In this section we collect a few notions and basic results on Gramians and volumes that are used in the remainder of the paper. We refer the reader to the books [10, 16] for details.

We recall that the Gramian (or the Gram determinant) $G = G(v_1, \ldots, v_p)$ of a set of vectors $v_1, \ldots, v_p \in \mathbb{R}^q$ is the determinant of the matrix of inner products $G_{ij} = \langle v_i, v_j \rangle$, using the standard inner product on $\mathbb{R}^q$. One can show that the Gramian G coincides with the square of the p-volume $\Gamma(v_1, \ldots, v_p)$ determined by the vectors $v_1, \ldots, v_p$, that is,

$$G(v_1, \ldots, v_p) = \Gamma(v_1, \ldots, v_p)^2.$$

In particular, the Gramian has the following properties:

1. $G(v_1, \dots, v_p) \geq 0$ for any vectors $v_1, \dots, v_p \in \mathbb{R}^q$;
2. $G(v_1, \dots, v_p) = 0$ if and only if $v_1, \dots, v_p$ are linearly dependent;
3. $G(v) = \|v\|^2$ and $G(v, w) = \|v\|^2\|w\|^2 - \langle v, w\rangle^2$.

By properties 1 and 3 we obtain as a particular case the Cauchy–Schwarz inequality $|\langle v, w\rangle| \leq \|v\| \cdot \|w\|$ (with equality if and only if v and w are colinear, in view of property 2). Moreover, we have the inequalities

$$G(v_1, \dots, v_p) \leq G(v_1, \dots, v_i)G(v_{i+1}, \dots, v_p)$$

and so also

$$\Gamma(v_1, \dots, v_p) \leq \Gamma(v_1, \dots, v_i)\Gamma(v_{i+1}, \dots, v_p),$$

for $i = 1, \dots, p-1$. In fact, these inequalities follow from a more general result in Proposition 2.1 below.

We also recall that the *angle* between two subspaces $E, F \subset \mathbb{R}^q$ is defined by

$$\angle(E, F) = \arccos\langle u_1, v_1\rangle \in [0, \pi/2],$$

where $u_1 \in E$ and $v_1 \in F$ are unit vectors such that

$$\langle u_1, v_1\rangle = \max\{\langle u, v\rangle : u \in E, v \in F, \|u\| = \|v\| = 1\}.$$

Now let $k = \dim E$, $l = \dim F$ and $p = \min\{k, l\}$. Set $\theta_i = \angle(E, F)$. The *principal angles*

$$\theta_1 \leq \theta_2 \leq \cdots \leq \theta_p$$

between E and F are defined recursively by

$$\theta_i = \arccos\langle u_i, v_i\rangle \in [0, \pi/2],$$

where $u_i \in E$ and $v_i \in F$ are unit vectors such that

$$\langle u_i, v_i\rangle = \max\{\langle u, v\rangle : u \in E \cap G_i^\perp, v \in F \cap H_i^\perp, \|u\| = \|v\| = 1\},$$

with

$$G_i = \operatorname{span}\{u_1, \dots, u_{i-1}\} \quad \text{and} \quad H_i = \operatorname{span}\{v_1, \dots, v_{i-1}\}.$$

Proposition 2.1 ([2]). *For any subspaces*

$$E = \operatorname{span}\{u_1, \dots, u_k\} \quad \text{and} \quad F = \operatorname{span}\{v_1, \dots, v_l\}$$

we have

$$G(u_1, \dots, u_k, v_1, \dots, v_l) = G(u_1, \dots, u_k)G(v_1, \dots, v_l)\prod_{i=1}^{p}\sin^2\theta_i,$$

where $\theta_1 \leq \theta_2 \leq \cdots \leq \theta_p$ are the principal angles between E and F.

When $l = 1$, there exists a single principal angle between E and F (which in fact is the angle between the two spaces). Hence, writing $E = \operatorname{span}\{u_1, \dots, u_k\}$ and $F = \operatorname{span}\{v\}$ we have

$$G(u_1, \dots, u_k, v) = G(u_1, \dots, u_k) G(v) \sin^2 \theta_1$$

or, equivalently,

$$\Gamma(u_1, \dots, u_k, v) = \Gamma(u_1, \dots, u_k) \|v\| \sin \angle(v, E). \tag{2.1}$$

Moreover, it follows from Proposition 2.1 that given $v_1, \dots, v_k \in \mathbb{R}^q$ and $i \in [1, k) \cap \mathbb{N}$, we have

$$G(v_1, \dots, v_k) \leq G(v_1, \dots, v_i) G(v_{i+1}, \dots, v_k) \leq \prod_{j=1}^{k} G(v_j).$$

In particular, taking $k = q$ and $v_i = e_i$ for $i = 1, \dots, q$, where $e_1, \dots, e_q$ is the canonical basis for $\mathbb{R}^q$, we obtain Hadamard's inequality

$$|\det A| \leq \prod_{i=1}^{q} \|Ae_i\| \tag{2.2}$$

(using the 2-norm on $\mathbb{R}^q$). This inequality can be seen as a consequence of the fact that $|\det A|$ gives the volume of the parallelepiped determined by the vectors $Ae_1, \dots, Ae_q$. For completeness we give an elementary derivation. Let U be the orthogonal matrix whose columns are obtained applying the Gram–Schmidt process to the basis $Ae_1, \dots, Ae_q$. Then

$$\operatorname{span}\{Ae_1, \dots, Ae_j\} = \operatorname{span}\{Ue_1, \dots, Ue_j\}$$

for each $j \leq q$ and writing $Ae_j = \sum_{i=1}^{j} \alpha_{ij} Ue_i$, we obtain $\langle Ae_j, Ue_i \rangle = \alpha_{ij}$ because U is orthogonal. Hence,

$$Ae_j = \sum_{i=1}^{j} \langle Ae_j, Ue_i \rangle Ue_i$$

and so also

$$\|Ae_j\|^2 = \sum_{i=1}^{j} |\langle Ae_j, Ue_i \rangle|^2 = \sum_{i=1}^{j} |\alpha_{ij}|^2. \tag{2.3}$$

Now let B be the upper-triangular matrix with entries $b_{ij} = \alpha_{ij}$ for $i \leq j$. Then $A = UB$ and since U is orthogonal, we obtain

$$\begin{aligned} |\det A|^2 &= \det(A^* A) = \det(B^* U^* U B) \\ &= \det(B^* B) = |\det B|^2 \\ &= \prod_{i=1}^{q} |\alpha_{ii}|^2 \leq \prod_{i=1}^{q} \|Ae_i\|^2, \end{aligned}$$

using (2.3) in the last inequality.

3 Criteria for Lyapunov regularity

In this section we describe several criteria for the Lyapunov regularity of a sequence of invertible $q \times q$ matrices with finite values of the Lyapunov exponent on $\mathbb{R}^q \setminus \{0\}$. We emphasize that the sequence need not be bounded or even tempered. All matrices are assumed to have real entries.

3.1 Basic notions

Without loss of generality we shall always consider the 2-norm $\|\cdot\|$ on $\mathbb{R}^q$ and for each $q \times q$ matrix A we consider its operator norm

$$\|A\| = \sup_{v \in \mathbb{R}^q \setminus \{0\}} \frac{\|Av\|}{\|v\|}.$$

We define the *Lyapunov exponent* $\lambda \colon \mathbb{R}^q \to [-\infty, +\infty]$ of a sequence of invertible $q \times q$ matrices $A = (A_n)_{n \in \mathbb{N}}$ by

$$\lambda(v) = \lambda_A(v) = \limsup_{n \to \infty} \frac{1}{n} \log\|\mathcal{A}_n v\|, \tag{3.1}$$

where

$$\mathcal{A}_n = \begin{cases} A_{n-1} \cdots A_1 & \text{if } n > 1, \\ \mathrm{Id} & \text{if } n = 1 \end{cases} \tag{3.2}$$

(with the convention that $\log 0 = -\infty$). We denote by $\mathcal{L}$ the set of all sequences of invertible $q \times q$ matrices whose Lyapunov exponent λ takes only finite values on $\mathbb{R}^q \setminus \{0\}$. By the theory of Lyapunov exponents (see [5]), for each $A \in \mathcal{L}$ the Lyapunov exponent λ can take at most q values on $\mathbb{R}^q \setminus \{0\}$, say

$$\lambda_1 < \cdots < \lambda_p$$

for some integer $p \leq q$, and the sets

$$E_i = \{ v \in \mathbb{R}^q : \lambda(v) \leq \lambda_i \}$$

are linear subspaces. We denote by

$$\lambda'_1 \leq \cdots \leq \lambda'_q \tag{3.3}$$

the values of λ counted with their multiplicities, that is, $\lambda'_j = \lambda_i$ for $j = \dim E_{i-1} + 1, \ldots, \dim E_i$ and $i = 1, \ldots, p$, with the convention that $E_0 = \{0\}$. A basis $v_1, \ldots, v_q$ for $\mathbb{R}^q$ is said to be *normal (with respect to the sequence A)* if for each $i = 1, \ldots, p$ there exists a basis for E_i composed of vectors in $\{v_1, \ldots, v_q\}$. Finally, a sequence of matrices $A \in \mathcal{L}$ is said to be *Lyapunov regular* if there exists a basis $v_1, \ldots, v_q$ for $\mathbb{R}^q$ such that

$$\liminf_{n \to \infty} \frac{1}{n} \log|\det \mathcal{A}_n| = \sum_{i=1}^q \lambda(v_i). \tag{3.4}$$

Equivalently, a sequence $A \in \mathcal{L}$ is Lyapunov regular if (3.4) holds for some *normal* basis $v_1, \ldots, v_q$ for $\mathbb{R}^q$ (see [5]). Moreover, by (2.2) we have

$$|\det(\mathcal{A}_n V)| \leq \prod_{i=1}^q \|\mathcal{A}_n v_i\|$$

for the matrix V with columns $v_1, \ldots, v_q$, and so

$$\limsup_{n \to \infty} \frac{1}{n} \log|\det \mathcal{A}_n| \leq \sum_{i=1}^q \lambda(v_i).$$

Hence, it follows from (3.4) that a sequence $A \in \mathcal{L}$ is Lyapunov regular if and only if

$$\lim_{n \to \infty} \frac{1}{n} \log|\det \mathcal{A}_n| = \sum_{i=1}^q \lambda(v_i)$$

for some basis $v_1, \dots, v_q$ for $\mathbb{R}^q$ (that is, if and only if the limit exists and is equal to the right-hand side).

