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Dynamics of a Leslie–Gower predator–prey system with cross-diffusion

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Abstract. A Leslie–Gower predator–prey system with cross-diffusion subject to Neumann boundary conditions is considered. The global existence and boundedness of solutions are shown. Some sufficient conditions ensuring the existence of nonconstant solutions are obtained by means of the Leray–Schauder degree theory. The local and global stability of the positive constant steady-state solution are investigated via eigenvalue analysis and Lyapunov procedure. Based on center manifold reduction and normal form theory, Hopf bifurcation direction and the stability of bifurcating time-periodic solutions are investigated and a normal form of Bogdanov–Takens bifurcation is determined as well.

Keywords: cross-diffusion, predator–prey system, global existence, stability, Hopf bifurcation, Bogdanov–Takens bifurcation.

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1 Introduction

In ecological systems, the interaction of predator and prey has abundant dynamical features although the investigations on predator-prey models has improved and lasted for several decades, which are based on the pioneering works of Lotka and Volterra [34]. Moreover, more realistic models are proposed in view of laboratory experiments and observations. Leslie and Gower [17] first proposed the following predator–prey model

$$\begin{cases} \dfrac{\mathrm{d}u}{\mathrm{d}t} = u(a_1 - u - c_1 v), \\ \dfrac{\mathrm{d}v}{\mathrm{d}t} = v\left(b_1 - \dfrac{d_1 v}{u}\right), \end{cases} \tag{1.1}$$

where $u(t)$ and $v(t)$ represent the densities of prey and predators at time t, respectively; the parameters a_1, b_1 c_1 and d_1 are positive constants; the term $d_1 v/u$ is called the Leslie–Gower

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terms, which measures the loss in the predator population due to rarity of its favorite food. System (1.1) is regarded as a prototypical predator-prey system in the ecological studies. But the interaction terms in (1.1) are unbounded, which is not reasonable in the real world. By using Holling type II functional response [13] in both prey and predator interaction terms, a Leslie–Gower predator–prey system with saturated functional responses is obtained and takes the form (see [4]):

$$\begin{cases} \dfrac{\mathrm{d}u}{\mathrm{d}t} = u(a_1 - b_1 u - \dfrac{c_1 v}{u + k_1}), \\ \dfrac{\mathrm{d}v}{\mathrm{d}t} = v\left(a_2 - \dfrac{c_2 v}{u + k_2}\right). \end{cases} \tag{1.2}$$

The model (1.2) is based on the biological fact that if the predator v is more capable of switching from its favorite food (the prey u) to other food options, then it has better ability to survive when the prey population is low; here a_1 and a_2 are the growth rates per capita of prey u and predator v, respectively; b_1 measures the strength of intraspecific competition among individuals of species u, and it is related to the carrying capacity of the prey; c_1 is the maximum value of the per capita reduction rate of u due to v, and c_2 is the maximum growth per capita of v due to predation of u; k_1 and k_2 measure the extent to which environment provides protection to prey u and predator v, respectively.

Non-monotonic responses appear at the microbial level; when the nutrient concentration reaches at a high level an inhibitory effect of the specific growth rate can occur [3, 6]. This may frequently be noticed when micro-organisms are used for waste decomposition or for water purification. Andrews [3] suggested a response function $p(u) = \frac{mu}{k_1 + k_2 u + u^2}$, known as Monod–Haldane response function, to model such an inhibitory effect at high concentrations. In particular, Sokol and Howell [31] derived a simplified Monod–Haldane type $p(u) = \frac{mu}{k_1 + u^2}$. A Leslie-Gower predator-prey system with a Monod–Haldane functional response takes the form:

$$\begin{cases} \dfrac{\mathrm{d}u}{\mathrm{d}t} = u\left(a_1 - b_1 u - \dfrac{mv}{k_1 + u^2}\right), \\ \dfrac{\mathrm{d}v}{\mathrm{d}t} = v\left(a_2 - \dfrac{dv}{k_2 + u^2}\right). \end{cases} \tag{1.3}$$

In mathematical ecology, population may be distributed non-homogeneously, and the predators and preys naturally develop strategies for survival. Thus, we may introduce diffusive structure, which can be illustrated as different concentration levels of predators and preys causing different movements. Diffusion means the movement of individuals from a higher to a lower concentration region, while cross diffusion implies the population fluxes of one species owing to the presence of the other species. In this paper, our concern is the following system with cross-diffusion rates

$$\begin{cases} \dfrac{\partial u}{\partial t} = d_1 \Delta u + u\left(a - u - \dfrac{v}{1 + u^2}\right) & \text{in } \Omega \times (0, \infty), \\ \dfrac{\partial v}{\partial t} = \Delta[(d_2 + \beta u)v] + v\left(b - \dfrac{v}{1 + u^2}\right) & \text{in } \Omega \times (0, \infty), \\ \dfrac{\partial u}{\partial \mathbf{n}} = \dfrac{\partial v}{\partial \mathbf{n}} = 0 & \text{on } \partial\Omega \times (0, \infty), \\ u(x, 0) = \varphi(x) \geq 0, \quad v(x, 0) = \psi(x) \geq 0, & \text{in } \Omega, \end{cases} \tag{1.4}$$

whose corresponding ordinary differential equations (ODEs) is (1.3) with all the parameters b_1, m, k_1 and k_2 equal to 1. Here Δ denotes the Laplacian operator on $\mathbb{R}^N$ ($N \geq 1$), Ω is a connected bounded open domain in $\mathbb{R}^N$, with a smooth boundary $\partial\Omega$, $\mathbf{n}$ is the outward unit normal vector on $\partial\Omega$. The homogeneous Neumann boundary condition means that the two species have zero flux across the boundary $\partial\Omega$. The diffusion terms d_j, $j = 1, 2$ stand for natural dispersive force of movement of an individual, while β describes the mutual interferences between individuals and is usually referred as the cross-diffusion pressure measuring the situation that the prey keeps away from the predator; a and b are the growth rates per capita of prey u and predator v. The parameters a, b, d_1 and d_2 are positive constants and β is non-negative constant.

In some cases, the quantity v is not influenced by any cross diffusion in the sense that the coefficient β in the second equation of (1.4) vanishes, that is, we ignore the population migration of predators due to the presence of preys. In this situation, Li et al. [20] considered the following reaction-diffusion system in the one-dimensional space domain $\Omega = (0, \pi)$:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u + u\left(a - u - \dfrac{v}{1+u^2}\right) & \text{in } \Omega \times (0,\infty), \\ \dfrac{\partial v}{\partial t} = d\Delta v + v\left(b - \dfrac{v}{1+u^2}\right) & \text{in } \Omega \times (0,\infty), \\ \dfrac{\partial u}{\partial \mathbf{n}} = \dfrac{\partial v}{\partial \mathbf{n}} = 0 & \text{on } \partial\Omega \times (0,\infty), \end{cases} \tag{1.5}$$

where d is the relative diffusion rate of predator v when the diffusion rate of prey is rescaled to 1. Li et al. [20] studied the Hopf bifurcation and steady-state bifurcation by taking d as the bifurcation parameter and described both the global structure of the steady-state bifurcation from simple eigenvalues and the local structure of the steady-state bifurcation from double eigenvalues by using space decomposition and the implicit function theorem.

The presence of the cross-diffusion term causes more abundant dynamic behaviors. For example, the effect of cross diffusion on dynamics of predator-prey models has been studied in [5, 7, 22, 24, 30, 35, 37, 43, 44, 47]. The relevant discussion is a bit difficult and requires more techniques than for models without cross-diffusion. In [5, 24, 43, 44], the researchers mainly obtained the non-existence and existence of non-constant positive steady-states (patterns) and showed cross diffusion can create non-constant steady states. Gambino et al. [7] analyzed the linear stability of the positive equilibrium of a competitive Lotka–Volterra system, and showed the cross-diffusion is the key mechanism for the formation of spatial patterns through Turing bifurcation. Liu et al. [22] not only obtained the global existence result of solutions under an appropriate parameter condition, but also gave explicit parameter ranges of the existence of non-constant positive steady-states.

For system (1.4), we first discuss the influence of the cross-diffusion coefficient β on the global existence of the solution. As far as global existence is concerned, many researchers have some relevant works, for example, [22, 26, 33, 41]. Wu et al. [41] and Tao [33] analyzed the predator-prey model with prey-taxis and discussed the effect of the prey-taxis term on the global existence of solutions of the system. Mu et al. [26] studied the global existence of classical solutions to a parabolic-parabolic chemotaxis system, but there are strict restrictions on functions in the system. Liu et al. [22] investigated the global existence of solutions of a parabolic-elliptic two-species competition model with cross diffusion.

Next, for a predator-prey system, what we are interested in is whether the various species can exist and takes the form of non-constant time-independent positive solutions. In [5, 8, 24, 25, 43, 44], the authors have established the existence of stationary patterns in some predator-prey models in the presence of self-diffusion and cross-diffusion. Our results are a little different from theirs. We not only prove the existence of non-constant solution of system (1.4) when the cross-diffusion β is sufficiently large, but also we find infinitely many intervals of $d_1 > 0$ near zero such that (1.4) admits at least one nonconstant solution if d_1 belongs to such intervals. Moreover, researchers have paid more attention to Hopf bifurcation and steady state bifurcation (cf. [9,10,15,18,19,36,42,46]), and investigated some predator-prey models without cross diffusion term. Only a few works [23, 45] have concentrated on the Bogdanov–Takens bifurcation phenomena of diffusive predator-prey systems with delay effect. In this paper, we study the Bogdanov–Takens bifurcation by regarding the cross-diffusion term β as one of bifurcation parameters.

The organization of the remaining part of the paper is as follows. In Section 2 we prove the global existence and boundedness results of solutions to (1.4) and in Section 3 we obtain a priori bounds of nonnegative steady state solutions. In Section 4 we deal with the non-existence of non-constant positive steady states for sufficient large diffusion coefficient and consider the existence of non-constant positive steady states for a small range of diffusion coefficient and sufficient large cross-diffusion coefficient by using the Leray–Schauder degree theory. Section 5 is devoted to the local and global stability of homogeneous steady states. Center manifold reduction and normal form theory are employed in Section 6 not only to discuss the existence of Hopf bifurcation but also to determine the Hopf bifurcation direction and the stability of bifurcating time-periodic solutions. In Section 7 we observe that system (1.4) exhibits Bogdanov–Takens bifurcation phenomena. Finally in Section 8, some conclusions are presented and numerical simulations are carried out to illustrate some previous theoretical results.

For convenience, we introduce the following notations. Let $H^k(\Omega)$ $(k \geq 0)$ be the Sobolev space of the L^2-functions f defined on Ω whose derivatives $f^{(n)}$ $(n = 1, \dots, k)$ also belong to $L^2(\Omega)$. Denote the spaces $\mathbb{X} = \{\phi \in H^2(\Omega) | \frac{\partial \phi}{\partial \mathbf{n}} = 0 \text{ on } \partial\Omega\}$ and $\mathbb{Y} = L^2(\Omega)$. For a space Z, we also define the complexification of Z to be $Z_{\mathbb{C}} \triangleq Z \oplus \mathrm{i}Z = \{x_1 + \mathrm{i}x_2 | x_1, x_2 \in Z\}$. Define an inner product on the complex-valued Hilbert space $\mathbb{Y}^2_{\mathbb{C}}$ by

$$\langle u, v \rangle = \int_\Omega \overline{u}(s)^T v(s) \mathrm{d}s \qquad \text{for } u, v \in \mathbb{Y}^2_{\mathbb{C}}. \tag{1.6}$$

2 Global existence and boundedness

In this section, we employ the method in [40] to obtain the global existence and boundedness of solutions of model (1.4). We need to establish some priori estimates. It is clear that the local existence of solutions to (1.4) was established by Amann [1]. This result can be summarized as follows.