Given a sequence of invertible $q \times q$ matrices $(A_n)_{n\in\mathbb{N}}$, we consider the new sequence $C_n = (A_n^*)^{-1}$, for $n \in \mathbb{N}$, where A_n^* denotes the transpose of A_n. In a similar manner to that in (3.2), we define

$$\mathcal{C}_n = (\mathcal{A}_n^*)^{-1} = \begin{cases} (A_{n-1}^*)^{-1} \cdots (A_1^*)^{-1} & \text{if } n > 1, \\ \text{Id} & \text{if } n = 1. \end{cases}$$

The Lyapunov exponent $\mu_A = \lambda_C$ of the sequence $C = (C_n)_{n\in\mathbb{N}}$ is given by

$$\mu_A(w) = \limsup_{n\to\infty} \frac{1}{n} \log\|\mathcal{C}_n w\|.$$

Moreover, in a similar manner to that in (3.3), we denote by

$$\mu_1' \geq \cdots \geq \mu_q'$$

the values of μ_A counted with their multiplicities.

A sequence of invertible $q \times q$ matrices $(V_n)_{n\in\mathbb{N}}$ is called a *Lyapunov coordinate change* if condition (1.5) holds, that is, if

$$\lim_{n\to\infty} \frac{1}{n} \log\|V_n\| = \lim_{n\to\infty} \frac{1}{n} \log\|V_n^{-1}\| = 0.$$

For the matrices $B_n = V_{n+1}^{-1} A_n V_n$, for $n \in \mathbb{N}$, we have $\mathcal{A}_n V_1 = V_n \mathcal{B}_n$, with $\mathcal{A}_n$ as in (3.2) and where

$$\mathcal{B}_n = \begin{cases} B_{n-1} \cdots B_1 & \text{if } n > 1, \\ \text{Id} & \text{if } n = 1. \end{cases}$$

Hence, it follows readily from (1.5) that

$$\begin{aligned} \lambda_A(V_1 v) &= \limsup_{n\to\infty} \frac{1}{n} \log\|\mathcal{A}_n V_1 v\| \\ &= \limsup_{n\to\infty} \frac{1}{n} \log\|\mathcal{B}_n v\| = \lambda_B(v) \end{aligned} \tag{3.5}$$

for any $v \in \mathbb{R}^q$. This shows that any Lyapunov coordinate change preserves the values of the Lyapunov exponent. In fact it also preserves Lyapunov regularity.

Proposition 3.1. *If the sequences $A = (A_n)_{n\in\mathbb{N}}$ and $B = (B_n)_{n\in\mathbb{N}}$ are in $\mathcal{L}$ and are related by $B_n = V_{n+1}^{-1} A_n V_n$, for each $n \in \mathbb{N}$, for some Lyapunov coordinate change $(V_n)_{n\in\mathbb{N}}$, then $\sigma(A) = \sigma(B)$. In particular, A is Lyapunov regular if and only if B is Lyapunov regular.*

Proof. Note that $\mathcal{B}_n = V_n^{-1} \mathcal{A}_n V_1$ and so

$$\begin{aligned} \sigma(B) &= \min \sum_{i=1}^{q} \lambda_B(v_i) - \liminf_{n\to\infty} \frac{1}{n} \log|\det \mathcal{B}_n| \\ &= \min \sum_{i=1}^{q} \lambda_A(V_1 v_i) - \liminf_{n\to\infty} \frac{1}{n} \log|\det \mathcal{A}_n|, \end{aligned}$$

with the minimum taken over all basis $v_1, \dots, v_q$ for $\mathbb{R}^q$. Since any basis for $\mathbb{R}^q$ can be written in the form $V_1 v_1, \dots, V_1 v_q$ for some basis $v_1, \dots, v_q$ for $\mathbb{R}^q$, we conclude that $\sigma(B) = \sigma(A)$. □

Now let $e_1, \dots, e_q$ be the canonical basis for $\mathbb{R}^q$.

Proposition 3.2. *For a Lyapunov coordinate change* $(V_n)_{n\in\mathbb{N}}$ *we have*

$$\lim_{n\to\infty} \frac{1}{n} \log|\det V_n| = 0 \quad \text{and} \quad \lim_{n\to\infty} \frac{1}{n} \log\|V_n e_i\| = 0, \text{ for } i = 1, \dots, q$$

(that is, the limits exist and are zero).

Proof. For the first statement, by (2.2) we have

$$|\det V_n| \le \prod_{i=1}^{q} \|V_n e_i\| \le \prod_{i=1}^{q} \|V_n\| = \|V_n\|^q. \tag{3.6}$$

Together with (1.5), this implies that

$$\limsup_{n\to\infty} \frac{1}{n} \log|\det V_n| \le 0. \tag{3.7}$$

In a similar manner, we have $|\det(V_n^{-1})| \le \|V_n^{-1}\|^q$ and so again by (1.5) we obtain

$$\limsup_{n\to\infty} \frac{1}{n} \log|\det(V_n^{-1})| \le 0.$$

Hence,

$$\liminf_{n\to\infty} \frac{1}{n} \log|\det V_n| = -\limsup_{n\to\infty} \frac{1}{n} \log|\det(V_n^{-1})| \ge 0,$$

with together with (3.7) yields the first statement in the proposition. For the second statement we first observe that

$$\|V_n\| \ge c\sqrt{\sum_{i=1}^{q} \|V_n e_i\|^2} \ge c\|V_n e_i\|$$

for some positive constant c (since all norms on a finite-dimensional space are equivalent). Thus, by (1.5) we obtain

$$\limsup_{n\to\infty} \frac{1}{n} \log\|V_n e_i\| \le 0. \tag{3.8}$$

On the other hand, proceeding as in (3.6) one can write

$$|\det V_n| \le \prod_{i=1}^{q} \|V_n e_i\| = \|V_n e_i\| \prod_{j\ne i} \|V_n e_j\| \le \|V_n e_i\| \cdot \|V_n\|^{q-1}.$$

Hence, by (1.5) and the first statement in the proposition, we obtain

$$\liminf_{n\to\infty} \frac{1}{n} \log\|V_n e_i\| \ge 0. \tag{3.9}$$

The second statement follows now readily from (3.8) and (3.9). □

3.2 Criteria for Lyapunov regularity

The following result describes several criteria for Lyapunov regularity. The emphasis is on sequences of matrices that need not be bounded, although their Lyapunov exponent takes only finite values on $\mathbb{R}^q \setminus \{0\}$. To the possible extent, the proofs are obtained by modifying existing arguments for bounded sequences, although we give a clean streamlined argument.

Theorem 3.3. *For a sequence of invertible $q \times q$ matrices $(A_n)_{n\in\mathbb{N}} \in \mathcal{L}$, the following properties are equivalent:*

1. *$(A_n)_{n\in\mathbb{N}}$ is Lyapunov regular;*
2. *$(C_n)_{n\in\mathbb{N}} = ((A_n^*)^{-1})_{n\in\mathbb{N}} \in \mathcal{L}$ and $\lambda'_k = -\mu'_{q-k+1}$ for $k = 1, \dots, q$;*
3. *there exist a Lyapunov coordinate change $(V_n)_{n\in\mathbb{N}}$ and a diagonal $q \times q$ matrix D such that $V_{n+1}^{-1} A_n V_n = D$ for all $n \in \mathbb{N}$;*
4. *given a normal basis $v_1, \dots, v_q$ for $\mathbb{R}^q$, we have*
$$\lambda(v_i) = \lim_{n\to\infty} \frac{1}{n} \log\|\mathcal{A}_n v_i\| \tag{3.10}$$
for $i = 1, \dots, q$ and
$$\lim_{n\to\infty} \frac{1}{n} \log \gamma_{jn} = 0 \tag{3.11}$$
for $j = 1, \dots, q-1$, where
$$\gamma_{jn} = \angle\big(\mathcal{A}_n v_j, \operatorname{span}\{\mathcal{A}_n v_{j+1}, \dots, \mathcal{A}_n v_q\}\big); \tag{3.12}$$
5. *there exists a basis $v_1, \dots, v_q$ for $\mathbb{R}^q$ such that properties (3.10) and (3.11) hold for $i = 1, \dots, q$ and $j = 1, \dots, q-1$.*

Proof. We separate the proof into several steps.

Step 1: $3 \Rightarrow 2$

Property 3 says that
$$V_{n+1}^{-1} A_n V_n = \operatorname{diag}(d_1, \dots, d_q), \tag{3.13}$$
for some Lyapunov coordinate change $(V_n)_{n\in\mathbb{N}}$ and some numbers $d_1, \dots, d_q$ in $\mathbb{R}$. Hence,
$$V_n^{-1} \mathcal{A}_n V_1 = \operatorname{diag}(d_1^{n-1}, \dots, d_q^{n-1}) \tag{3.14}$$
and so
$$\det \mathcal{A}_n \det V_1 = \det V_n \prod_{i=1}^{q} d_i^{n-1},$$
which by Proposition 3.2 yields the identity
$$\lim_{n\to\infty} \frac{1}{n} \log|\det \mathcal{A}_n| = \sum_{i=1}^{q} \log|d_i|.$$
Moreover,
$$\mathcal{A}_n V_1 e_i = d_i^{n-1} V_n e_i \quad \text{and so} \quad \|\mathcal{A}_n V_1 e_i\| = |d_i|^{n-1} \|V_n e_i\|.$$

Again it follows from Proposition 3.2 that

$$\lambda_A(V_1 e_i) = \lim_{n\to\infty} \frac{1}{n} \log\|\mathcal{A}_n V_1 e_i\| = \log|d_i|. \tag{3.15}$$

Now we consider the sequence of matrices $C_n = (A_n^*)^{-1}$, for $n \in \mathbb{N}$. Let $U_n = (V_n^*)^{-1}$. It follows from (3.13) and (3.14) that

$$U_{n+1}^{-1} C_n U_n = \operatorname{diag}(d_1^{-1}, \dots, d_q^{-1})$$

and so

$$U_n^{-1} \mathcal{C}_n U_1 = \operatorname{diag}(d_1^{-n+1}, \dots, d_q^{-n+1}).$$

Therefore,

$$\det \mathcal{C}_n \det U_1 = \det U_n \prod_{i=1}^q d_i^{-n+1},$$

which by Proposition 3.2 yields the identity

$$\lim_{n\to\infty} \frac{1}{n} \log|\det \mathcal{C}_n| = -\sum_{i=1}^q \log|d_i|.$$

Moreover,

$$\mathcal{C}_n U_1 e_i = d_i^{-n+1} U_n e_i \quad \text{and so} \quad \|\mathcal{C}_n U_1 e_i\| = |d_i|^{-n+1} \|U_n e_i\|.$$

Again by Proposition 3.2 we obtain

$$\mu_A(U_1 e_i) = \lim_{n\to\infty} \frac{1}{n} \log\|\mathcal{C}_n U_1 e_i\| = -\log|d_i|. \tag{3.16}$$

Since $e_1, \dots, e_q$ is a normal basis with respect to any constant of sequence of diagonal matrices, it follows from (3.15) that $\lambda_i' = \log|d_i|$ for $i = 1, \dots, q$ and it follows from (3.16) that $\mu_i' = -\log|d_{q-i+1}|$ for $i = 1, \dots, q$.

Step 2: $2 \Rightarrow 1$

Property 2 says that the numbers $\mu_1' \geq \cdots \geq \mu_q'$ are finite and coincide, respectively, with

$$-\lambda_q' \geq \cdots \geq -\lambda_1'.$$

For any normal basis $v_1, \dots, v_q$ for $\mathbb{R}^q$ with respect to the sequence $A = (A_n)_{n\in\mathbb{N}}$ we have

$$|\det(\mathcal{A}_n V)| \leq \prod_{i=1}^q \|\mathcal{A}_n v_i\|, \tag{3.17}$$

where V is the matrix whose columns are $v_1, \dots, v_q$. This follows readily from Hadamard's inequality in (2.2). It follows from (3.17) that

$$\begin{aligned} \limsup_{n\to\infty} \frac{1}{n} \log|\det \mathcal{A}_n| &\leq \sum_{i=1}^q \limsup_{n\to\infty} \frac{1}{n} \log\|\mathcal{A}_n v_i\| \\ &= \sum_{i=1}^q \lambda_A(v_i) = \sum_{i=1}^q \lambda_i'. \end{aligned} \tag{3.18}$$

In a similar manner, for any normal basis $w_1, \dots, w_q$ for $\mathbb{R}^q$ with respect to the sequence $(C_n)_{n\in\mathbb{N}}$ we have

$$-\liminf_{n\to\infty} \frac{1}{n} \log|\det \mathcal{A}_n| = \limsup_{n\to\infty} \frac{1}{n} \log|\det \mathcal{C}_n| \le \sum_{i=1}^{q} \mu_A(w_i) = \sum_{i=1}^{q} \mu_i'.$$

Therefore, it follows from property 2 that

$$\liminf_{n\to\infty} \frac{1}{n} \log|\det \mathcal{A}_n| \ge -\sum_{i=1}^{q} \mu_i' = \sum_{i=1}^{q} \lambda_i'$$

and so, by (3.18),

$$\lim_{n\to\infty} \frac{1}{n} \log|\det \mathcal{A}_n| = \sum_{i=1}^{q} \lambda_i'.$$

This shows that the sequence A is Lyapunov regular.

Step 3: $1 \Rightarrow 4$

Consider a sequence $(A_n)_{n\in\mathbb{N}}$ satisfying property 1. This corresponds to assume that the numbers $\lambda_1' \le \cdots \le \lambda_q'$ satisfy

$$\liminf_{n\to\infty} \frac{1}{n} \log|\det \mathcal{A}_n| = \sum_{i=1}^{q} \lambda_i'.$$

We claim that each number λ_i' is a limit, that is,

$$\lambda_i' = \lim_{n\to\infty} \frac{1}{n} \log\|\mathcal{A}_n v_i\| \tag{3.19}$$

for $i = 1, \dots, q$ and any normal basis $v_1, \dots, v_q$ with $\lambda(v_1) \le \cdots \le \lambda(v_q)$. We proceed by contradiction. Assume that there exists a vector $v \ne 0$ for which $\lambda(v)$ is not a limit, that is,

$$\lim_{k\to\infty} \frac{1}{n_k} \log\|\mathcal{A}_{n_k} v\| < \lambda(v)$$

along some sequence $(n_k)_{k\in\mathbb{N}} \nearrow +\infty$. Now we consider any normal basis $v_1, \dots, v_q$ such that $v_j = v$ for some j. Then

$$|\det \mathcal{A}_n| \le \|\mathcal{A}_n v\| \prod_{i\ne j} \|\mathcal{A}_n v_i\| \tag{3.20}$$

and so, by (3.20), we have

$$\begin{aligned}\sum_{i=1}^{q} \lambda(v_i) &\le \limsup_{k\to\infty} \frac{1}{n_k} \log|\det \mathcal{A}_{n_k}| \\ &\le \limsup_{k\to\infty} \frac{1}{n_k} \log\|\mathcal{A}_{n_k} v\| + \sum_{i\ne j} \lambda(v_i) \\ &< \lambda(v) + \sum_{i\ne j} \lambda(v_i) = \sum_{i=1}^{q} \lambda(v_i).\end{aligned}$$

This contradiction shows that (3.19) holds.

To establish (3.11) we consider an arbitrary normal basis $v_1, \dots, v_q$. Let V be the matrix whose columns are the vectors $v_1, \dots, v_q$. We claim that

$$|\det(\mathcal{A}_n V)| = \prod_{i=1}^{q} \|\mathcal{A}_n v_i\| \prod_{i=1}^{q-1} \sin \gamma_{in}, \tag{3.21}$$

with the angles γ_{in} as in (3.12). First observe that

$$|\det(\mathcal{A}_n V)|^2 = G(\mathcal{A}_n v_1, \dots, \mathcal{A}_n v_q).$$

By Proposition 2.1 we have

$$G(\mathcal{A}_n v_1, \dots, \mathcal{A}_n v_q) = G(\mathcal{A}_n v_i) G(\mathcal{A}_n v_{i+1}, \dots, \mathcal{A}_n v_q) \sin^2 \gamma_{in}$$

for each $i \in [1, q) \cap \mathbb{N}$. Indeed, since $\mathcal{A}_n v_j$ generates a space E of dimension 1, there is a single principal angle between E and

$$F = \operatorname{span}\{\mathcal{A}_n v_{i+1}, \dots, \mathcal{A}_n v_q\},$$

which is simply the angle between E and F. Proceeding by induction we obtain

$$|\det(\mathcal{A}_n V)|^2 = \prod_{i=1}^{q} G(\mathcal{A}_n v_i) \prod_{i=1}^{q-1} \sin^2 \gamma_{in},$$

which yields identity (3.21) since $G(\mathcal{A}_n v_i) = \|\mathcal{A}_n v_i\|^2$.

Since the basis $v_1, \dots, v_q$ is normal and the numbers $\lambda(v_i) = \lambda'_i$ are limits, it follows from (3.21) that

$$\begin{aligned}
\sum_{i=1}^{q} \lambda(v_i) &= \lim_{n\to\infty} \frac{1}{n} \log|\det \mathcal{A}_n| \\
&= \sum_{i=1}^{q} \lim_{n\to\infty} \frac{1}{n} \log\|\mathcal{A}_n v_i\| + \lim_{n\to\infty} \sum_{i=1}^{q-1} \frac{1}{n} \log \sin \gamma_{in} \\
&= \sum_{i=1}^{q-1} \lambda(v_i) + \lim_{n\to\infty} \sum_{i=1}^{q-1} \frac{1}{n} \log \sin \gamma_{in}
\end{aligned}$$

and so

$$\lim_{n\to\infty} \sum_{i=1}^{q-1} \frac{1}{n} \log \sin \gamma_{in} = 0. \tag{3.22}$$

Given $j \in \{1, \dots, q-1\}$, we take a sequence $(n_k)_{k\in\mathbb{N}} \nearrow +\infty$ such that

$$\liminf_{n\to\infty} \frac{1}{n} \log \sin \gamma_{jn} = \lim_{k\to\infty} \frac{1}{n_k} \log \sin \gamma_{jn_k}.$$

Since $\sin \gamma_{jn_k} \le 1$, it follows from (3.22) that

$$\begin{aligned}
0 = \lim_{n\to\infty} \sum_{i=1}^{q-1} \frac{1}{n} \log \sin \gamma_{in} &= \lim_{k\to\infty} \sum_{i=1}^{q-1} \frac{1}{n_k} \log \sin \gamma_{in_k} \\
&\le \liminf_{n\to\infty} \frac{1}{n} \log \sin \gamma_{jn} \le \limsup_{n\to\infty} \frac{1}{n} \log \sin \gamma_{jn} \le 0
\end{aligned}$$

and so

$$\lim_{n\to\infty} \frac{1}{n} \log \sin \gamma_{jn} = 0, \quad \text{for } j = 1, \dots, q-1.$$

Since $2x/\pi \le \sin x \le x$ for $x \in [0, \pi/2]$, this implies that

$$\lim_{n\to\infty} \frac{1}{n} \log \gamma_{jn} = 0, \quad \text{for } j = 1, \dots, q-1.$$

Step 4: $4 \Rightarrow 5$

It is immediate that property 4 implies property 5.

Step 5: $5 \Rightarrow 3$

It follows from property 5 and (3.21) that the limit

$$\begin{aligned}\lim_{n\to\infty}\frac{1}{n}\log|\det\mathcal{A}_n| &= \sum_{i=1}^{q}\lim_{n\to\infty}\frac{1}{n}\log\|\mathcal{A}_n v_i\| + \sum_{i=1}^{q-1}\lim_{n\to\infty}\frac{1}{n}\log\sin\gamma_{in}\\ &= \sum_{i=1}^{q}\lambda(v_i)\end{aligned}$$

exists. Hence, by (3.18), the sequence of matrices $(A_n)_{n\in\mathbb{N}}$ is Lyapunov regular. One can now apply Theorem 2 in [8] to conclude that property 3 holds. This completes the proof of the theorem. □

The equivalence between properties 1 and 2 in Theorem 3.3 was obtained in [7, Theorem 9], following to the possible extent the case of continuous time in Section 1.3 of [5] (the results are formulated for a smaller class of linear dynamics although the arguments apply to the more general case considered here). It was shown in [8, Theorem 2] that property 1 implies property 3 (the converse is immediate). Moreover, it was shown in [6, Theorem 1.3.11] that properties 1 and 4 are equivalent. It is also simple to show that properties 4 and 5 are also equivalent. A version of Theorem 3.3 for continuous time was obtained earlier by Barabanov and Konyukh in [3].

4 Triangular reduction

In this section we discuss how the reduction of a sequence of matrices to a sequence of upper-triangular matrices via a Lyapunov coordinate change relates to Lyapunov regularity. It turns out that unlike in the case of bounded sequences and, more generally, tempered sequences, certain related properties are no longer equivalent. We refer the reader to [3] for corresponding earlier work of Barabanov and Konyukh in the case of continuous time.

4.1 Necessary condition for regularity

As noted in the introduction, for a tempered sequence of upper-triangular matrices, it follows for example from Theorem 1.3.12 in [6] that if the limits in (1.7) exist and are finite, then the sequence is Lyapunov regular. On the other hand, the example of a nontempered sequence of upper-triangular matrices in (1.9) shows that the existence and finiteness of those limits is not a sufficient condition for Lyapunov regularity.

The following result shows that the former condition (that is, the requirement that the limits in (1.7) exist and are finite) is always necessary for Lyapunov regularity, even for non-tempered sequences. We recall that the values of the Lyapunov exponent λ in (3.1), counted with their multiplicities, are denoted by $\lambda'_1, \dots, \lambda'_q$ (see (3.3)).