Lemma 2.1. *For each fixed $p > N$, assume that the initial data $(\varphi, \psi) \in (W^{1,p}(\Omega))^2$ satisfies $\varphi \geq 0$ and $\psi \geq 0$, then there exists a positive constant $T_{\max}$ (the maximal existence time) such that (φ, ψ) determines a unique nonnegative classical solution $(u(x,t), v(x,t))$ of system (1.4) satisfying $(u, v) \in (C([0, T_{\max}), W^{1,p}(\Omega)) \cap C^{2,1}(\bar{\Omega} \times (0, T_{\max})))^2$ and*

$$0 \leq u(x,t) \leq c \triangleq \max\left\{\max_{\bar{\Omega}} \varphi(x), a\right\}, \quad v(x,t) \geq 0 \tag{2.1}$$

for all $(x,t) \in \bar{\Omega} \times [0, T_{\max})$.

Proof. (i) The local existence of the solution to (1.4) follows from [1]. Denote by $T_{\max}$ the maximal existence time of the solution. Next, we shall prove (2.1). On account of (1.4), we know that $v(x,t)$ satisfies

$$\begin{cases} \dfrac{\partial v}{\partial t} = \Delta[(d_2+\beta u)v] + v\left(b - \dfrac{v}{1+u^2}\right) & \text{in } \Omega\times(0,\infty), \\ \dfrac{\partial v}{\partial \mathbf{n}} = 0 & \text{on } \partial\Omega\times(0,\infty), \\ v(x,0)=\psi(x)\geq 0 & \text{in } \Omega. \end{cases} \tag{2.2}$$

Clearly, $\underline{v}\equiv 0$ is a sub-solution to problem (2.2). Hence, we can apply the maximum principle for parabolic equations to obtain that $v(x,t)\geq 0$. Similarly, we can obtain $u(x,t)\geq 0$. Also from (1.4) and $v\geq 0$, we obtain that

$$\begin{cases} \dfrac{\partial u}{\partial t} = d_1\Delta u + u\left(a-u-\dfrac{v}{1+u^2}\right) \leq u(a-u) & \text{in } \Omega\times(0,\infty), \\ \dfrac{\partial u}{\partial \mathbf{n}} = 0 & \text{on } \partial\Omega\times(0,\infty), \\ u(x,0)=\varphi(x)\geq 0 & \text{in } \Omega. \end{cases}$$

Then from comparison principle of parabolic equations, it is easy to verify $u(x,t)\leq c$, where c is given in (2.1). This completes the proof of Lemma 2.1. □

The above lemma means that, in the space $W^{1,p}(\Omega)$ each pair of the initial values φ and ψ can determine a unique nonnegative classical solution $(u(x,t),v(x,t))$, which is twice continuously differentiable with respect to $x\in\overline{\Omega}$ and continuously differentiable with respect to $t\in[0,T_{\max})$. Moreover, $u(\cdot,t)$, $v(\cdot,t)\in W^{1,p}(\Omega)$ can be regarded as a continuous mapping with respect to $t\in[0,T_{\max})$.

According to Amann's results [2], we need to establish the L^∞ bound of (u,v) in order to show its global existence. Based on Lemma 2.1, it is enough to establish the L^∞ bound of $v(x,t)$. Firstly, we shall show that the solution $v(x,t)$ is bounded in $L^1(\Omega)$. In the proof, we need to use the following elementary inequality [39].

Lemma 2.2. *Assume that $z(t)\geq 0$ satisfy*

$$\begin{cases} z'(t)\leq -a_1 z^r(t)+a_2 z(t)+a_3, & t>0, \\ z(0)=z_0, \end{cases}$$

where $a_1,a_2,a_3>0$ and $r>1$. Then there exist constants $c_1(a_1,a_2,a_3,r)$ and $c_2(z_0)$ such that

$$z(t)\leq \max\{c_1(a_1,a_2,a_3,r),c_2(z_0)\}.$$

Lemma 2.3. *There exists a constant $C_0>0$ such that the second component of the solution of (1.4) satisfies the following estimate*

$$\int_\Omega v(x,t)\mathrm{d}x \leq C_0 \quad \text{for all } t\in(0,T_{\max}). \tag{2.3}$$

Proof. Let

$$U(t)=\int_\Omega u(x,t)\mathrm{d}x, \quad V(t)=\int_\Omega v(x,t)\mathrm{d}x.$$

Then we have

$$\dot{U}(t)+\dot{V}(t)=\int_\Omega (au+bv)\mathrm{d}x - \int_\Omega\left(u^2+\frac{uv}{1+u^2}+\frac{v^2}{1+u^2}\right)\mathrm{d}x.$$

In view of Lemma 2.1, we know that $0 \leq u(x,t) \leq c$ for all $(x,t) \in \bar{\Omega} \times [0, T_{\max})$ and hence that

$$u^2 + \frac{uv}{1+u^2} + \frac{v^2}{1+u^2} \geq \frac{(u+v)^2}{2(1+u^2)} \geq \frac{(u+v)^2}{2(1+c^2)},$$

which, together with the Hölder inequality, implies that

$$\begin{aligned}\dot{U}(t) + \dot{V}(t) &\leq \int_\Omega r(u+v)\mathrm{d}x - \int_\Omega \frac{(u+v)^2}{2(1+c^2)}\mathrm{d}x \\ &\leq r\int_\Omega (u+v)\mathrm{d}x - \frac{1}{2(1+c^2)|\Omega|}\left[\int_\Omega (u+v)\mathrm{d}x\right]^2 \\ &= r[U(t)+V(t)] - \frac{[U(t)+V(t)]^2}{2(1+c^2)|\Omega|}\end{aligned}$$

with $r = \max\{a, b\}$. It follows from Lemma 2.2 that there exists a positive constant M such that $U(t) + V(t) \leq M$ for all $t \in (0, T_{\max})$, and hence that there exists a positive constant C_0 such that (2.3) holds. The proof is completed. □

Secondly, we will establish L^p estimates for $v(x,t)$ by using a weight function $\phi(u)$ similar to that in [32,38,41]. We now present some basic inequalities which will be used in the sequel (see [14,27]). In several places we shall need the following Poincaré's inequality:

$$\|u\|_{1,p} \leq C_4(\|\nabla u\|_p + \|u\|_q) \quad \text{for all } u \in W^{1,p}(\Omega)$$

with arbitrary $p > 1$ and $q > 0$. Also, an essential role will be played by Gagliardo-Nirenberg interpolation inequality

$$\|u\|_p \leq C_3 \|u\|_{1,q}^{\eta} \|u\|_m^{1-\eta} \quad \text{for all } u \in W^{1,p}(\Omega),$$

which holds for all $1 \leq p, q \leq \infty$ satisfying $p(n-q) < nq$ and all $m \in (0, p)$ with

$$\eta = \frac{\frac{n}{m} - \frac{n}{p}}{\frac{n}{m} + 1 - \frac{n}{q}} \in (0, 1).$$

Lemma 2.4. *Let $(u(x,t), v(x,t))$ be a solution of (1.4), then for every $p \in [2, \infty)$, there exists a positive constant $E > 0$ such that*

$$\|v(x,t)\|_p \leq E \quad \text{for } t \in (0, T_{\max})$$

if

$$\beta \in \left[0, \frac{d_1 d_2}{2\sqrt{2}(d_1+d_2)pc}\right]. \tag{2.4}$$

Proof. Let

$$\alpha = \frac{d_1 d_2 (p-1)}{4(d_1+d_2)^2 pc^2}, \tag{2.5}$$

and consider a weight function

$$\phi(u(x,t)) = e^{\alpha u^2(x,t)} \quad \text{when } 0 \leq u(x,t) \leq c. \tag{2.6}$$

Denote $\phi(u(x,t))$ by $\phi(u)$, then we have

$$1 \leq \phi(u) = e^{\alpha u^2} \leq e^{\alpha c^2} = h \quad \text{and} \quad ,1 \leq \phi'(u) = 2\alpha u e^{\alpha u^2} \leq 2\alpha c e^{\alpha c^2}, \qquad 0 \leq u \leq c. \tag{2.7}$$

It follows from system (1.4) that

$$\begin{aligned}\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega v^p\phi(u)\mathrm{d}x &= \int_\Omega v^{p-1}\phi(u)\frac{\partial v}{\partial t}\mathrm{d}x + \frac{1}{p}\int_\Omega v^p\phi'(u)\frac{\partial u}{\partial t}\mathrm{d}x \\ &= \int_\Omega v^{p-1}\phi(u)[\Delta(d_2+\beta u)v]\mathrm{d}x + \int_\Omega v^p\phi(u)\left(b-\frac{v}{1+u^2}\right)\mathrm{d}x \\ &\quad + \frac{1}{p}\int_\Omega v^p\phi'(u)\left[d_1\Delta u + u\left(a-u-\frac{v}{1+u^2}\right)\right]\mathrm{d}x \\ &\le -(p-1)\int_\Omega v^{p-2}(d_2+\beta u)\phi(u)|\nabla v|^2\mathrm{d}x \\ &\quad -(p-1)\beta\int_\Omega v^{p-1}\phi(u)\nabla u\cdot\nabla v\mathrm{d}x - \int_\Omega v^{p-1}\phi'(u)(d_2+\beta u)\nabla u\cdot\nabla v\mathrm{d}x \\ &\quad -\beta\int_\Omega v^p\phi'(u)|\nabla u|^2\mathrm{d}x + b\int_\Omega v^p\phi(u)\mathrm{d}x - d_1\int_\Omega v^{p-1}\phi'(u)\nabla u\cdot\nabla v\mathrm{d}x \\ &\quad -\frac{d_1}{p}\int_\Omega v^p\phi''(u)|\nabla u|^2\mathrm{d}x + \frac{ac}{p}\int_\Omega v^p\phi'(u)\mathrm{d}x,\end{aligned}$$

which implies that

$$\begin{aligned}&\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega v^p\phi(u)\mathrm{d}x + (p-1)d_2\int_\Omega v^{p-2}\phi(u)|\nabla v|^2\mathrm{d}x + \frac{d_1}{p}\int_\Omega v^p\phi''(u)|\nabla u|^2\mathrm{d}x \\ &\le -\int_\Omega(d_2+\beta u)v^{p-1}\phi'(u)\nabla u\cdot\nabla v\mathrm{d}x - \beta(p-1)\int_\Omega v^{p-1}\phi(u)\nabla u\cdot\nabla v\mathrm{d}x \\ &\quad - d_1\int_\Omega v^{p-1}\phi'(u)\nabla u\cdot\nabla v\mathrm{d}x + b\int_\Omega v^p\phi(u)\mathrm{d}x + \frac{ac}{p}\int_\Omega v^p\phi'(u)\mathrm{d}x.\end{aligned}\tag{2.8}$$

In virtue of (2.7), we know that $\phi'(u),\phi(u)>0$. Combining with $v(x,t)\ge 0$, it is easy to see that

$$\begin{aligned}&-(d_1+d_2)\int_\Omega v^{p-1}\phi'(u)\nabla u\cdot\nabla v\mathrm{d}x \\ &\le \int_\Omega \frac{\sqrt{\phi(u)d_2(p-1)}v^{\frac{p-2}{2}}|\nabla v|}{\sqrt{2}}\cdot\frac{\sqrt{2}(d_1+d_2)v^{\frac{p}{2}}\phi'(u)|\nabla u|}{\sqrt{\phi(u)d_2(p-1)}}\mathrm{d}x.\end{aligned}$$

Furthermore, using Young's inequality, we obtain

$$\begin{aligned}&-(d_1+d_2)\int_\Omega v^{p-1}\phi'(u)\nabla u\cdot\nabla v\mathrm{d}x \\ &\le \frac{d_2(p-1)}{4}\int_\Omega v^{p-2}\phi(u)|\nabla v|^2\mathrm{d}x + \frac{(d_1+d_2)^2}{d_2(p-1)}\int_\Omega v^p\frac{\phi'^2(u)}{\phi(u)}|\nabla u|^2\mathrm{d}x.\end{aligned}\tag{2.9}$$

Similar to the above, we obtain

$$\begin{aligned}&-\beta(p-1)\int_\Omega v^{p-1}\phi(u)\nabla u\cdot\nabla v\mathrm{d}x \\ &\le \frac{d_2(p-1)}{4}\int_\Omega v^{p-2}\phi(u)|\nabla v|^2\mathrm{d}x + \frac{\beta^2(p-1)}{d_2}\int_\Omega v^p\phi(u)|\nabla u|^2\mathrm{d}x.\end{aligned}\tag{2.10}$$

Together with $0\le u\le c$, we similarly have

$$\begin{aligned}&-\beta\int_\Omega uv^{p-1}\phi'(u)\nabla u\cdot\nabla v\mathrm{d}x \\ &\le \frac{d_2(p-1)}{4}\int_\Omega v^{p-2}\phi(u)|\nabla v|^2\mathrm{d}x + \frac{\beta^2(p-1)}{d_2}\int_\Omega u^2v^p\frac{\phi'^2(u)}{\phi(u)}|\nabla u|^2\mathrm{d}x \\ &\le \frac{d_2(p-1)}{4}\int_\Omega v^{p-2}\phi(u)|\nabla v|^2\mathrm{d}x + \frac{\beta^2(p-1)c^2}{d_2}\int_\Omega v^p\frac{\phi'^2(u)}{\phi(u)}|\nabla u|^2\mathrm{d}x.\end{aligned}\tag{2.11}$$

Substituting (2.9), (2.10) and (2.11) into (2.8), we have

$$\begin{aligned}
&\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega v^p\phi(u)\mathrm{d}x+\frac{d_2(p-1)}{4}\int_\Omega v^{p-2}\phi(u)|\nabla v|^2\mathrm{d}x+\frac{d_1}{p}\int_\Omega v^p\phi''(u)|\nabla u|^2\mathrm{d}x\\
&\quad\le\left[\frac{(d_1+d_2)^2}{d_2(p-1)}+\frac{\beta^2c^2(p-1)}{d_2}\right]\int_\Omega v^p\frac{\phi'^2(u)}{\phi(u)}|\nabla u|^2\mathrm{d}x\\
&\qquad+\frac{\beta^2(p-1)}{d_2}\int_\Omega v^p\phi(u)|\nabla u|^2\mathrm{d}x+b\int_\Omega v^p\phi(u)\mathrm{d}x+\frac{ac}{p}\int_\Omega v^p\phi'(u)\mathrm{d}x.
\end{aligned}\tag{2.12}$$

Clearly,

$$\frac{\phi'^2(u)}{\phi(u)}=4\alpha^2u^2\phi(u)\quad\text{and}\quad\phi''(u)=(2\alpha+4\alpha^2u^2)\phi(u).$$

By a direct calculation, we obtain

$$\begin{aligned}
&\frac{2a_2(u)}{a_1(u)}\le\frac{4(d_1+d_2)^2c^2p}{d_1d_2(p-1)}\alpha=1,\\
&\frac{4a_3(u)}{a_1(u)}\le\frac{2\beta^2(p-1)p}{d_1d_2\alpha}=\frac{8\beta^2c^2p^2(d_1+d_2)^2}{d_1^2d_2^2}\le1,\\
&\frac{4a_4(u)}{a_1(u)}\le\frac{4c^2\beta^2p(p-1)}{d_1d_2}=\frac{4p(p-1)}{d_1d_2}\cdot\frac{d_1^2d_2^2}{8p^2(d_1+d_2)^2}=\frac{d_1d_2(p-1)}{2p(d_1+d_2)^2}<1,
\end{aligned}\tag{2.13}$$

for $0\le u\le c$, where β and α satisfy (2.4) and (2.5) respectively, and

$$\begin{aligned}
a_1(u)&=\frac{d_1}{p}\phi''(u),\\
a_2(u)&=\frac{(d_1+d_2)^2}{d_2(p-1)}\cdot\frac{\phi'^2(u)}{\phi(u)},\\
a_3(u)&=\frac{\beta^2(p-1)}{d_2}\phi(u),\\
a_4(u)&=\frac{\beta^2c^2(p-1)}{d_2}\cdot\frac{\phi'^2(u)}{\phi(u)}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\frac{\beta^2(p-1)}{d_2}\int_\Omega v^p\phi(u)|\nabla u|^2\mathrm{d}x+\frac{(d_1+d_2)^2}{d_2(p-1)}\int_\Omega v^p\frac{\phi'^2(u)}{\phi(u)}|\nabla u|^2\mathrm{d}x\\
&\quad+\frac{\beta^2c^2(p-1)}{d_2}\int_\Omega v^p\frac{\phi'(u)^2}{\phi'(u)}|\nabla u|^2\mathrm{d}x\le\frac{d_1}{p}\int_\Omega v^p\phi''(u)|\nabla v|^2\mathrm{d}x.
\end{aligned}\tag{2.14}$$

It follows from (2.14) that (2.12) is simplified to be

$$\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega v^p\phi(u)\mathrm{d}x+\frac{d_2(p-1)}{4}\int_\Omega v^{p-2}\phi(u)|\nabla v|^2\le C_1\int_\Omega v^p\phi(u)\mathrm{d}x,\tag{2.15}$$

where $C_1=(bp+2\alpha ac^2)/p$. By the Gagliardo–Nirenberg and Poincaré's inequality and (2.7) and (2.3), we have

$$\begin{aligned}\int_\Omega v^p\phi(u)\mathrm{d}x &\le h\int_\Omega v^p\mathrm{d}x = h\|v^{\frac{p}{2}}\|_2^2 \le C_2\|v^{\frac{p}{2}}\|_{1,2}^{2\eta}\|v^{\frac{p}{2}}\|_{\frac{2}{p}}^{2(1-\eta)}\\ &\le hC_3\left(C_4\left(\frac{2}{p}\right)\right)^{2\eta}\left(\|\nabla v^{\frac{p}{2}}\|_2+\|v^{\frac{p}{2}}\|\right)^{2\eta}\|v^{\frac{p}{2}}\|_{\frac{2}{p}}^{2(1-\eta)}\\ &= hC_3\left(C_4\left(\frac{2}{p}\right)\right)^{2\eta}\left(\|\nabla v^{\frac{p}{2}}\|_2+\|v\|_1^{\frac{p}{2}}\right)^{2\eta}\|v\|_1^{p(1-\eta)}\\ &\le C_5\left(\|\nabla v^{\frac{p}{2}}\|_2^2+1\right)^{\eta},\end{aligned} \tag{2.16}$$

where

$$\eta = \frac{pn-n}{2-n+pn}\in(0,1).$$

Now from (2.7) and (2.16), we have

$$\begin{aligned}\int_\Omega v^{p-2}\phi(u)|\nabla v|^2\mathrm{d}x &\ge \int_\Omega v^{p-2}|\nabla v|^2\mathrm{d}x = \frac{4}{p^2}\int_\Omega|\nabla v^{\frac{p}{2}}|^2\mathrm{d}x\\ &\ge \frac{4}{p^2C_5^{\frac{1}{\eta}}}\left(\int_\Omega v^p\phi(u)\mathrm{d}x\right)^{\frac{1}{\eta}}-\frac{4}{p^2}.\end{aligned} \tag{2.17}$$

Hence from (2.15) and (2.17) we obtain

$$\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega v^p\phi(u)\mathrm{d}x \le -\frac{d_2(p-1)}{p^2C_5^{\frac{1}{\eta}}}\left(\int_\Omega v^p\phi(u)\mathrm{d}x\right)^{\frac{1}{\eta}}+C_1\int_\Omega v^p\phi(u)\mathrm{d}x+\frac{d_2(p-1)}{p^2}$$

for all $t\in(0,T_{\max})$, where $\frac{1}{\eta}>1$. By using Lemma 2.2 and (2.7), we conclude that there exists $E>0$ such that

$$\|v(\cdot,t)\|_p \le \left(\int_\Omega v^p\phi(u)\mathrm{d}x\right)^{\frac{1}{p}} \le E \quad \text{for } t\in(0,T_{\max}),$$

which is the desired result. □

Finally, we establish the L^∞ bound of $v(x,t)$ using Lemma 2.4.

Lemma 2.5. *If β satisfies (2.4) and let $(u(x,t),v(x,t))$ be a solution of (1.4). Then there exists a positive constant A such that*

$$\|v(\cdot,t)\|_\infty \le A \quad \text{for } t\in(0,T_{\max}).$$

Proof. Define

$$f(u,v) = u\left(a-u-\frac{v}{1+u^2}\right), \qquad g(u,v) = \beta v\Delta u + v\left(b-\frac{v}{1+u^2}\right)$$

for $(u,v)\in(C([0,T_{\max}),W^{1,p}(\Omega)\cap C^{2,1}(\bar\Omega\times(0,T_{\max})))^2$. It follows from Lemmas 2.4 and 2.1 that there exists a positive constant A_1 such that

$$\|f\|_{L^p(\Omega)} \le A_1 < +\infty \quad \text{for all } t\in(0,T_{\max}). \tag{2.18}$$

In virtue of (2.18) and the first equation of system (1.4) and the L^p-estimate for parabolic equations, we obtain

$$\|u(\cdot,t)\|_{W_p^2(\Omega)} \le A_1 \quad \text{for all } t\in(0,T_{\max}). \tag{2.19}$$

This, together with the Sobolev embedding theorem (see [16]), yields

$$\|\nabla u(\cdot,t)\|_{L^\infty(\Omega)} \le A_1 \quad \text{for all } t \in (0, T_{\max}). \tag{2.20}$$

We now turn to the second equation of (1.4), which can be rewritten as the non-divergence form:

$$\frac{\partial v}{\partial t} = (d_2 + \beta u)\Delta v + 2\beta \nabla u \cdot \nabla v + g(u,v). \tag{2.21}$$