Theorem 4.1. *For any reduction of a Lyapunov regular sequence $(A_n)_{n\in\mathbb{N}}$ to a sequence of upper-triangular matrices $B_n = (b_{ij}(n))_{1\le i\le j\le q}$ via a Lyapunov coordinate change $(V_n)_{n\in\mathbb{N}}$, the limits*

$$d_i := \lim_{n\to\infty} \frac{1}{n} \log \prod_{l=1}^{n-1} |b_{ii}(l)| \tag{4.1}$$

exist and are finite, for $i = 1, \ldots, q$, and $(d_1, \ldots, d_q)$ is a permutation of $(\lambda'_1, \ldots, \lambda'_q)$.

Proof. Let $(A_n)_{n\in\mathbb{N}}$ be a Lyapunov regular sequence and let $(V_n)_{n\in\mathbb{N}}$ be a Lyapunov coordinate change such that $B_n = V_{n+1}^{-1} A_n V_n$ is upper-triangular for $n \in \mathbb{N}$. Since $\mathcal{B}_n = V_n^{-1} \mathcal{A}_n V_1$, we have

$$\det \mathcal{A}_n = \det \mathcal{B}_n \det V_n \det(V_1^{-1}). \tag{4.2}$$

Moreover, since $(A_n)_{n\in\mathbb{N}}$ is Lyapunov regular, it follows from Proposition 3.2 together with (3.5) and (4.2) that $(B_n)_{n\in\mathbb{N}}$ is also Lyapunov regular and

$$\lim_{n\to\infty} \frac{1}{n} \log|\det \mathcal{B}_n| = \lim_{n\to\infty} \frac{1}{n} \log|\det \mathcal{A}_n| = \sum_{i=1}^{q} \lambda'_i. \tag{4.3}$$

Now let

$$c_i = \limsup_{n\to\infty} \frac{1}{n} \log \prod_{l=1}^{n-1} |b_{ii}(l)| = \limsup_{n\to\infty} \frac{1}{n} \sum_{l=1}^{n-1} \log|b_{ii}(l)|.$$

We have

$$\mathcal{B}_n e_i = \left(\ldots, \prod_{l=1}^{n-1} b_{ii}(l), 0, \ldots, 0 \right)^*,$$

with the term $\prod_{l=1}^{n-1} b_{ii}(l)$ at the ith position, and so

$$\lambda_B(e_i) = \limsup_{n\to\infty} \frac{1}{n} \log\|\mathcal{B}_n e_i\| \ge c_i$$

for $i = 1, \ldots, q$. Since $(B_n)_{n\in\mathbb{N}}$ is Lyapunov regular, its Lyapunov exponent takes only finite values on $\mathbb{R}^q \setminus \{0\}$ and so $c_i \le \lambda_B(e_i) < +\infty$ for $i = 1, \ldots, q$.

To show that c_i is not $-\infty$, we consider the diagonal sequence

$$D_n = \operatorname{diag}(b_{11}(n), \ldots, b_{qq}(n)).$$

Then the matrices

$$\mathcal{D}_n = \begin{cases} D_{n-1} \cdots D_1 & \text{if } n > 1, \\ \mathrm{Id} & \text{if } n = 1 \end{cases}$$

are given explicitly by

$$\mathcal{D}_n = \operatorname{diag}\left(\prod_{l=1}^{n-1} b_{11}(l), \ldots, \prod_{l=1}^{n-1} b_{qq}(l) \right). \tag{4.4}$$

Now assume that along some sequence $(n_k)_{k\in\mathbb{N}} \nearrow +\infty$ we have

$$\lim_{k\to\infty} \frac{1}{n_k} \log\|\mathcal{D}_{n_k} e_j\| = -\infty$$

for some $j \in \{1, \ldots, q\}$. Since

$$|\det \mathcal{D}_n| \le \prod_{i=1}^{q} \|\mathcal{D}_n e_i\|,$$

we have

$$\sum_{i=1}^{q} \lambda_i' = \lim_{n\to\infty} \frac{1}{n} \log|\det \mathcal{B}_n| = \lim_{n\to\infty} \frac{1}{n} \log|\det \mathcal{D}_n|$$
$$= \lim_{k\to\infty} \frac{1}{n_k} \log|\det \mathcal{D}_{n_k}| \le \sum_{i=1}^{q} \limsup_{k\to\infty} \frac{1}{n_k} \log\|\mathcal{D}_{n_k} e_i\|.$$

But the last inequality cannot hold since the right-hand side is $-\infty$ while the numbers λ_i' are finite. This contradiction shows that $c_i > -\infty$ for $i = 1, \ldots, q$.

Now let $c_1' \le \cdots \le c_q'$ be the numbers $c_1, \ldots, c_q$ written in increasing order. It follows from the general theory that there exists an upper-triangular matrix $(f_{ij})_{1\le i\le j\le q}$ with unit diagonal such that the vectors

$$w_i = e_i + f_{i,i+1} e_{i+1} + \cdots + f_{iq} e_q, \quad \text{for } i = 1, \ldots, q,$$

form a normal basis with respect to the sequence $B = (B_n)_{n\in\mathbb{N}}$. The numbers f_{ij} can be obtained as follows. Take $w_q = e_q$. Now we proceed by induction. After having $w_{i+1}, \ldots, w_q$ we construct w_i as follows. Take numbers f_{ij} for $j = i+1, \ldots, q$ such that $\lambda_B(w_i)$ takes the smallest possible value. Then $w_1, \ldots, w_q$ is a normal basis with respect to B. This is a variation of Lyapunov's construction of a normal basis (see Section 1.2 in [5]). By (4.4) we have

$$\mathcal{D}_n e_i = \left(0, \ldots, 0, \prod_{l=1}^{n-1} b_{ii}(l), 0, \ldots, 0\right)^*,$$

with the term $\prod_{l=1}^{n-1} b_{ii}(l)$ at the ith position, and so

$$\limsup_{n\to\infty} \frac{1}{n} \log\|\mathcal{D}_n w_i\| \ge c_i, \quad \text{for } i = 1, \ldots, q.$$

Since $w_1, \ldots, w_q$ is a normal basis, the values λ_i' of the Lyapunov exponent of the sequence $(B_n)_{n\in\mathbb{N}}$ (that are the same as those of $(A_n)_{n\in\mathbb{N}}$) satisfy

$$\lambda_k' = \min\max\left\{\limsup_{n\to\infty} \frac{1}{n} \log\|\mathcal{B}_n w_{i_1}\|, \ldots, \limsup_{n\to\infty} \frac{1}{n} \log\|\mathcal{B}_n w_{i_k}\|\right\},$$

where the minimum is taken over all collections of numbers $i_1 < \cdots < i_k$ in the set $\{1, \ldots, q\}$. Since

$$\max\left\{\limsup_{n\to\infty} \frac{1}{n} \log\|\mathcal{B}_n w_{i_1}\|, \ldots, \limsup_{n\to\infty} \frac{1}{n} \log\|\mathcal{B}_n w_{i_k}\|\right\} \ge c_k'$$

for any such set, we have $\lambda_k' \ge c_k'$ for $k = 1, \ldots, q$. In particular,

$$\sum_{i=1}^{q} \lambda_i' \ge \sum_{i=1}^{q} c_i' = \sum_{i=1}^{q} c_i. \tag{4.5}$$

Finally, we show that each number c_i is a limit. Since the matrices D_n are diagonal, the canonical basis is a normal basis with respect to the sequence $D = (D_n)_{n\in\mathbb{N}}$ and the finite numbers $c_1, \ldots, c_q$ are the values of the Lyapunov exponent of this sequence. Therefore,

$$\sum_{i=1}^{q} c_i = \sum_{i=1}^{q} \limsup_{n\to\infty} \frac{1}{n} \log \prod_{l=1}^{n-1} |b_{ii}(l)| \ge \limsup_{n\to\infty} \frac{1}{n} \sum_{i=1}^{q} \log \prod_{l=1}^{n-1} |b_{ii}(l)|$$
$$= \limsup_{n\to\infty} \frac{1}{n} \log \prod_{i=1}^{q} \prod_{l=1}^{n-1} |b_{ii}(l)| = \limsup_{n\to\infty} \frac{1}{n} \log|\det \mathcal{D}_n|$$
$$\ge \liminf_{n\to\infty} \frac{1}{n} \log|\det \mathcal{D}_n| = \lim_{n\to\infty} \frac{1}{n} \log|\det \mathcal{B}_n| = \sum_{i=1}^{q} \lambda_i',$$

using (4.3) in the last line. It follows from (4.5) that

$$\liminf_{n\to\infty}\frac{1}{n}\log|\det \mathcal{D}_n| = \sum_{i=1}^{q} c_i = \sum_{i=1}^{q}\lambda_D(e_i)$$

and so the sequence $(D_n)_{n\in\mathbb{N}}$ is Lyapunov regular. Hence, it follows from property 4 in Theorem 3.3 that each number c_i is a limit. Together with (4.5) this establishes the last property in the theorem. $\square$

4.2 Lyapunov regularity and triangularization

Now we provide an even more detailed information on the relation between the Lyapunov regularity of a sequence of matrices and its reduction to sequences of upper-triangular matrices via Lyapunov coordinate changes.

We first introduce three classes of matrices:

1. $\mathcal{S}_1$ is the set of all sequences of invertible $q\times q$ matrices that are Lyapunov regular;

2. $\mathcal{S}_3$ is the set of all sequences of invertible $q\times q$ matrices $(A_n)_{n\in\mathbb{N}}$ such that after a reduction to a sequence of upper-triangular matrices $B_n = (b_{ij}(n))_{1\le i\le j\le q}$ via a Lyapunov coordinate change $(V_n)_{n\in\mathbb{N}}$ the limits in (4.1), that is,

$$d_i := \lim_{n\to\infty}\frac{1}{n}\log\prod_{l=1}^{n-1}|b_{ii}(l)|$$

   exist and are finite for $i=1,\dots,q$;

3. $\mathcal{S}_2$ is the set of all sequences of invertible $q\times q$ matrices $(A_n)_{n\in\mathbb{N}}\in\mathcal{S}_3$ such that, up to a permutation, the vector $(d_1,\dots,d_q)$ is the same for any Lyapunov coordinate change.

The following result clarifies the relation between these classes of matrices.

Theorem 4.2. *We have $\mathcal{S}_1\subset\mathcal{S}_2\subset\mathcal{S}_3\subset\mathcal{L}$ and these inclusions are proper.*

Proof. We divide the proof of the theorem into steps.

Step 1: Auxiliary results I

We start with two auxiliary results. We recall that all Gramians are nonnegative and that a Gramian $G(v_1,\dots,v_k)$ vanishes if and only if the vectors $v_1,\dots,v_k$ are linearly dependent.