In virtue of Lemmas 2.4 and 2.1 and (2.19), we have

$$\|g(u,v)\|_{L^p(\Omega)} \le A_1 \quad \text{and} \quad \|d_2 + \beta u\|_{L^\infty(\Omega)} \le A_1 \quad \text{for all } t \in (0, T_{\max}). \tag{2.22}$$

Using (2.21), (2.20) and (2.22) and the L^p-estimate for parabolic equations, we have

$$\|v(\cdot,t)\|_{W^2_p(\Omega)} \le A_1 \quad \text{for all } t \in (0, T_{\max}).$$

Again, taking p to be sufficiently large and combing with the Sobolev embedding theorem (see [16]), we have

$$\|v(\cdot,t)\|_{L^\infty(\Omega)} \le A \quad \text{for all } t \in (0, T_{\max}).$$

Hence, this proof is completed. □

Obviously, from Lemmas 2.1 and 2.5 and [2], we conclude that $T_{\max} = \infty$ and $\|v(\cdot,t)\|_\infty + \|v(\cdot,t)\|_\infty \le M(\varphi,\psi)$ for all $t \in [0,\infty)$, where $M(\varphi,\psi)$ depends on the initial value (φ,ψ). Notice that in the proof of Lemma 2.1, for any positive constant ε_0, there exists $t_1 > 0$ such that

$$\|u(\cdot,t)\|_{L^\infty} \le a + \varepsilon_0 \quad \text{for all } t \in (t_1, \infty). \tag{2.23}$$

Hence we can replace c by $a + \varepsilon_0$ for $t \in (t_1,\infty)$. Similarly in Lemma 2.3, C_0 can be chosen to be independent of (φ,ψ). So $\int_\Omega u(x,t) \le C_0$ for $t \in (t_2,\infty)$ with $t_2 > t_1$. Again in the proof of Lemmas 2.4 and 2.5, we can also replace c by $a + \varepsilon_0$ and then we can find $t_0 > t_2$ such that

$$\|v(\cdot,t)\|_p \le E \quad \text{for all } t \in (t_0, \infty)$$

and

$$\|v(\cdot,t)\|_\infty \le A \quad \text{for all } t \in (t_0, \infty) \tag{2.24}$$

if

$$\beta \in \left[0, \frac{d_1 d_2}{2\sqrt{2}(d_1+d_2)pa}\right], \tag{2.25}$$

where E and A are independent of (φ,ψ). In view of (2.23) and (2.24), there exists a constant M_1 such that

$$\|v(\cdot,t)\|_\infty + \|v(\cdot,t)\|_\infty \le M_1 \quad \text{for all } t \in (t_0, \infty),$$

where M_1 is independent of (φ,ψ). Therefore, we have the following theorem.

Theorem 2.6. *Suppose that $p > N$ and β satisfies (2.4), then every initial value $(\varphi,\psi) \in (W^{1,p}(\Omega))^2$ satisfying $\varphi(x) \ge 0$ and $\psi(x) \ge 0$ for all $x \in \Omega$, determines a unique global classical solution $(u(x,t), v(x,t))$ of system (1.4), which satisfies $(u,v) \in (C([0,\infty); W^{1,p}(\Omega)) \cap C^{2,1}(\Omega \times [0,\infty)))^2$. Moreover, (u,v) is uniformly bounded in $\Omega \times (0,\infty)$, that is, there exists a constant $M(\varphi,\psi) > 0$, depending on the initial (φ,ψ), such that $\|u(\cdot,t)\|_\infty + \|v(\cdot,t)\|_\infty \le M$ for all $t \in [0,\infty)$. Furthermore, if β satisfies (2.25), then there exist two positive constants M_1, independent of (φ,ψ), and $t_0 > 0$, such that $\|u(\cdot,t)\|_\infty + \|v(\cdot,t)\|_\infty \le M_1$ for all $t \in (t_0,\infty)$.*

3 A priori estimates

Steady-state solutions of (1.4) satisfy the following system:

$$\begin{cases} d_1\Delta u + u\left(a - u - \dfrac{v}{1+u^2}\right) = 0 & \text{in } \Omega, \\ \Delta[(d_2+\beta u)v] + v\left(b - \dfrac{v}{1+u^2}\right) = 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \mathbf{n}} = \dfrac{\partial v}{\partial \mathbf{n}} = 0 & \text{on } \partial\Omega, \\ u(x,0) = \varphi(x) \geq 0, \quad v(x,0) = \psi(x) \geq 0, & \text{in } \Omega. \end{cases} \tag{3.1}$$

It is easy to see that system (1.4) has a positive constant steady-state solution $\mathbf{e} = (u^*, v^*)^T$ if and only if $a > b$, where $u^* = \theta \triangleq a - b$, $v^* = b(1+\theta^2)$.

Next, we study the asymptotic behavior of positive solutions of (3.1) as d_1 is small or β is sufficiently large. For the first step of the asymptotic analysis, we derive a priori positive upper and lower bounds for positive solutions to (3.1).

Lemma 3.1. *Suppose that (u,v) is a solution of (3.1) and $a \neq b$, then there exists a positive constant $\check{C}$ such that (u,v) satisfies*

$$\check{C} \leq u(x) \leq a, \qquad \frac{d_2 b}{d_2+\beta} \leq v(x) \leq \kappa \triangleq \frac{b}{d_2}(d_2+\beta a)(1+a^2)$$

for all $x \in \bar{\Omega}$.

Proof. Let $x_0 \in \bar{\Omega}$ be a maximum point of u, i.e., $u(x_0) = \max_{x\in\bar{\Omega}} u(x)$. Then by using the maximum principle [24] to the first equation of (3.1), one has $a - u(x_0) - \frac{v(x_0)}{1+u^2(x_0)} \geq 0$ and hence $u \leq a$.

By setting $w = (d_2+\beta u)v$, we can reduce the second equation of (3.1) with the boundary condition to

$$\begin{cases} \Delta w + v\left(b - \dfrac{v}{1+u^2}\right) = 0 & \text{in } \Omega, \\ \dfrac{\partial w}{\partial \mathbf{n}} = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.2}$$

Let $x_1 \in \bar{\Omega}$ be a maximum point of w, i.e., $w(x_1) = \max_{x\in\bar{\Omega}} w(x)$. Applying the maximum principle [24] to (3.2), we get $v(x_1) \leq b(1+u^2(x_1)) = b(1+a^2)$. Note that $0 \leq u(x_1) \leq a$ and $v(x_1) \leq b(1+a^2)$, then we have $\max_{\bar{\Omega}} w(x) = w(x_1) \leq b(d_2+\beta a)(1+a^2)$, which in turn implies that

$$\max_{\bar{\Omega}} v(x) \leq \frac{1}{d_2}\max_{\bar{\Omega}} w(x) = \frac{1}{d_2}[d_2+\beta u(x_1)]v(x_1) \leq \frac{b}{d_2}(d_2+\beta a)(1+a^2) = \kappa.$$

To obtain the lower bound for v, we define $w(y_0) = \min_{\bar{\Omega}} w(x)$. Similarly, applying the maximum principle [24] to (3.2) yields $v(y_0) \geq b(1+u^2(y_0))$. According to the definition of w, we obtain

$$\min_{\bar{\Omega}} v \geq \frac{\min_{\bar{\Omega}} w}{d_2+\beta\max_{\bar{\Omega}} u} = \frac{w(y_0)}{d_2+\beta u(x_0)} = \frac{d_2+\beta u(y_0)}{d_2+\beta u(x_0)}v(y_0) \geq \frac{d_2 b}{d_2+\beta a}.$$

Now, denote $u(y_1) = \min_{\bar{\Omega}} u(x)$ for some $y_1 \in \bar{\Omega}$. It follows from the maximum principle [24] that

$$u(y_1) \geq a - \frac{v(y_1)}{1 + u^2(y_1)}. \tag{3.3}$$

Note that

$$\frac{v(y_1)}{1 + u^2(y_1)} \leq v(y_1) \leq \max_{\bar{\Omega}} v(x) \leq \kappa. \tag{3.4}$$

This, together with (3.3) and (3.4), implies that

$$u(y_1) \geq a - \frac{v(y_1)}{1 + u^2(y_1)} \geq a - \kappa.$$

If $a > \kappa$ then $u(x) \geq a - \kappa$ for all $x \in \overline{\Omega}$ and hence the proof of Lemma 3.1 is completed.

In what follows, we shall show that $u(x) \geq \check{C}$ in the case where $a \leq \kappa$. Let

$$c_1(x) = a - u(x) - \frac{v(x)}{1 + u^2(x)},$$

then

$$|c_1(x)| \leq 2a + \kappa.$$

From Harnack's inequality (see [21]), there exists a positive constant C^* such that

$$\max_{\bar{\Omega}} u(x) \leq C^* \min_{\bar{\Omega}} u(x).$$

Hence, it remains to prove that there is a positive constant ε such that $\max_{\bar{\Omega}} u(x) > \varepsilon$. Suppose this is not true, then there exists a sequence $\{(d_{1n}, d_{2n}, \beta_n)\}_{n=1}^{\infty}$ such that the corresponding positive solutions (u_n, v_n) of problem (3.1) with $(d_1, d_2, \beta) = (d_{1n}, d_{2n}, \beta_n)$ satisfy $\max_{\bar{\Omega}} u_n \to 0$ as $n \to \infty$.

From the Sobolev embedding theorem and elliptic estimates, there exists a subsequence of $\{(u_n, v_n)^T\}_{n=1}^{\infty}$, which we still denote by $\{(u_n, v_n)\}_{n=1}^{\infty}$, such that $u_n \to u_\infty$ and $v_n \to v_\infty$ in $C^2(\overline{\Omega})$ as $n \to \infty$. From the assumption, we have $u_\infty \equiv 0$ and (u_∞, v_∞) satisfies (3.1). Then the second equation of (3.1) implies

$$-d_2 \Delta v_\infty = v_\infty (b - v_\infty) \quad \text{in } \Omega, \qquad \frac{\partial v}{\partial \mathbf{n}} = 0 \quad \text{on } \partial\Omega.$$

By the property of solutions of the logistic equation and $\min v_n \geq d_2 b/(d_2 + \beta)$, we have $v_\infty = b$. Denote by $\tilde{u}_n = u_n / \|u_n\|_{L^\infty}$ the L^∞ normalization of u_n. Then by dividing the first equation of (3.1) by $\|u_n\|_{L^\infty}$, we know that $\{\tilde{u}_n\}$ forms a sequence of positive solutions of

$$-d_1 \Delta \tilde{u}_n = \tilde{u}_n \left(a - u_n - \frac{v_n}{1 + u_n^2} \right) \quad \text{in } \Omega, \qquad \frac{\partial \tilde{u}_n}{\partial \mathbf{n}} = 0 \quad \text{on } \partial\Omega. \tag{3.5}$$

Note that $\|\tilde{u}_n\|_{L^\infty} = 1$ for $n \in \mathbb{N}$, then it follows from the elliptic regularity theory and the Sobolev embedding theorem that there exists a nonnegative function $\tilde{u}_\infty \in C^1(\bar{\Omega})$ such that $\lim_{n \to \infty} \tilde{u}_n = \tilde{u}_\infty$ in $C^1(\bar{\Omega})$. This, combining with $\|\tilde{u}_\infty\|_{L^\infty} = 1$, yields that $\tilde{u}_\infty > 0$. On the other hand, by integrating the first equation in (3.5) over Ω, we observe that

$$\int_\Omega \tilde{u}_n \left(a - u_n - \frac{v_n}{1 + u_n^2} \right) \mathrm{d}x = 0.$$

Let $n \to \infty$, and note that $\tilde{u}_\infty > 0$, $u_\infty = 0$ and $v_\infty = b$, then we have $a = b$, which contradicts our assumption. Therefore, we complete the proof of Lemma 3.1. □

4 Existence/nonexistence of nonconstant solutions

Throughout the remaining part of this paper, we always assume that $a > b$.

Lemma 4.1. *Every sequence $\{(u_n, v_n)\}_{n=1}^{\infty}$ of positive solutions of (3.1) with $a > b$ and $d_1 = d_{1n} \to \infty$ as $n \to \infty$ satisfies*

$$\|u_n - u^*\|_{L^\infty} + \|v_n - v^*\|_{L^\infty} \to 0 \quad \text{as } n \to \infty,$$

where $\mathbf{e} = (u^*, v^*)$ is the unique positive constant solution.

Proof. For fixed a, b, β and Ω, Lemma 3.1 and standard regularity arguments tell that $\{(u_n, v_n)\}_{n=1}^{\infty}$ has a convergent subsequence, which we still denote by $\{(u_n, v_n)\}_{n=1}^{\infty}$. According to the argument by Lou and Ni [24], we can obtain a positive constant K, which is independent of n, such that

$$\|u_n - \bar{u}_n\|_{L^\infty} \le \frac{K}{d_{1n}} \quad \text{with } \bar{u}_n = \frac{1}{|\bar{\Omega}|} \int_\Omega u_n \mathrm{d}x$$

for $n \in \mathbb{N}$. Together with Lemma 3.1, we can find a constant $\bar{u} \in [0, a]$ such that $\lim_{n\to\infty} u_n = \bar{u}$ uniformly in $\bar{\Omega}$. Lemma 3.1 and the standard L^p-estimate for elliptic equations mean that both $\{u_n\}_{n=1}^{\infty}$ and $\{v_n\}_{n=1}^{\infty}$ are uniformly bounded in $W^{2,p}(\Omega)$. Thus, the usual compactness argument implies

$$\lim_{n\to\infty} u_n = \bar{u} \quad \text{in } C^1(\bar{\Omega}), \tag{4.1}$$

passing to subsequence. We can similarly get a nonnegative function $\bar{v}$ such that

$$\lim_{n\to\infty} v_n = \bar{v} \quad \text{in } C^1(\bar{\Omega}), \tag{4.2}$$

passing to a subsequence. By setting $n \to \infty$ in the weak form of the second equation of (3.1) and using the elliptic regularity theory, we know that $\bar{v}$ satisfies

$$(d_2 + \beta\bar{u})\Delta\bar{v} = \bar{v}\left(b - \frac{\bar{v}}{1+\bar{u}^2}\right) \quad \text{in } C^1(\bar{\Omega}), \qquad \frac{\partial\bar{v}}{\partial\mathbf{n}} = 0 \quad \text{on } \partial\bar{\Omega}.$$

Since $\bar{u} \in [0, a]$ is constant, the well-known property of the logistic equation implies that $\bar{v}$ is also constant and satisfies

$$\bar{v} = 0 \quad \text{or} \quad b - \frac{\bar{v}}{1+\bar{u}^2} = 0. \tag{4.3}$$

Integrating the first equation of (3.1) yields

$$\int_\Omega u_n \left(a - u_n - b - \frac{v_n}{1+u_n^2}\right) \mathrm{d}x = 0, \qquad n \in \mathbb{N}. \tag{4.4}$$

By (4.1) and (4.2), letting $n \to \infty$ in (4.4) implies

$$\bar{u} = 0 \quad \text{or} \quad a - \bar{u} - \frac{\bar{v}}{1+\bar{u}^2} = 0$$

because $\bar{u}$ and $\bar{v}$ are constants. Suppose for contradiction that $a - \bar{u} - \frac{\bar{v}}{1+\bar{u}^2} \neq 0$. Hence (4.1) and (4.2) imply $a - u_n - \frac{v_n}{1+u_n^2} \neq 0$ in Ω for sufficiently large $n \in \mathbb{N}$. Together with $u_n > 0$ in Ω, we obtain

$$\int_\Omega u_n \left(a - u_n - \frac{v_n}{1+u_n^2}\right) \mathrm{d}x \neq 0$$

for sufficiently large $n \in \mathbb{N}$. However, this contradicts (4.4). Then we obtain $a - \bar{u} - \frac{\bar{v}}{1+\bar{u}^2} = 0$. Using a similar argument, we have $b - \frac{\bar{v}}{1+\bar{u}^2} = 0$. Therefore, $(\bar{u}, \bar{v}) = (u^*, v^*)$. □

Theorem 4.2. *For any fixed $(d_2, \beta, a, b, \Omega)$ satisfying $a > b$, there exists a large positive constant D such that (3.1) with $d_1 \geq D$ has no nonconstant solutions.*

Proof. Assume that (u, v) is a non-negative solution of (3.1) and denote

$$\bar{u} = \frac{1}{|\Omega|}\int_\Omega u \mathrm{d}x, \qquad \bar{v} = \frac{1}{|\Omega|}\int_\Omega v \mathrm{d}x.$$

Then, multiplying $u - \bar{u}$ the first equation in (3.1) by and integrating over Ω yield

$$\begin{aligned} d_1 \int_\Omega |\nabla u|^2 \mathrm{d}x &= \int_\Omega u(a - u - \frac{v}{1+u^2})(u - \bar{u})\mathrm{d}x \\ &= \int_\Omega \left(a - u - \bar{u} - \frac{v}{1+u^2} + \frac{\bar{u}\bar{v}(u+\bar{u})}{(1+u^2)(1+\bar{u}^2)}\right)(u-\bar{u})^2 \mathrm{d}x \\ &\quad - \int_\Omega \frac{\bar{u}(u-\bar{u})(v-\bar{v})}{1+u^2}\mathrm{d}x \\ &\leq \int_\Omega (a + 2a^2\kappa)(u-\bar{u})^2\mathrm{d}x + \frac{\bar{u}}{2}\left[\int_\Omega \frac{(u-\bar{u})^2}{1+u^2}\mathrm{d}x + \int_\Omega \frac{(v-\bar{v})^2}{1+u^2}\mathrm{d}x\right] \\ &\leq \left(\frac{3a}{2} + 2a^2\kappa\right)\int_\Omega (u-\bar{u})^2\mathrm{d}x + \frac{a}{2}\int_\Omega (v-\bar{v})^2\mathrm{d}x, \end{aligned} \tag{4.5}$$

where the last inequality comes from Lemma 3.1. Recall the Poincaré–Wirtinger inequality $\lambda_1\|U - \bar{U}\|_{L^2}^2 \leq \|\nabla U\|_{L^2}^2$ for any $U \in H^1(\Omega)$, where λ_1 is the least positive eigenvalue of $-\Delta$ with homogeneous Neumann boundary condition on $\partial\Omega$. Then it follows from (4.5) that

$$\left[1 - \frac{a}{\lambda_1 d_1}\left(\frac{3}{2} + 2a\kappa\right)\right]\|\nabla u\|_{L^2}^2 \leq \frac{a}{2\lambda_1 d_1}\|\nabla v\|_{L^2}^2. \tag{4.6}$$

Similarly, multiplying by $v - \bar{v}$ the second equation of (3.1) and integrating the resulting expression lead us to

$$\begin{aligned} \int_\Omega (d_2 + \beta u)|\nabla v|^2 \mathrm{d}x &= \int_\Omega v\left(b - \frac{v}{1+u^2}\right)(v-\bar{v})\mathrm{d}x - \beta\int_\Omega v\nabla u\cdot\nabla v \mathrm{d}x \\ &= \int_\Omega \left(b - \frac{v}{1+u^2} - \frac{\bar{v}}{1+u^2}\right)(v-\bar{v})^2\mathrm{d}x \\ &\quad + \int_\Omega \frac{\bar{v}^2(u-\bar{u})(v-\bar{v})(u+\bar{u})}{(1+u^2)(1+\bar{u}^2)}\mathrm{d}x - \beta\int_\Omega v\nabla u\cdot\nabla v\mathrm{d}x. \end{aligned}$$

By Lemma 3.1 and Young's inequality, for any $\varepsilon > 0$, one can find a positive constant K such that

$$\begin{aligned} \int_\Omega (d_2 + \beta u)|\nabla v|^2\mathrm{d}x \leq& \int_\Omega \left(b - \frac{v}{1+u^2} - \frac{\bar{v}}{1+u^2}\right)(v-\bar{v})^2\mathrm{d}x \\ &+ 2a\kappa^2\left[\int_\Omega \frac{K}{\varepsilon}(u-\bar{u})^2\mathrm{d}x + \int_\Omega \varepsilon(v-\bar{v})^2\mathrm{d}x\right] \\ &+ \beta\kappa\left[\int_\Omega \frac{K}{\varepsilon}|\nabla u|^2\mathrm{d}x + \int_\Omega \varepsilon|\nabla v|^2\mathrm{d}x\right], \end{aligned} \tag{4.7}$$

where κ is the positive number given in Lemma 3.1. Then the Poincaré–Wirtinger inequality implies

$$\begin{aligned} &\left[1 - \varepsilon\kappa\left(\frac{2a\kappa}{d_2\lambda_1} + \frac{\beta}{d_2}\right)\right]\|\nabla v\|_{L^2}^2 \\ &\leq \frac{1}{d_2}\int_\Omega \left(b - \frac{v}{1+u^2} - \frac{\bar{v}}{1+u^2}\right)(v-\bar{v})^2\mathrm{d}x + \frac{\kappa K}{\varepsilon}\left(\frac{2a\kappa}{d_2\lambda_1} + \frac{\beta}{d_2}\right)\|\nabla u\|_{L^2}^2. \end{aligned} \tag{4.8}$$

Note that

$$b - \frac{v}{1+u^2} - \frac{\bar{v}}{1+u^2} = b - \frac{\bar{v}}{1+\bar{u}^2} - \left(\frac{\bar{v}}{1+u^2} - \frac{\bar{v}}{1+\bar{u}^2}\right) - \frac{v}{1+u^2},$$

then it follows from Lemma 4.1 that

$$b - \frac{v}{1+u^2} - \frac{\bar{v}}{1+u^2} < \varepsilon \quad \text{if } d_1 > 0 \text{ is sufficiently large.} \tag{4.9}$$

Thus, when $d_1 > 0$ is large, (4.6) and (4.8) enable us to find a positive constant K_1 such that

$$\|\nabla u\|_{L^2}^2 \leq \frac{K_1}{d_1}\|\nabla u\|_{L^2}^2,$$

which implies that u is a constant if d_1 is large enough. Combining with (4.8) and (4.9), we deduce that (u, v) is a constant solution if $d_1 > 0$ is sufficiently large. Then the proof of Theorem 4.2 is completed. □

Remark 4.3. The conclusion of Theorem 4.2 is still valid in the case where $\beta = 0$, that is, for any fixed (d_2, a, b, Ω) with $a > b$, there exists a large positive constant D such that (3.1) with $\beta = 0$ and $d_1 \geq D$ has no nonconstant solutions.