Lemma 4.3. *If $(V_n)_{n\in\mathbb{N}}$ is a sequence of invertible $q\times q$ matrices and $B_n = V_{n+1}^{-1}A_nV_n$ is upper-triangular for each $n\in\mathbb{N}$, then for the vectors $v_i = V_1e_i$, for $i=1,\dots,q$, we have*

$$\frac{G(\mathcal{A}_nv_1,\dots,\mathcal{A}_nv_k)}{G(\mathcal{A}_nv_1,\dots,\mathcal{A}_nv_{k-1})} = \left(\prod_{l=1}^{n-1}b_{kk}(l)\right)^2\frac{G(V_ne_1,\dots,V_ne_k)}{G(V_ne_1,\dots,V_ne_{k-1})} \tag{4.6}$$

for $k=1,\dots,q$, where $B_n=(b_{ij}(n))_{1\le i\le j\le q}$.

Proof of the lemma. We have $V_n^{-1}\mathcal{A}_n V_1 = \mathcal{B}_n$ and so

$$\begin{aligned}\mathcal{A}_n V_1 &= V_n\mathcal{B}_n = (V_n e_1 \cdots V_n e_q)\mathcal{B}_n \\ &= (V_n e_1 \cdots V_n e_q)\begin{pmatrix} c_{11}(n) & c_{12}(n) & \cdots & c_{1q}(n) \\ 0 & c_{22}(n) & \cdots & c_{2q}(n) \\ 0 & 0 & \ddots & \vdots \\ 0 & 0 & 0 & c_{qq}(n), \end{pmatrix},\end{aligned}$$

where $\mathcal{B}_n = (c_{ij}(n))_{1\le i\le j\le q}$. In particular, we have $c_{ii}(n) = \prod_{l=1}^{n-1} b_{ii}(l)$ for each i. Now write $V_1 e_i = v_i$ for $i = 1,\dots,q$. Then

$$\mathcal{A}_n v_i = \mathcal{A}_n V_1 e_i = \sum_{j=1}^{i} c_{ji}(n) V_n e_j, \quad \text{for } i = 1,\dots,q.$$

Since the Gramian is the determinant of a matrix of inner products and $\mathcal{A}_n v_i - c_{ii}(n)V_n e_i$ is a linear combination of the vectors $V_n e_1,\dots,V_n e_{i-1}$, one can show that

$$G(\mathcal{A}_n v_1,\dots,\mathcal{A}_n v_k) = G\big(c_{11}(n)V_n e_1, c_{22}(n)V_n e_2,\dots,c_{kk}(n)V_n e_k\big). \tag{4.7}$$

For completeness we detail the argument. Consider the $q\times q$ matrix M with entries $m_{ij} = \langle \mathcal{A}_n v_i, \mathcal{A}_n v_j\rangle$. Multiplying the first column of M by $-c_{12}(n)/c_{11}(n)$ and adding it to the second column corresponds to replace the entries in this column by

$$\langle \mathcal{A}_n v_i, \mathcal{A}_n v_2\rangle + \left\langle \mathcal{A}_n v_i, -\frac{c_{12}(n)}{c_{11}(n)}\mathcal{A}_n v_1\right\rangle = \langle \mathcal{A}_n v_i, c_{22}(n)V_n e_2\rangle.$$

Similarly, multiplying the first row of M by $-c_{12}(n)/c_{11}(n)$ and adding it to the second row corresponds to replace the entries in this row by

$$\langle \mathcal{A}_n v_2, \mathcal{A}_n v_i\rangle + \left\langle -\frac{c_{12}(n)}{c_{11}(n)}\mathcal{A}_n v_1, \mathcal{A}_n v_i\right\rangle = \langle c_{22}(n)V_n e_2, \mathcal{A}_n v_i\rangle.$$

Now we apply successively these two operations to the matrix M, after which we apply successively similar operations to the remaining columns and rows. Namely, for $i = 3,\dots,q$ (in this order) we multiply each kth column with $k < i$ by $-c_{ki}(n)/c_{kk}(n)$ and we add it to the ith column. Then, for $i = 3,\dots,q$ (again in this order) we multiply each kth row with $k < i$ by $-c_{ki}/c_{kk}(n)$ and we add it to the ith row. After all these operations we obtain the matrix of inner products

$$\langle c_{ii}(n)V_n e_i, c_{jj}(n)V_n e_j\rangle.$$

Since none of the former operations changes the determinant, we obtain identity (4.7). Therefore,

$$G(\mathcal{A}_n v_1,\dots,\mathcal{A}_n v_k) = \left(\prod_{i=1}^{k}\prod_{l=1}^{n-1} b_{ii}(l)\right)^2 G(V_n e_1,\dots,V_n e_k). \tag{4.8}$$

Identity (4.6) follows now immediately from (4.8). □

Lemma 4.4. *A sequence of invertible $q\times q$ matrices $(V_n)_{n\in\mathbb{N}}$ is a Lyapunov coordinate change if and only if the sequence of matrices $A_n = V_{n+1}V_n^{-1}$ is Lyapunov regular and all values of its Lyapunov exponent on $\mathbb{R}^q\setminus\{0\}$ are zero. In this case we have*

$$\lim_{n\to\infty}\frac{1}{n}\log G(V_n e_1,\dots,V_n e_k) = 0, \quad \textit{for } k = 1,\dots,q. \tag{4.9}$$

Proof of the lemma. Assume that $(V_n)_{n\in\mathbb{N}}$ is a Lyapunov coordinate change. Then the matrices $A_n = V_{n+1}V_n^{-1}$ satisfy $V_{n+1}^{-1}A_nV_n = \mathrm{Id}$ and by Theorem 3.3 (see property 3), the sequence $(A_n)_{n\in\mathbb{N}}$ is Lyapunov regular. Moreover, the values of its Lyapunov exponent on $\mathbb{R}^q\setminus\{0\}$ are zero (because the constant diagonal matrix D is the identity matrix).

In the other direction, if the sequence of matrices $A_n = V_{n+1}V_n^{-1}$ is Lyapunov regular and all values of its Lyapunov exponent on $\mathbb{R}^q\setminus\{0\}$ are zero, then it follows from Theorem 3.3 that there exists a Lyapunov coordinate change $(U_n)_{n\in\mathbb{N}}$ such that $U_{n+1}^{-1}A_nU_n = D$ is a fixed diagonal matrix for $n\in\mathbb{N}$, with entries ±1 in the main diagonal. Therefore,

$$\mathcal{A}_n = V_nV_1^{-1} = U_nD^{n-1}U_1^{-1}.$$

Since $(U_n)_{n\in\mathbb{N}}$ is a Lyapunov coordinate change, the same is true for the sequence

$$V_n = U_nD^{n-1}U_1^{-1}V_1.$$

Now we establish the last statement in the lemma. It follows from Theorem 1.3.11 in [6] that for $k=1,\dots,q$ the limit

$$\lim_{n\to\infty}\frac{1}{n}\log G(V_ne_1,\dots,V_ne_k)$$

exists and is equal to a sum of values of the Lyapunov exponent of the sequence $A_n = V_{n+1}^{-1}V_n$. Since all these values are zero, property (4.9) holds. □

Step 2: Auxiliary results II

Now we use the former results to show that the limits in (1.7) can be obtained considering smaller classes of Lyapunov coordinate changes. This will be crucial later on in the proof of the theorem.

Let $\mathcal{E}$ be the set of all Lyapunov coordinate changes $(V_n)_{n\in\mathbb{N}}$ such that

$$V_{n+1}^{-1}A_nV_n = B_n$$

is upper-triangular for all $n\in\mathbb{N}$. On the other hand, given a normal basis $v_1,\dots,v_q$ for $\mathbb{R}^q$ with respect to the sequence $(A_n)_{n\in\mathbb{N}}$, let U_n be the orthogonal matrix whose columns are obtained applying the Gram–Schmidt process to the basis $\mathcal{A}_nv_1,\dots,\mathcal{A}_nv_q$. Then

$$U_{n+1}^{-1}A_nU_n = C_n \tag{4.10}$$

is upper-triangular for all $n\in\mathbb{N}$ (see [7, Theorem 7]) and so the set $\mathcal{F}$ of all such sequences of orthogonal matrices $(U_n)_{n\in\mathbb{N}}$ satisfies $\mathcal{F}\subset\mathcal{E}$. We write

$$B_n = (b_{ij}(n))_{1\le i\le j\le q} \quad\text{and}\quad C_n = (c_{ij}(n))_{1\le i\le j\le q}.$$

Finally, let

$$\underline{b}_V = (\underline{b}_1,\dots,\underline{b}_q),\quad \overline{b}_V = (\overline{b}_1,\dots,\overline{b}_q),$$

where

$$\underline{b}_i = \liminf_{n\to\infty}\frac{1}{n}\log\prod_{l=1}^{n-1}|b_{ii}(l)|,\quad \overline{b}_i = \limsup_{n\to\infty}\frac{1}{n}\log\prod_{l=1}^{n-1}|b_{ii}(l)|,$$

and let

$$\underline{c}_U = (\underline{c}_1,\dots,\underline{c}_q),\quad \overline{c}_U = (\overline{c}_1,\dots,\overline{c}_q),$$

where

$$\underline{c}_i = \liminf_{n\to\infty}\frac{1}{n}\log\prod_{l=1}^{n-1}|c_{ii}(l)|,\quad \overline{c}_i = \limsup_{n\to\infty}\frac{1}{n}\log\prod_{l=1}^{n-1}|c_{ii}(l)|. \tag{4.11}$$

Lemma 4.5. *For a sequence of invertible $q \times q$ matrices $A = (A_n)_{n\in\mathbb{N}} \in \mathcal{L}$ we have*

$$\{(\underline{b}_V, \overline{b}_V) : V \in \mathcal{E}\} = \{(\underline{c}_U, \overline{c}_U) : U \in \mathcal{F}\}.$$

Proof of the lemma. Since $\mathcal{F} \subset \mathcal{E}$, we have

$$\{(\underline{b}_V, \overline{b}_V) : V \in \mathcal{E}\} \supset \{(\underline{c}_U, \overline{c}_U) : U \in \mathcal{F}\}.$$

For the reverse inclusion, take $(V_n)_{n\in\mathbb{N}} \in \mathcal{E}$ and let $v_1, \ldots, v_q$ be the columns of the matrix V_1. By Lemmas 4.3 and 4.4 we obtain

$$\limsup_{n\to\infty} \frac{1}{n} \log \frac{G(\mathcal{A}_n v_1, \ldots, \mathcal{A}_n v_i)}{G(\mathcal{A}_n v_1, \ldots, \mathcal{A}_n v_{i-1})} = 2 \limsup_{n\to\infty} \frac{1}{n} \log \prod_{l=1}^{n-1} |b_{ii}(l)| \tag{4.12}$$

and

$$\liminf_{n\to\infty} \frac{1}{n} \log \frac{G(\mathcal{A}_n v_1, \ldots, \mathcal{A}_n v_i)}{G(\mathcal{A}_n v_1, \ldots, \mathcal{A}_n v_{i-1})} = 2 \liminf_{n\to\infty} \frac{1}{n} \log \prod_{l=1}^{n-1} |b_{ii}(l)|. \tag{4.13}$$