Recall that $-\Delta$ under Neumann boundary condition has eigenvalues $0 = \lambda_0 < \lambda_1 < \cdots < \lambda_n < \cdots$ with $\lim_{n\to\infty} \lambda_n = +\infty$. Let $\mathbb{S}_i$ be the eigenspace associated with λ_i with multiplicity n_i. Let ϕ_{ij}, $1 \leq j \leq n_i$, be the normalized eigenfunctions corresponding to λ_i. Then the set $\{\phi_{ij} | \, i \geq 0, 1 \leq j \leq n_i\}$ forms a complete orthonormal basis of the Lebesgue space $L^2(\overline{\Omega})$ of integrable functions defined on Ω, $\phi_0(x) > 0$ for all $x \in \Omega$. Let $\mathbb{X}_{ij} = \{c\phi_{ij} | \, c \in \mathbb{R}^2\}$, and $\{\phi_{ij} | \, 1 \leq j \leq \dim \mathbb{S}_i\}$ be an orthonormal basis of $\mathbb{S}_i$. For $i \geq 0$, it can be observed that

$$\mathbb{X} = \bigoplus_{i=1}^{\infty} \mathbb{X}_i \quad \text{and} \quad \mathbb{X}_i = \bigoplus_{j=1}^{\dim \mathbb{S}_i} \mathbb{X}_{ij}. \tag{4.10}$$

Next, we study the linearization of (3.1) at (u^*, v^*), where $\mathbf{e} = (u^*, v^*)$ is the unique positive constant solution of (1.4). Let $\Phi(U) = (d_1 u, d_2 v + \beta uv)^T$ and

$$G(U) = \begin{bmatrix} u\left(a - u - \dfrac{v}{1+u^2}\right) \\ v\left(b - \dfrac{v}{1+u^2}\right) \end{bmatrix}$$

for $U = (u, v)^T$. Then (3.1) can be rewritten as

$$\begin{cases} -\Delta\Phi(U) = G(U) & \text{in } \Omega, \\ \dfrac{\partial U}{\partial \mathbf{n}} = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.11}$$

Define

$$\mathbb{X}^+ = \{U \in \mathbb{X} \mid u > 0,\, v > 0 \text{ on } \bar{\Omega}\}$$

and

$$\mathbb{B} = \left\{U \in \mathbb{X} \;\middle|\; \frac{1}{C} < u < C,\, \frac{1}{C} < v < C\right\},$$

where C is a positive constant whose existence is guaranteed by Lemma 3.1. Note that the derivative $\Phi_U(U)$ of $\Phi(U)$ with respect to U is a positive operator for all non-negative U, then

$\Phi_U^{-1}(U)$ exists and is a positive operator as well. Hence, U is a positive solution to (4.11) if and only if

$$F(U) \triangleq U - (I-\Delta)^{-1}\left\{\Phi_U^{-1}(U)[G(U) + \nabla U \Phi_{UU}(U)\nabla U^T] + U\right\} = 0 \quad \text{in } \mathbb{X}^+,$$

where $(I-\Delta)^{-1}$ is the inverse of $I-\Delta$ in X. As $F(\cdot)$ is a compact perturbation of the identity operator, the Leray–Schauder degree $\deg(F(\cdot), 0, \mathbb{B})$ is well-defined if $F(U) \neq 0$ on $\partial\mathbb{B}$. Note that

$$D_U F(\mathbf{e}) = I - (I-\Delta)^{-1}\{\Phi_U^{-1}(\mathbf{e})G_U(\mathbf{e}) + I\},$$

and that the index of F at $\mathbf{e}$ is defined as $\text{index}(F(\cdot), \mathbf{e}) = (-1)^\gamma$ provided that $D_U F(\mathbf{e})$ is invertible, where $\gamma = \sum m_\mu$ and m_μ is the multiplicity of any negative eigenvalue μ of $D_U F(\mathbf{e})$; see [28] for more details.

We now consider the eigenvalues of $D_U F(\mathbf{e})$. First, for every integer $i \geq 0$ and $1 \leq j \leq \dim\mathbb{S}_i$, $\mathbb{X}_{ij}$ is invariant under $D_U F(\mathbf{e})$, and μ is an eigenvalue of $D_U F(\mathbf{e})$ on $\mathbb{X}_{ij}$ if and only if it is an eigenvalue of the matrix

$$I - \frac{1}{1+\lambda_i}\left[\Phi_U^{-1}(\mathbf{e})G_U(\mathbf{e}) + I\right] = \frac{1}{1+\lambda_i}\left[\lambda_i I - \Phi_U^{-1}(\mathbf{e})G_U(\mathbf{e})\right].$$

Thus, $D_U F(\mathbf{e})$ is invertible if and only if, for all $i \geq 0$, the above matrix is nonsingular. To calculate γ, we first define

$$H(\lambda) = \det\{\lambda I - \Phi_U^{-1}(\mathbf{e})G_U(\mathbf{e})\}. \tag{4.12}$$

If $H(\lambda_i) \neq 0$, then for each $1 \leq j \leq \dim\mathbb{S}_i$, the number of negative eigenvalues of $D_U F(\mathbf{e})$ on $\mathbb{X}_{ij}$ is odd if and only if $H(\lambda_i) < 0$. In conclusion, we have the following lemma (see [29]), which gives the explicit formula of calculating the index.

Lemma 4.4. *If $a > b$ and $H(\lambda_i) \neq 0$ for all $i \geq 0$, then*

$$\text{index}(F(\cdot), \mathbf{e}) = (-1)^\gamma \quad \textit{with} \quad \gamma = \sum_{i\geq 0, H(\lambda_i)<0} n_i(\lambda_i),$$

where $n_i(\lambda_i)$ is the algebraic multiplicity of λ_i.

To facilitate our computation of $\text{index}(F(\cdot), \mathbf{e})$, we will consider the sign of $H(\lambda_i)$. Notice that our aim is to investigate the effect of the cross-diffusion coefficient β and diffusion coefficient d_1 on the existence of stationary patterns. Then we will concentrate on the dependence of $H(\lambda_i)$ on β and d_1. Note that

$$\lambda I - \Phi_U^{-1}(\mathbf{e})G_U(\mathbf{e}) = \begin{bmatrix} \lambda - \frac{2\theta^2 b}{d_1(1+\theta^2)} + \frac{\theta}{d_1} & \frac{\theta}{d_1(1+\theta^2)} \\ -\frac{\beta\theta b(1+\theta^2)}{d_1(d_2+\beta\theta)} + \frac{2\beta\theta^2 b^2}{d_1(d_2+\beta\theta)} - \frac{2b\theta^2}{d_2+\beta\theta} & \lambda - \frac{\beta\theta b}{d_1(d_2+\beta\theta)} + \frac{b}{d_2+\beta\theta} \end{bmatrix}.$$

Then, we have

$$H(\lambda) = \lambda^2 - \left[\frac{\beta\theta b}{d_1(d_2+\beta\theta)} + \frac{2\theta^2 b}{d_1(1+\theta^2)} - \frac{\theta}{d_1} - \frac{b}{d_2+\beta\theta}\right]\lambda + \frac{b\theta}{d_1(d_2+\beta\theta)}. \tag{4.13}$$

If

$$\Lambda(\beta, d_1) \triangleq \left[\frac{\beta\theta b}{d_1(d_2+\beta\theta)} + \frac{2\theta^2 b}{d_1(1+\theta^2)} - \frac{\theta}{d_1} - \frac{b}{d_2+\beta\theta}\right]^2 - \frac{4b\theta}{d_1(d_2+\beta\theta)} > 0,$$

then $H(\lambda) = 0$ has two roots $\lambda = \lambda^{+}(\beta, d_1)$ and $\lambda = \lambda^{-}(\beta, d_1)$, where

$$\lambda^{\pm}(\beta, d_1) = \frac{1}{2}\left[\frac{\beta\theta b}{d_1(d_2+\beta\theta)} + \frac{2\theta^2 b}{d_1(1+\theta^2)} - \frac{\theta}{d_1} - \frac{b}{d_2+\beta\theta} \pm \sqrt{\Lambda(\beta, d_1)}\right].$$

We first consider the dependence of $H(\lambda)$ on β. When β is large enough, we have $\Lambda > 0$ and the two roots of $H(\lambda)$ satisfy

$$\lim_{\beta\to\infty} \lambda^{-}(\beta, d_1) = 0$$

and

$$\lim_{\beta\to\infty} \lambda^{+}(\beta, d_1) = \frac{(1+3\theta^2)b - \theta(1+\theta^2)}{d_1(1+\theta^2)} \triangleq \bar{\lambda} \quad \text{if } b > \frac{(1+\theta^2)\theta}{1+3\theta^2}. \tag{4.14}$$

Thus, we have the following existence result about the non-constant steady state solution:

Theorem 4.5. *Assume that $a > b$, $b > \frac{(1+\theta^2)\theta}{1+3\theta^2}$ and $\bar{\lambda} \in (\lambda_n, \lambda_{n+1})$ for some $n \geq 1$ and $\sum_{i=1}^{n} n_i(\lambda_i)$ is odd, then there exists a positive number β^* such that system (3.1) with $\beta \geq \beta^*$ has at least one non-constant positive solution.*

Proof. In virtue of (4.14) and $\bar{\lambda} \in (\lambda_n, \lambda_{n+1})$, there exists a positive constant β^* such that, if $\beta \geq \beta^*$ then

$$0 < \lambda^{-}(\beta, d_1) < \lambda_1 \quad \text{and} \quad \lambda^{+}(\beta, d_1) \in (\lambda_n, \lambda_{n+1}). \tag{4.15}$$

We argue by contradiction. Assume that system (3.1) with $\beta \geq \beta^*$ has no non-constant positive solutions. For $s \in [0, 1]$, define

$$\Psi(s, U) = ((sd_1 + (1-s)d_1^*)u, (d_2 + s\beta u)v)^T,$$

where d_1^* is a positive constant such that $d_1^* \geq D$ and $\frac{2\theta^2 b}{d_1^*(1+\theta^2)} - \frac{\theta}{d_1^*} - \frac{b}{d_2} < 0$. Obviously, $\Psi(1, \cdot) = \Phi(\cdot)$. Consider the following system

$$\begin{cases} -\Delta\Psi(s, U) = G(U) & \text{in } \Omega, \\ \dfrac{\partial U}{\partial \mathbf{n}} = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.16}$$

Then U is a positive non-constant solution of (3.1) if and only if it is a solution to (4.16) with $s = 1$. It is obvious that $\mathbf{e}$ is the unique constant positive solution of (4.16) for all $0 \leq s \leq 1$. U is a positive solution of (4.16) if and only if

$$\mathcal{F}(s, U) \triangleq U - (I-\Delta)^{-1}\left\{\Psi_U^{-1}(s, U)[G(U) + \nabla U \Psi_{s,UU}(U)\nabla U^T] + U\right\} = 0 \quad \text{in } \mathbb{X}^+.$$

It is obvious that $\mathcal{F}(1, U) = F(U)$. Remark 4.3 says that $\mathcal{F}(0, U) = 0$ has only one positive solution $\mathbf{e}$ in $\mathbb{X}^+$. By a direct computation, we have

$$D_U\mathcal{F}(s, \mathbf{e}) = I - (I-\Delta)^{-1}\{\Psi_U^{-1}(s, \mathbf{e})G_U(\mathbf{e}) + I\}.$$

In particular,

$$D_U\mathcal{F}(0, \mathbf{e}) = I - (I-\Delta)^{-1}\{\Psi_U^{-1}(0, \mathbf{e})G_U(\mathbf{e}) + I\},$$
$$D_U\mathcal{F}(1, \mathbf{e}) = I - (I-\Delta)^{-1}\{\Phi_U^{-1}(\mathbf{e})G_U(\mathbf{e}) + I\} = D_U F(\mathbf{e}),$$

where $\Psi_U(0, \cdot) = \mathrm{diag}(d_1^*, d_2)$. From the previous analysis, we know that the key point is to determine the sign of

$$\mathcal{H}(s, \lambda) = \det\{\lambda I - \Psi_U^{-1}(s, \mathbf{e})G_U(\mathbf{e})\}. \tag{4.17}$$

By direction calculation, we have $\mathcal{H}(0,\lambda) = \lambda^2 - \left(\frac{2\theta^2 b}{d_1^*(1+\theta^2)} - \frac{\theta}{d_1^*} - \frac{b}{d_2}\right)\lambda + \frac{b\theta}{d_1^* d_2}$ and hence

$$\mathcal{H}(0,\lambda_i) > 0 \quad \text{for all } i \geq 0.$$

Clearly, $\mathcal{H}(1,\lambda) = H(\lambda)$. Therefore, in view of (4.14) and (4.15), we can get

$$\begin{cases} H(\lambda_0) = H(0) > 0, & \\ H(\lambda_i) < 0 & \text{when } 1 \leq i \leq n, \\ H(\lambda_i) > 0 & \text{when } i \geq n+1. \end{cases}$$

Therefore, zero is not an eigenvalue of $\lambda_i I - \Phi_U^{-1}(\mathbf{e})G_U(\mathbf{e})$ for all $i \geq 0$, and

$$\sum_{i\geq 1, H(\lambda_i)<0} \dim \mathbb{S}_i = \sum_{i=1}^{n} n_i(\lambda_i), \quad \text{which is odd.}$$

Thanks to Lemma 4.4, we have

$$\begin{aligned} \text{index}(\mathcal{F}(1,\cdot),\mathbf{e}) &= (-1)^{\gamma} = (-1)^{\sum_{i=1}^{n} n_i(\lambda_i)} = -1, \\ \text{index}(\mathcal{F}(0,\cdot),\mathbf{e}) &= (-1)^{\gamma} = (-1)^0 = 1. \end{aligned}$$

Now, by Lemma 3.1, we know that every positive solution of system (3.1) lies in $\mathbb{B}$ and $\mathcal{F}(t,\cdot) \neq 0$ on $\partial\mathbb{B}$. So $\deg(\mathcal{F}(s,\cdot),\mathbb{B},0)$ is well defined. By the homotopy invariance of topological degree, we have

$$\deg(\mathcal{F}(1,\cdot),\mathbb{B},0) = \deg(F(0,\cdot),\mathbb{B},0). \tag{4.18}$$

On the other hand, from our assumption, both equations $\mathcal{F}(1,\mathbf{e}) = 0$ and $\mathcal{F}(0,\mathbf{e}) = 0$ have only one positive solution $\mathbf{e}$ in $\mathbb{B}$, then we have

$$\begin{aligned} \deg(\mathcal{F}(1,\cdot),\mathbb{B},0) &= \text{index}(\mathcal{F}(1,\cdot),\mathbf{e}) = -1, \\ \deg(\mathcal{F}(0,\cdot),\mathbb{B},0) &= \text{index}(\mathcal{F}(0,\cdot),\mathbf{e}) = 1, \end{aligned}$$

which is a contradiction with (4.18). So the proof is completed. □

Next we consider the dependence of $H(\lambda)$ on d_1. From the previous analysis, it follows that the roots of $H(\lambda) = 0$ are all negative if $\frac{2\theta b}{1+\theta^2} + \frac{\beta b}{d_2+\beta\theta} - 1 < 0$ and $\Lambda(\beta, d_1) > 0$. So, in this case, we can't obtain the existence of non-constant positive solutions by using the method of degree theory.

We begin with the case $\frac{2\theta b}{1+\theta^2} + \frac{\beta b}{d_2+\beta\theta} - 1 > 0$. By straightforward computations, one can get $\Lambda(\beta, d_1) > 0$ and the two roots of $H(\lambda) = 0$ satisfy $0 < \lambda^-(\beta, d_1) < \lambda^+(\beta, d_1)$ if $d_1 \in (0, d^*)$ and $\frac{2\theta b}{1+\theta^2} + \frac{\beta b}{d_2+\beta\theta} > 1$, where

$$d^* = \left\{ \frac{1}{b}\sqrt{\theta b(d_2+\beta\theta)} \left(-1 + \sqrt{\left(\frac{\beta b}{d_2+\beta\theta} + \frac{2\theta b}{1+\theta^2} \right)} \right) \right\}^2.$$

Furthermore, one can verify that $\lambda^-(\beta, d_1)$ is monotone increasing and $\lambda^+(\beta, d_1)$ is monotone decreasing with respect to $d_1 \in (0, d^*)$. Moreover, $\lambda^+(\beta, d_1)$ and $\lambda^-(\beta, d_1)$ satisfy

$$\begin{aligned} &\lim_{d_1\to 0} \lambda^-(\beta, d_1) = \frac{b}{(d_2+\beta\theta)} \left(-1 + \frac{2\theta b}{1+\theta^2} + \frac{\beta b}{d_2+\beta\theta} \right)^{-1} \triangleq \eta, \\ &\lim_{d_1\to 0} \lambda^+(\beta, d_1) = +\infty, \\ &\lim_{d_1\to d^*} \lambda^-(\beta, d_1) = \lim_{d_1\to d^*} \lambda^+(\beta, d_1) = \frac{b}{(d_2+\beta\theta)} \left(-1 + \sqrt{\frac{2\theta b}{1+\theta^2} + \frac{\beta b}{d_2+\beta\theta}} \right)^{-1}. \end{aligned}$$

In order to state the structure of nonconstant solutions, we introduce the following two natural numbers j_0 and k_0 by

$$j_0 \triangleq \min\left\{ j \in \mathbb{N} \,\middle|\, \frac{b}{(d_2+\beta\theta)}\left(-1+\frac{2\theta b}{1+\theta^2}+\frac{\beta b}{d_2+\beta\theta}\right)^{-1} < \lambda_j \right\},$$

$$k_0 \triangleq \min\left\{ k \in \mathbb{N} \,\middle|\, \frac{b}{(d_2+\beta\theta)}\left(-1+\sqrt{\frac{2\theta b}{1+\theta^2}+\frac{\beta b}{d_2+\beta\theta}}\right)^{-1} \leq \lambda_k \right\} (\geq j_0).$$

Since $\lambda^+(\beta, d_1)$ is monotone decreasing with respect to d_1 and $\lim_{d_1\to 0}\lambda^+(\beta, d_1) = +\infty$, there are positive numbers

$$d_{1k} = \sup\{d_1 > 0 |\, \lambda^+(\beta, d_1) > \lambda_k\} \quad \text{for } k = k_0, k_0+1, \ldots. \tag{4.19}$$

Solving $\lambda^+(\beta, d_1) = \lambda_k$ for d_1, we get the solution d_{1k} $(k = k_0, k_0+1, \ldots)$ with

$$d_{1k} = \left(-1+\frac{2\theta b}{1+\theta^2}+\frac{\beta b}{d_2+\beta\theta}-\frac{b\theta}{\lambda_k(d_2+\beta\theta)}\right)\left(\lambda_k+\frac{b}{d_2+\beta\theta}\right)^{-1}.$$

Therefore, the sequence $\{d_{1k}\}_{k=k_0}^{\infty}$ defined by (4.19) satisfies

$$0 \leftarrow \cdots < d_{1k} < \cdots < d_{1k_0+1} < d_{1k_0} < d^* \triangleq d_{1k_0-1}. \tag{4.20}$$

If $k_0 > j_0$, we define

$$\tilde{d}_{1j} \triangleq \inf\{d_1 > 0 |\, \lambda^-(\beta, d_1) > \lambda_j\} \quad \text{for } j = j_0, j_0+1, \cdots, k_0-1.$$

Similarly, it follows from $\lambda^-(\beta, d_1) = \lambda_j$ that $\tilde{d}_{1j} = d_{1j}$. Hence the monotone increasing property of $\lambda^-(\beta, d_1)$ for $d_1 \in (0, d^*)$ induces the monotone increasing property of $\{\tilde{d}_{1j}\}_{j=j_0}^{k_0-1}$ as

$$(\tilde{d}_{j_0-1} \triangleq) 0 < \tilde{d}_{j_0} < \tilde{d}_{j_0+1} < \cdots < \tilde{d}_{k_0-1} < d^*.$$

Therefore, we have the following conclusions:

Theorem 4.6. *Assume that $a > b$ and $\frac{2\theta b}{1+\theta^2}+\frac{\beta b}{d_2+\beta\theta} > 1$, then the following* (i) *and* (ii) *hold true:*

(i) In case where $k_0 > j_0$, there exists at least one nonconstant solution of (3.1) *provided that $d_1 \in (\tilde{d}_{1j}, \tilde{d}_{1j+1}) \cap (d_{1k+1}, d_{1k})$ and $\sum_{i=j+1}^{k} n_i(\lambda_i)$ is odd or $d_1 \in (d_{1k+1}, d_{1k}) \cap (\tilde{d}_{k_0-1}, d^*)$ and $\sum_{i=k_0}^{k} n_i(\lambda_i)$ is odd.*

(ii) In case where $k_0 = j_0$, there exists at least one nonconstant solution of (3.1) *provided that $d_1 \in (d_{1k+1}, d_{1k})$ and $\sum_{i=k_0}^{k} n_i(\lambda_i)$ is odd.*

Proof. In the case where $\frac{2\theta b}{1+\theta^2}+\frac{\beta b}{d_2+\beta\theta} > 1$, suppose for contradiction that there is no nonconstant solution of (3.1). According to Lemma 3.1, we know every positive solution of system (3.1) lies in $\mathbb{B}$ and $F(U) \neq 0$ on $\partial\mathbb{B}$. Then the homotopy invariance of topological degree implies

$$\deg(F(\cdot), \mathbb{B}, 0) \quad \text{is constant for all } d_1 > 0. \tag{4.21}$$

In view of Theorem 4.2, we recall that if $d_1 \geq D$, then $F(U) = 0$ has a unique solution $\mathbf{e}$ in $\mathbb{X}^+$. Therefore, we know that

$$\deg(F(\cdot), \mathbb{B}, 0) = \text{index}(F(\cdot), \mathbf{e}) \quad \text{for } d_1 \geq D.$$

It is easy to verify that $\lambda^+(\beta, d_1)$ is monotone decreasing with respect to d_1 and satisfies $\lim_{d_1\to\infty} \lambda^+(\beta, d_1) = 0$. Together with $H(\lambda_0) > 0$ and $\lambda^-(\beta, d_1) < \lambda^+(\beta, d_1)$, we obtain $H(\lambda_i) > 0$ for all $i \geq 0$ when d_1 is sufficiently large. It follows from Lemma 4.4 that if $d_1 > 0$ is large enough,

$$\deg(F(\cdot), \mathbb{B}, 0) = \operatorname{index}(F(\cdot), \mathbf{e}) = (-1)^\gamma = (-1)^0 = 1. \tag{4.22}$$