Now observe that there exists an upper-triangular matrix B with unit diagonal such that the columns $u_1, \ldots, u_q$ of $V_1 B$ form a normal basis with respect to A. Let U_n be the matrix whose columns are obtained applying the Gram–Schmidt process to the basis $\mathcal{A}_n u_1, \ldots, \mathcal{A}_n u_q$. Then $(U_n)_{n\in\mathbb{N}} \in \mathcal{F}$ (see [7, Theorem 7]). Moreover, let $w_i = U_1 e_i$ be the columns of U_1, for $i = 1, \ldots, q$. Again by Lemmas 4.3 and 4.4 we obtain

$$\limsup_{n\to\infty} \frac{1}{n} \log \frac{G(\mathcal{A}_n w_1, \ldots, \mathcal{A}_n w_i)}{G(\mathcal{A}_n w_1, \ldots, \mathcal{A}_n w_{i-1})} = 2 \limsup_{n\to\infty} \frac{1}{n} \log \prod_{l=1}^{n-1} |c_{ii}(l)| \tag{4.14}$$

and

$$\liminf_{n\to\infty} \frac{1}{n} \log \frac{G(\mathcal{A}_n w_1, \ldots, \mathcal{A}_n w_i)}{G(\mathcal{A}_n w_1, \ldots, \mathcal{A}_n w_{i-1})} = 2 \liminf_{n\to\infty} \frac{1}{n} \log \prod_{l=1}^{n-1} |c_{ii}(l)|. \tag{4.15}$$

On the other hand, using the properties of the Gramian, one can show that

$$G(\mathcal{A}_n w_1, \ldots, \mathcal{A}_n w_i) = \rho_i G(\mathcal{A}_n u_1, \ldots, \mathcal{A}_n u_i) = \rho_i G(\mathcal{A}_n v_1, \ldots, \mathcal{A}_n v_i)$$

for some constants ρ_i independent of n, for $i = 1, \ldots, q$. This follows as in the proof of Lemma 4.3. Indeed, note that

$$u_i = v_i + \sum_{j=1}^{i-1} b_{ij} v_j,$$

denoting by b_{ij} the entries of B. Since $\mathcal{A}_n u_i - \mathcal{A}_n v_i$ is a linear combination of the vectors $b_{ij}\mathcal{A}_n v_j$ with $j < i$, we obtain

$$G(\mathcal{A}_n u_1, \ldots, \mathcal{A}_n u_i) = G(\mathcal{A}_n v_1, \ldots, \mathcal{A}_n v_i).$$

Similarly, since

$$w_i = \sum_{j=1}^{i} c_{ij} u_j = c_{ii} u_i + \sum_{j=1}^{i-1} c_{ij} u_j$$

for some constants c_{ij} with $c_{ii} \neq 0$, it follows as before that

$$\begin{aligned} G(\mathcal{A}_n w_1, \ldots, \mathcal{A}_n w_i) &= G(c_{11}\mathcal{A}_n u_1, \ldots, c_{ii}\mathcal{A}_n u_i) \\ &= \prod_{j=1}^{i} |c_{jj}|^2 G(\mathcal{A}_n u_1, \ldots, \mathcal{A}_n u_i). \end{aligned}$$

This shows that

$$\rho_i = \prod_{j=1}^{i} |c_{jj}|^2.$$

Hence, it follows from (4.12) and (4.13) together with (4.14) and (4.15) that for these particular sequences we have $\underline{c}_U = \underline{b}_V$ and $\overline{c}_U = \overline{b}_V$. Therefore,

$$\{(\underline{b}_V, \overline{b}_V) : V \in \mathcal{E}\} \subset \{(\underline{c}_U, \overline{c}_U) : U \in \mathcal{F}\}$$

and the lemma is proved. □

Now we turn to the proof of the statement in the theorem. We consider each inclusion separately.

Step 3: $\mathcal{S}_1 \subset \mathcal{S}_2$

This inclusion is the content of Theorem 4.1. To show that it is strict, for each $n \geq 1$ let

$$A_n = \operatorname{diag}\left(\begin{pmatrix} e & e^n \\ 0 & 1 \end{pmatrix}, \mathrm{Id}_{q-2} \right),$$

where Id_{q-2} denotes the $(q-2) \times (q-2)$ identity matrix. Then

$$\mathcal{A}_n = \operatorname{diag}\left(\begin{pmatrix} e^{n-1} & (n-1)e^{n-1} \\ 0 & 1 \end{pmatrix}, \mathrm{Id}_{q-2} \right). \tag{4.16}$$

Clearly, the values of the Lyapunov exponent are limits and are equal to 1 (with multiplicity 2) and 0 (with multiplicity $q-2$). The sum of these values is 2 while

$$\lim_{n\to\infty} \frac{1}{n} \log|\det \mathcal{A}_n| = 1$$

and so $A \notin \mathcal{S}_1$. It remains to show that $A \in \mathcal{S}_2$.

In view of Lemma 4.5, it suffices to show that the limits d_i in (4.1) exist, are finite, and that up to a permutation the vector $(d_1, \ldots, d_q)$ is the same, considering instead of general upper-triangular matrices B_n only those upper-triangular matrices C_n as in (4.10) obtained from a normal basis $v_1, \ldots, v_q$ or, without loss of generality, from a normal orthonormal basis $u_1, \ldots, u_q$ (it is easy to verify that when the limits in (4.1) exist for some matrices C_n they also exist for any particular matrices obtained from a normal orthonormal basis). More precisely, when computing the numbers d_i, instead of considering $\mathcal{B}_n$ we can consider the matrices $U_n^{-1}\mathcal{A}_n U_1$, where U_n is the orthogonal matrix whose columns are obtained applying the Gram–Schmidt process to the basis $\mathcal{A}_n u_1, \ldots, \mathcal{A}_n u_q$. Moreover, in view of Lemma 4.4 we may simply consider the matrices $\mathcal{A}_n U_1$, where U_1 is the matrix with columns $u_1, \ldots, u_q$.

A normal basis $u_1, \ldots, u_q$ with respect to A has $q-2$ vectors that are in

$$E = \operatorname{span}\{e_3, \ldots, e_q\} \tag{4.17}$$

and vectors $x_i = (c_1^i, \ldots, c_q^i)^*$, for $i = 1, 2$, where c_j^i are constants such that $\theta = c_1^1 c_2^2 - c_2^1 c_1^2 \neq 0$. For simplicity of the notation, we shall write

$$u_i(n) = \mathcal{A}_n u_i \quad \text{and} \quad x_i(n) = \mathcal{A}_n x_i.$$

In particular,

$$x_i(n) = \left((c_1^i + c_2^i(n-1))e^{n-1}, c_2^i, \ldots, c_q^i\right)^*.$$

Without loss of generality, we assume that $u_l = x_1$ and $u_m = x_2$ for some $l < m$. Since c_1^i and c_2^i cannot be zero simultaneously, there exists $D \geq 1$ such that

$$D^{-1}e^n \leq \|x_i(n)\| \leq Dne^n, \quad \text{for } n \in \mathbb{N}, i = 1, 2. \tag{4.18}$$

Before proceeding we establish two auxiliary results.

Lemma 4.6. *There exists* $C > 0$ *such that*

$$\alpha_n := \angle(x_2(n), x_1(n)) \leq Ce^{-n}, \quad \text{for } n \in \mathbb{N}. \tag{4.19}$$

Proof of the lemma. The triangle inequality for a trihedral angle says that

$$\langle u, v \rangle \leq \angle(u, w) + \angle(w, v) \tag{4.20}$$

for any vectors $u, v, w \in \mathbb{R}^q \setminus \{0\}$ (this follows readily from considering the space spanned by u, v, w, and using the triangle inequality for a spherical triangle). Hence, letting $\alpha_i(n) = \angle(x_i(n), e_1)$ we obtain

$$\alpha_n \leq \angle(x_1(x), e_1) + \angle(x_2(n), e_1) = \alpha_1(n) + \alpha_2(n), \tag{4.21}$$

for $n \in \mathbb{N}$. On the other hand,

$$\cos \alpha_i(n) = \frac{\langle x_i(n), e_1 \rangle}{\|x_i(n)\|} = \frac{(c_1^i + c_2^i(n-1))e^{n-1}}{\sqrt{(c_1^i + c_2^i(n-1))^2 e^{2(n-1)} + \sum_{k=2}^q (c_k^i)^2}}$$

and so

$$\sin \alpha_i(n) = \sqrt{1 - \cos^2 \alpha_i(n)} = \frac{(\sum_{k=2}^q (c_k^i)^2)^{1/2}}{\|x_i(n)\|}.$$

By (4.18), there exists $K > 0$ such that

$$\sin \alpha_i(n) \leq Ke^{-n}, \quad \text{for } n \in \mathbb{N}, i = 1, 2.$$

Since $x/\sin x \to 1$ when $x \to 0$, this implies that there exists $K' > 0$ such that $\alpha_i(n) \leq K'e^{-n}$ for all $n \in \mathbb{N}$ and $i = 1, 2$. Hence, it follows from (4.21) that property (4.19) holds. □

Furthermore, since $\alpha_i(n) = \angle(x_i(n), e_1) \to 0$ when $n \to \infty$ and $e_1 \perp E$ with E as in (4.17), there exists $K_1 > 0$ such that

$$\angle(x_1(n), W) \geq \angle(x_1(n), E) > K_1 \tag{4.22}$$

for any $n \in \mathbb{N}$ and any subspace $W \neq \{0\}$ of E.

Now let $W(n)$ be the space generated by the set

$$\{u_1(n), \ldots, u_q(n)\} \setminus \{u_m(n)\}.$$

Lemma 4.7. *There exists* $K_2 > 0$ *such that*

$$\beta_n := \angle(x_2(n), W(n)) \geq K_2 n^{-2} e^{-n}, \quad \text{for } n \in \mathbb{N}. \tag{4.23}$$

Proof of the lemma. Consider the vector

$$w(n) = \left(-c_2^1, (c_1^1 + c_2^1(n-1))e^{n-1}, 0, \dots, 0\right)^*$$

that is orthogonal to $W(n)$. Since $\dim W(n) = q-1$ for all $n \in \mathbb{N}$, we have

$$\beta_n = \frac{\pi}{2} - \gamma_n, \quad \text{where } \gamma_n = \angle(x_2(n), w(n)).$$

Moreover, since $\langle x_2(n), w(n)\rangle = \theta e^{n-1}$ with $\theta = c_1^1 c_2^2 - c_2^1 c_1^2$, we obtain

$$\sin\beta_n = \frac{|\theta| e^{n-1}}{\|x_2(n)\| \cdot \|w(n)\|}.$$

Clearly, $\|w(n)\| \le K_3 n e^n$ for some constant $K_3 > 0$. Hence, by (4.18), there exists $K_4 > 0$ such that

$$\sin\beta_n \ge K_4 n^{-2} e^{-n}, \quad \text{for } n \in \mathbb{N},$$

thus yielding property (4.23). □

Given a finite set $R \subset \{v_1, \dots, v_q\}$, we denote by $V_R(n)$ the vector space spanned by the vectors $\mathcal{A}_n v$ with $v \in R$ and by $\Gamma_R(n)$ the square root of the Gramian of the vectors $\mathcal{A}_n v$ with $v \in R$. We use the former lemmas to estimate the Gramians $\Gamma_R(n)$ for some subsets of the basis. Then identity (4.6) together with Lemma 4.4 will allow us to compute the limits d_i in (4.1). Let

$$R_k = \{u_1, \dots, u_k\} \setminus \{u_l, u_m\}$$

(recall that $u_l = x_1$ and $u_m = x_2$). Moreover, let

$$S_k = R_k \cup \{x_1\} \quad \text{and} \quad T_k = R_k \cup \{x_1, x_2\} \tag{4.24}$$

for $k = 1, \dots, q$. Note that $W(n) = V_{R_q}(n)$.

Lemma 4.8. *There exist $D_1, D_2 > 0$ such that for each $k = 1, \dots, q$ and $n \in \mathbb{N}$ we have*

$$D_1^{-1} e^n \le \Gamma_{S_k}(n) \le D_1 n e^n \tag{4.25}$$

and

$$D_2^{-1} n^{-2} e^n \le \Gamma_{T_k}(n) \le D_2 n^2 e^n. \tag{4.26}$$

Proof of the lemma. Since $R_k \subset E$ (see (4.17)), we have $\Gamma_{R_k}(n) = \Gamma_{R_k}(1)$ for all n and it follows from (4.22) that

$$\angle(x_1(n), V_{R_k}(n)) \ge K_1, \quad \text{for } n \in \mathbb{N}. \tag{4.27}$$

On the other hand, by (2.1) we have

$$\begin{aligned}\Gamma_{S_k}(n) &= \Gamma_{R_k}(n)\sqrt{G(x_1(n))}\sin\angle(x_1(n), V_{R_k}(n)) \\ &= \Gamma_{R_k}(1)\|x_1(n)\|\sin\angle(x_1(n), V_{R_k}(n)),\end{aligned}$$

and so in view of (4.18) and (4.27) there exists $D_1 \ge 1$ such that

$$D_1^{-1} e^n \le \Gamma_{S_k}(n) \le D_1 n e^n, \quad \text{for } n \in \mathbb{N}. \tag{4.28}$$

Similarly, by (2.1) we have

$$\Gamma_{T_k}(n) = \Gamma_{S_k}(n)\|x_2(n)\| \sin \angle(x_2(n), V_{S_k}(n)).$$

Since $V_{S_k}(n) \subset W(n)$ and $x_1(n) \in W(n)$, it follows readily from the definitions of α_n and β_n in (4.19) and (4.23) that

$$\angle(x_2(n), V_{S_k}(n)) \geq \beta_n \quad \text{and} \quad \angle(x_2(n), V_{S_k}(n)) \leq \alpha_n.$$

Again by (4.19) and (4.23) this implies that

$$K_2 n^{-2} e^{-n} \leq \angle(x_2(n), V_{S_k}(n)) \leq Ce^{-n}$$

for $n \in \mathbb{N}$. Hence, by (4.18) and (4.28) there exists $D_2 \geq 1$ such that

$$D_2^{-1} n^{-2} e^n \leq \Gamma_{T_k}(n) \leq D_2 n^2 e^n, \quad \text{for } n \in \mathbb{N}.$$

This completes the proof of the lemma. □

Now let

$$\Gamma_k(n) = \sqrt{G(\mathcal{A}_n u_1, \dots, \mathcal{A}_n u_k)}, \quad \text{for } k = 1, \dots, q.$$

For $k < l$ we have $u_1, \dots, u_k \in E$ (see (4.17)) and so it follows from the form of the matrix $\mathcal{A}_n$ in (4.16) that $\mathcal{A}_n u_i = u_i$ for $i \leq k$. Therefore,

$$\Gamma_k(n) = \sqrt{G(u_1, \dots, u_k)} = \Gamma_k(1).$$

On the other hand, it follows from the definition of S_k and T_k in (4.24) that

$$\Gamma_k(n) = \Gamma_{S_k}(n), \quad \text{for } k = l, \dots, m-1,$$

and

$$\Gamma_k(n) = \Gamma_{T_k}(n), \quad \text{for } k = m, \dots, q.$$

Summing up, we have

$$\Gamma_k(n) = \begin{cases} \Gamma_k(1) & \text{if } k = 1, \dots, l-1, \\ \Gamma_{S_k}(n) & \text{if } k = l, \dots, m-1, \\ \Gamma_{T_k}(n) & \text{if } k = m, \dots, q. \end{cases}$$

In particular, by (4.25) and (4.26) we obtain

$$D_1^{-1} e^n \leq \Gamma_k(n) \leq D_1 n e^n$$

for $k = l, \dots, m-1$ and

$$D_2^{-1} n^{-2} e^n \leq \Gamma_k(n) \leq D_2 n^2 e^n$$

for $k = m, \dots, q$, which readily implies that

$$\lim_{n\to\infty} \frac{1}{n} \log \frac{\Gamma_k(n)}{\Gamma_{k-1}(n)} = \begin{cases} 0 & \text{if } k \neq l, \\ 1 & \text{if } k = l. \end{cases}$$

Hence, it follows from Lemma 4.3 that

$$\underline{c}_i = \overline{c}_i = \lim_{n\to\infty} \frac{1}{n} \log \prod_{l=1}^{n-1} |c_{ii}(l)| = \begin{cases} 0 & \text{if } k \neq l, \\ 1 & \text{if } k = l \end{cases}$$

for $i = 1, \dots, q$ (see (4.11)). As detailed in the beginning of Step 3, in view of Lemma 4.5 this readily implies that $A \in \mathcal{S}_2$.

Step 4: $\mathcal{S}_2 \subset \mathcal{S}_3$

This inclusion is clear from the definitions of the sets $\mathcal{S}_2$ and $\mathcal{S}_3$. To show that it is strict, for $n \geq 1$ let

$$A_n = \begin{pmatrix} 1 & 0_{q-2} & e^n - e^{n-1} \\ 0^*_{q-2} & \mathrm{Id}_{q-2} & 0^*_{q-2} \\ 0 & 0_{q-2} & 1 \end{pmatrix},$$

where 0_{q-2} denotes the $(q-2)$-vector $(0, \ldots, 0)$ and 0^*_{q-2} denotes its transpose. Then

$$\mathcal{A}_n = \begin{pmatrix} 1 & 0_{q-2} & e^{n-1} - 1 \\ 0^*_{q-2} & \mathrm{Id}_{q-2} & 0^*_{q-2} \\ 0 & 0_{q-2} & 1 \end{pmatrix}. \tag{4.29}$$

Clearly, the values of the Lyapunov exponent are 1 (with multiplicity 1) and 0 (with multiplicity $q-1$). We will show that for any reduction by a Lyapunov coordinate change to a sequence of upper-triangular matrices B_n the limits

$$d_i := \lim_{n\to\infty} \frac{1}{n} \log \prod_{l=1}^{n-1} |b_{ii}(l)|, \quad \text{for } i = 1, \ldots, q,$$

exist and are finite, but consist of either q zeros or $q-2$ zeros, 1 and -1. Moreover, we will show that both possibilities occur, and so $A \notin \mathcal{S}_2$. It remains to show that $A \in \mathcal{S}_3$.

As in Step 3, in view of Lemma 4.5, it suffices to show that the limits d_i in (4.1) exist, are finite, and that up to a permutation the vector $(d_1, \ldots, d_q)$ is the same, considering instead of general upper-triangular matrices B_n only those upper-triangular matrices C_n as in (4.10) obtained from a normal basis $v_1, \ldots, v_q$. Moreover, in view of Lemma 4.4 we may simply consider the matrices $\mathcal{A}_n V_1$, where V_1 is the matrix with columns $v_1, \ldots, v_q$.

We start with an auxiliary result. We shall write $\varphi_n \approx \psi_n$ if there exist $C_1, C_2 > 0$ such that

$$C_1 \varphi_n \leq \psi_n \leq C_2 \varphi_n$$

for all $n \in \mathbb{N}$. Let

$$F = \mathrm{span}\{e_1, \ldots, e_{q-1}\} \tag{4.30}$$

and

$$x_n = \mathcal{A}_n (c_1, \ldots, c_q)^* = \big(c_1 + c_q(e^{n-1} - 1), c_2, \ldots, c_q\big)^*, \tag{4.31}$$

with $c_1, \ldots, c_{q-1} \in \mathbb{R}$ and $c_q \neq 0$.

Lemma 4.9. *If $V, W \subset F$ are vector spaces such that $e_1 \notin V$ and $e_1 \in W$, then there exists a constant $D_V > 0$ such that*

$$\angle(x_n, V) \geq D_V \quad \text{and} \quad \angle(x_n, W) \approx e^{-n}. \tag{4.32}$$

Proof of the lemma. We have

$$\cos \angle(x_n, e_1) = \frac{\langle x_n, e_1 \rangle}{\|x_n\|} = \frac{c_1 + c_q(e^{n-1} - 1)}{\sqrt{(c_1 + c_q(e^{n-1} - 1))^2 + \sum_{i=2}^q c_i^2}}$$

and

$$\sin \angle(x_n, e_1) = \frac{(\sum_{i=2}^q c_i^2)^{1/2}}{\sqrt{(c_1 + c_q(e^{n-1} - 1))^2 + \sum_{i=2}^q c_i^2}}.$$

Since

$$(c_1 + c_q(e^{n-1} - 1))^2 + \sum_{i=2}^{q} c_i^2 = c_q^2 e^{2(n-1)} a_n$$

for some sequence $a_n \to 1$ when $n \to \infty$, we obtain

$$\angle(x_n, e_1) \approx e^{-n}. \tag{4.33}$$

By (4.20) we have

$$\angle(e_1, v) \leq \angle(x_n, e_1) + \angle(x_n, v)$$

for all $v \in V$ and so

$$\angle(e_1, v) = \inf_{v \in V} \angle(e_1, v) \leq \angle(x_n, e_1) + \angle(x_n, V).$$

This implies that there exists $\tilde{c} > 0$ such that

$$\angle(x_n, V) \geq \angle(e_1, V) - \angle(x_n, e_1) \geq \angle(e_1, V) - \tilde{c}e^{-n}. \tag{4.34}$$

Since $\angle(e_1, V) \neq 0$, one can take $p \in \mathbb{N}$ such that

$$\angle(e_1, v) - \tilde{c}e^{-n} \geq \frac{1}{2}\angle(e_1, v), \quad \text{for } n \geq p.$$

Therefore, by (4.34), we obtain

$$\angle(x_n, v) \geq \min\left\{\min_{m \leq p} \angle(x_m, V), \frac{1}{2}\angle(e_1, v)\right\} =: D_V > 0. \tag{4.35}$$

Moreover, since $\angle(x_n, W) \leq \angle(x_n, e_1)$, it follows from (4.33) that there exists $C_1 > 0$ such that

$$\angle(x_n, W) \leq C_1 e^{-n}, \quad \text{for } n \in \mathbb{N}.$$

Finally, since $\angle(x_n, W) \geq \angle(x_n, F)$ with F as in (4.30), it follows from (4.33) and (4.35) with

$$V = \operatorname{span}\{e_2, \ldots, e_{q-1}\}$$

that $\angle(x_n, F) \geq C_2 e^{-n}$ for some $C_2 > 0$. This establishes property (4.32). □

Note that any normal basis with respect to A is of the form $v_1, \ldots, v_q$, with all vectors but one in the space F in (4.30). Assume that v_l is equal to the vector $x_1 = (c_1, \ldots, c_q)^*$ in (4.31). Note that $x_1 \notin F$ since $c_q \neq 0$. Let $V_{l-1} = \operatorname{span}\{v_1, \ldots, v_{l-1}\}$ and

$$V_k = \operatorname{span}\{v_1, \ldots, v_{l-1}, v_{l+1}, \ldots, v_k\}$$

for $k \geq l + 1$. Moreover, let

$$\Gamma_k(n) = \sqrt{G(\mathcal{A}_n v_1, \ldots, \mathcal{A}_n v_k)}$$

for $k \leq l$ and

$$\Gamma'_k(n) = \sqrt{G(\mathcal{A}_n v_1, \ldots, \mathcal{A}_n v_{l-1}, \mathcal{A}_n v_{l+1}, \ldots, \mathcal{A}_n v_k)}$$

for $k \geq l + 1$.

Lemma 4.10. *If $e_1 \in V_{l-1}$, then*

$$\lim_{n \to \infty} \frac{1}{n} \log \frac{\Gamma_k(n)}{\Gamma_{k-1}(n)} = 0, \quad \textit{for } k = 1, \ldots, q. \tag{4.36}$$

Proof of the lemma. For $k = 1, \dots, l-1$, the volume $\Gamma_k(n)$ is independent of n since the same is true for $\mathcal{A}_n v_i$ for $i \neq l$ (then $v_i \in F$ and so it follows from the form for $\mathcal{A}_n$ in (4.29) that $\mathcal{A}_n v_i = v_i$ for all n). Therefore,

$$\Gamma_k(n) = \Gamma_k(1) \quad \text{for } k = 1, \dots, l-1. \tag{4.37}$$

On the other hand, by (2.1) we have

$$\Gamma_l(n) = \Gamma_{l-1}(n) \|x_n\| \sin \angle(x_n, V_{l-1}). \tag{4.38}$$

Moreover, by (4.31) we have $\|x_n\| \approx e^n$ and it follows from (4.32) that

$$\sin \angle(x_n, V_{l-1}) \approx e^{-n}$$

(because $e_1 \in V_{l-1}$). Hence, there exist constants $c, d > 0$ such that

$$ce^n \leq \|x_n\| \leq de^n, \quad ce^{-n} \leq \sin \angle(x_n, V_{l-1}) \leq de^{-n},$$

and so it follows from (4.38) that

$$c^2 \Gamma_{l-1}(1) \leq \Gamma_l(n) \leq \Gamma_{l-1}(1) d^2 \tag{4.39}$$

for all $n \in \mathbb{N}$, that is, $\Gamma_l(n) \approx 1$.

In a similar manner, by (2.1) we have

$$\Gamma_k(n) = \Gamma'_k(n) \|x_n\| \sin \angle(x_n, V_k)$$

for $k \geq l+1$. As in (4.37), for $i \neq l$ we have $v_i \in F$ and so it follows from (4.29) that $\mathcal{A}_n v_i = v_i$, which implies that $\Gamma'_k(n) = \Gamma'_k(1)$. Since $\|x_n\| \approx e^n$ and

$$\sin \angle(x_n, V_k) \approx e^{-n}$$

(in view of (4.32), because $e_1 \in V_k$), it follows as in (4.39) that $\Gamma_k(n) \approx 1$.

Summing up, we showed that $\Gamma_k(n) = \Gamma_k(1)$ for $k < l$ and that $\Gamma_k(n) \approx 1$ for $k \geq l$. Hence, there exist constants $c, d > 0$ such that

$$c \leq \Gamma_k(n) \leq d, \quad \text{for } n \in \mathbb{N}, k = 1, \dots, q.$$

This readily yields property (4.36). □

Now we consider the complimentary case.

Lemma 4.11. *If $e_1 \notin V_{l-1}$, then there exists $m > l$ such that*

$$\lim_{n \to \infty} \frac{1}{n} \log \frac{\Gamma_k(n)}{\Gamma_{k-1}(n)} = \begin{cases} 0 & \text{if } k \neq l \text{ and } k \neq m, \\ 1 & \text{if } k = l, \\ -1 & \text{if } k = m \end{cases} \tag{4.40}$$

for $k = 1, \dots, q$.

Proof of the lemma. First note that there exists $m \geq l+1$ such that $e_1 \in V_m$ and $e_1 \notin V_{m-1}$. As in the proof of Lemma 4.10, we have $\Gamma_k(n) \approx 1$ for $k < l$ and by (2.1) we obtain

$$\Gamma_l(n) = \begin{cases} \|x_n\| & \text{if } l = 1, \\ \Gamma_{l-1}(n)\|x_n\| \sin \angle(x_n, V_{l-1}) & \text{if } l > 1. \end{cases}$$

By (4.31) we obtain $\|x_n\| \approx e^n$ and so it follows from (4.32) that $\Gamma_l(n) \approx e^n$. Indeed, for $l = 1$ we have $\Gamma_l(n) = \|x_n\| \approx e^n$. For $l > 1$, by (4.37) we have $\Gamma_{l-1}(n) = \Gamma_{l-1}(1)$. Moreover,

$$ce^n \leq \|x_n\| \leq de^n \tag{4.41}$$

for some constants $c, d > 0$ and since $e_1 \notin V_{l-1}$, it follows from (4.32) that

$$D_{V_{l-1}} \leq \angle(x_n, V_{l-1}) \leq \frac{\pi}{2}.$$

Hence,

$$\Gamma_{l-1}(1)cD_{V_{l-1}}e^n \leq \Gamma_l(n) \leq \Gamma_{l-1}(1)\frac{\pi d}{2}de^n$$

and so $\Gamma_l(n) \approx e^n$.

Finally, we have

$$\Gamma_k(n) = \Gamma'_k(n)\|x_n\| \sin \angle(x_n, V_k), \quad \text{for } k \geq l+1.$$

Since $e_1 \in V_m$ but $e_1 \notin V_{m-1}$, it follows from (4.32) that

$$\angle(x_n, V_k) \approx \begin{cases} 1 & \text{if } k < m, \\ e^{-n} & \text{if } k \geq m. \end{cases}$$

On the other hand, we have $\Gamma'_k(n) = \Gamma'_k(1)$ for $k \geq l+1$ and so it follows from (4.41) that

$$\Gamma_k(n) \approx e^n, \quad \text{for } l+1 \leq k \leq m-1$$

and that $\Gamma_k(n) \approx 1$ for $k \geq m$. Summing up, we have

$$\Gamma_k(n) \approx \begin{cases} 1 & \text{if } k \notin [l, m), \\ e^n & \text{if } k \in [l, m) \end{cases}$$

and so

$$\frac{\Gamma_k(n)}{\Gamma_{k-1}(n)} \approx \begin{cases} 1 & \text{if } k \neq l \text{ and } k \neq m, \\ e^n & \text{if } k = l, \\ e^{-n} & \text{if } k = m. \end{cases}$$

This readily yields property (4.40). □

Proceeding as in Step 3, it follows from Lemma 4.3 together with properties (4.36) and (4.40) that $A \in \mathcal{S}_3$.

Step 5: $\mathcal{S}_3 \subset \mathcal{L}$

Assume that $(A_n)_{n\in\mathbb{N}} \in \mathcal{S}_3$. Given a vector $v \neq 0$, let $v_1, \dots, v_q$ be a basis for $\mathbb{R}^q$ with $v_1 = v$. Moreover, let V_n be the matrix whose columns are obtained applying the Gram–Schmidt process to the basis $\mathcal{A}_n v_1, \dots, \mathcal{A}_n v_q$. Then $B_n = V_{n+1}^{-1} A_n V_n$ is upper-triangular for all $n \in \mathbb{N}$. We have $V_1 e_1 = v/\|v\|$ and so

$$\begin{aligned}\frac{\|\mathcal{A}_n v\|}{\|v\|} &= \|\mathcal{A}_n V_1 e_1\| = \|V_n^{-1}\mathcal{A}_n V_1 e_1\| \\ &= \|\mathcal{B}_n e_1\| = \prod_{l=1}^{n-1} |b_{11}(l)|.\end{aligned}$$

Hence,

$$\lambda_A(v) = \lim_{n\to\infty} \frac{1}{n} \log \prod_{l=1}^{n-1} |b_{11}(l)| < +\infty,$$

which shows that $\mathcal{S}_3 \subset \mathcal{L}$.

To show that the inclusion is strict, for $n \geq 1$ let

$$A_n = \begin{pmatrix} e^{n\sin n - (n-1)\sin(n-1)} & 0_{q-1} \\ 0^*_{q-1} & \mathrm{Id}_{q-1} \end{pmatrix}.$$

Then

$$\mathcal{A}_n = \begin{pmatrix} e^{(n-1)\sin(n-1)} & 0_{q-1} \\ 0^*_{q-1} & \mathrm{Id}_{q-1} \end{pmatrix}.$$

Clearly, the values of the Lyapunov exponents are finite. Moreover, they are equal to 1 (with multiplicity 1) and 0 (with multiplicity $q-1$). In particular, $A \in \mathcal{L}$. On the other hand (A_n is triangular itself, so $(U_n)_{n\in\mathbb{N}} = \mathrm{Id}$ is a Lyapunov coordinate change) the limit

$$\begin{aligned}\lim_{n\to\infty} \frac{1}{n} \log \prod_{l=1}^{n-1} |a_{11}(l)| &= \lim_{n\to\infty} \frac{1}{n} \log e^{(n-1)\sin(n-1)} \\ &= \lim_{n\to\infty} \frac{(n-1)\sin(n-1)}{n}\end{aligned}$$

does not exist and so the sequence $A = (A_n)_{n\in\mathbb{N}} \notin \mathcal{S}_3$. This concludes the proof of the theorem. □

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