On the other hand, if $d_1 \in (\tilde{d}_{1j}, \tilde{d}_{1j+1}) \cap (d_{k+1}, d_k)$, then (4.19) and (4.20) imply that $\lambda_j < \lambda^-(\beta, d_1) < \lambda_{j+1}$ and $\lambda^+(\beta, d_1) > \lambda_k$. Hence, if $k_0 > j_0$, we can get

$$\begin{cases} H(\lambda_0) = H(0) > 0, \\ H(\lambda_i) < 0 & \text{when } j+1 \leq i \leq k, \\ H(\lambda_i) > 0 & \text{when } i \geq k. \end{cases}$$

By Lemma 4.4, we have

$$\operatorname{index}(F(\cdot), \mathbf{e}) = (-1)^\gamma = (-1)^{\sum_{i=j+1}^{k} n_i(\lambda_i)}.$$

If $\sum_{i=j+1}^{k} n_i(\lambda_i)$ is odd , then

$$\deg(F(\cdot), B, 0) = \operatorname{index}(F(\cdot), \mathbf{e}) = (-1)^\gamma = (-1)^{\sum_{i=j+1}^{k} n_i(\lambda_i)} = -1,$$

which is a contradiction with (4.22). Consequently, by the contradiction argument, we obtain at least one nonconstant solution if $d_1 \in (\tilde{d}_{1j}, \tilde{d}_{1j+1}) \cap (d_{k+1}, d_k)$ and $\sum_{i=j+1}^{k} n_i(\lambda_i)$ is odd. Similarly, we have $\lambda(k_0 - 1) < \lambda^-(\beta, d_1) < \lambda^-(\beta, d^*) \leq \lambda(k_0)$ and $\lambda^+(\beta, d_1) > \lambda(k)$ if $d_1 \in (\tilde{d}_{k_0-1}, d^*) \cap (d_{k+1}, d_k)$. Therefore,

$$\begin{cases} H(\lambda_0) = H(0) > 0, \\ H(\lambda_i) < 0, & \text{when } k_0 \leq i \leq k, \\ H(\lambda_i) > 0, & \text{when } i \geq k \end{cases}$$

if $k_0 > j_0$. Through similar calculations, we can get a contradiction with (4.22) if $\sum_{i=k_0}^{k} n_i(\lambda_i)$ is odd. So the proof for the statement (i) is completed. The proof for statement (ii) can be carried out by a similar manner. □

Remark 4.7. In particular, assume that $j_0 = k_0 = 1$, namely,

$$\frac{b}{(d_2 + \beta\theta)} \left(-1 + \sqrt{\frac{2\theta b}{1 + \theta^2} + \frac{\beta b}{d_2 + \beta\theta}} \right)^{-1} \leq \lambda_1.$$

If $a > b$ and

$$\frac{2\theta b}{1 + \theta^2} + \frac{\beta b}{d_2 + \beta\theta} > 1,$$

then there exists a sequence $\{d_{1k}\}_{j=0}^{\infty}$ such that $0 \leftarrow \cdots < d_{1k} < \cdots < d_{12} < d_{11}$ and (3.1) admits at least one nonconstant solution if $d_1 \in (d_{1k+1}, d_{1k})$ and $\sum_{i=1}^{k} n_i(\lambda_i)$ is odd.

5 Stability of the positive constant solution

In this section, we firstly analyze the stability of the positive constant steady-state solution by eigenvalue analysis. And then, we will investigate the global stability of the positive constant steady-state solution. To investigate the local dynamical behavior of system (1.4) near the positive constant solution $\mathbf{e}$, we need to consider the linearized operator $\mathcal{L}_{\alpha_1,\beta}$ of (1.4) with respect to (u,v) at (u^*,v^*). Note that

$$\mathcal{L}_{\alpha_1,\beta} = \begin{bmatrix} d_1\Delta + \alpha_1 & -\alpha_2 \\ \beta b(1+\theta^2)\Delta + \alpha_3 & (d_2+\beta\theta)\Delta - b \end{bmatrix},$$

where

$$\theta = a-b, \quad \alpha_1 = -\theta + \frac{2b\theta^2}{1+\theta^2}, \quad \alpha_2 = \frac{\theta}{1+\theta^2}, \quad \alpha_3 = 2b^2\theta.$$

The characteristic equation is $\mathcal{L}_{\alpha_1,\beta}(\tilde{\phi}_1,\tilde{\phi}_2) = \sigma(\tilde{\phi}_1,\tilde{\phi}_2)$. Let $\tilde{\phi}_1 = \sum_{0\le i\le\infty} a_i\varphi_i$, $\tilde{\phi}_2 = \sum_{0\le i\le\infty} b_i\varphi_i$. Notice that $\{\varphi_i\}_{i=0}^{\infty}$ is a complete orthogonal base of $\mathbb{X}$. Substituting them into the characteristic equation yields

$$\sum_{0\le i\le\infty} \mathcal{M}(\sigma,\alpha_1,\beta,\lambda_i)(a_i,b_i)^T\varphi_i = 0,$$

where

$$\mathcal{M}(\sigma,\alpha_1,\beta,\lambda_i) = \begin{bmatrix} -d_1\lambda_i + \alpha_1 - \sigma & -\alpha_2 \\ -\beta b(1+\theta^2)\lambda_i + \alpha_3 & -(d_2+\beta\theta)\lambda_i - b - \sigma \end{bmatrix}.$$

To investigate the stability of the positive steady-state solution, it suffices to study the characteristic equation $\det\mathcal{M}(\sigma,\alpha_1,\beta,\lambda_i) = 0$, that is,

$$\sigma^2 - T_i(\alpha_1,\beta)\sigma + D_i(\alpha_1,\beta) = 0, \qquad i = 0,1,2,\ldots, \tag{5.1}$$

where

$$\begin{aligned} T_i(\alpha_1,\beta) &= -(d_1+d_2+\beta\theta)\lambda_i + \alpha_1 - b, \\ D_i(\alpha_1,\beta) &= d_1(d_2+\beta\theta)\lambda_i^2 + [bd_1 - (d_2+\beta\theta)\alpha_1 - b\beta(1+\theta^2)\alpha_2]\lambda_i + b\theta. \end{aligned}$$

It is easy to know that two solutions of equation (5.1) have negative real parts if $T_i(\alpha_1,\beta) < 0$ and $D_i(\alpha_1,\beta) > 0$ for all $i \ge 0$. Thus, we have the following results.

Lemma 5.1. *If $a > b$, then all eigenvalues of $\mathcal{L}_{\alpha_1,\beta}$ have negative real parts, or equivalently, the homogenous steady-state $\mathbf{e} = (\theta, b(1+\theta^2))$ is locally asymptotically stable, provided that one of the following conditions is satisfied:*

(i) either $\alpha_1 < -b$ or $-b < \alpha_1 < 0$ and $\beta < \frac{bd_1 - d_2\alpha_1}{(\alpha_1+b)\theta}$ or $0 < \alpha_1 < \min\{b, \frac{bd_1}{d_2}\}$ and $\beta \le \frac{bd_1-d_2\alpha_1}{(\alpha_1+b)\theta}$;

(ii) $0 < \alpha_1 < \min\{b, \frac{bd_1}{d_2}\}$ and $\beta > \frac{bd_1-d_2\alpha_1}{(\alpha_1+b)\theta}$ and $[(bd_1 - d_2\alpha_1) - (\alpha_1+b)\beta\theta]^2 < 4d_1(d_2+\beta\theta)b\theta$;

(iii) $d_1 < d_2$ and $\frac{d_1 b}{d_2} < \alpha_1 < b$ and $[(bd_1 - d_2\alpha_1) - (\alpha_1+b)\beta\theta]^2 < 4d_1(d_2+\beta\theta)b\theta$.

Now, we consider the global stability of $\mathbf{e}$.

Lemma 5.2. *If $a > b$, $2b(a+c) < 1$ and $\beta \le \frac{2}{c}\sqrt{\frac{d_1d_2\theta}{b(1+\theta^2)}}$, then $\mathbf{e}$ is globally asymptotically stable.*

Proof. We discuss the global stability of **e** by Lyapunov method. Define

$$L(u(x,t),v(x,t)) = \int_\Omega \int_{u^*}^{u} \frac{\xi - u^*}{\xi} \mathrm{d}\xi \mathrm{d}x + \int_\Omega \int_{v^*}^{v} \frac{\eta - u^*}{\eta} \mathrm{d}\eta \mathrm{d}x.$$

Then

$$\begin{aligned} L'(t) &= \int_\Omega \frac{u-u^*}{u}\frac{\partial u}{\partial t}\mathrm{d}x + \int_\Omega \frac{v-v^*}{v}\frac{\partial v}{\partial t}\mathrm{d}x \\ &= d_1 \int_\Omega \frac{u-u^*}{u}\Delta u \mathrm{d}x + \int_\Omega \frac{v-v^*}{v}[\Delta(d_2+\beta u)v]\mathrm{d}x \\ &\quad + \int_\Omega (u-u^*)\left(a-u-\frac{v}{1+u^2}\right)\mathrm{d}x + \int_\Omega (v-v^*)\left(b-\frac{v}{1+u^2}\right)\mathrm{d}x \\ &\triangleq I_1 + I_2, \end{aligned}$$

where

$$\begin{aligned} I_1 &= -d_1\int_\Omega \frac{u^*}{u^2}|\nabla u|^2\mathrm{d}x - \int_\Omega (d_2+\beta u)\frac{v^*}{v^2}|\nabla v|^2\mathrm{d}x - \beta\int_\Omega \frac{v^*}{v}|\nabla u||\nabla v|\mathrm{d}x \\ I_2 &= -\int_\Omega (u-u^*)\left(u-u^*+\frac{v}{1+u^2}-\frac{v^*}{1+(u^*)^2}\right)\mathrm{d}x - \int_\Omega (v-v^*)\left(\frac{v}{1+u^2}-\frac{v^*}{1+(u^*)^2}\right)\mathrm{d}x. \end{aligned}$$

It is easy to see that $4d_1(d_2+\beta u)u^* \geq \beta^2 u^2 v^*$ when $\beta \leq \frac{2}{c}\sqrt{\frac{d_1 d_2 \theta}{b(1+\theta^2)}}$. Hence, we have $I_1 \leq 0$. Further computation gives

$$\begin{aligned} I_2 = &-\int_\Omega \left[(u-u^*)^2 - (u-u^*)\left(\frac{v}{1+u^2}-\frac{v^*}{1+u^2}+\frac{v^*}{1+u^2}-\frac{v^*}{1+(u^*)^2}\right)\right]\mathrm{d}x \\ &-\int_\Omega (v-v^*)\left(\frac{v}{1+u^2}-\frac{v^*}{1+u^2}+\frac{v^*}{1+u^2}-\frac{v^*}{1+(u^*)^2}\right)\mathrm{d}x \\ = &\int_\Omega \left(-1+\frac{2v^*(u+u^*)}{(1+(u^*)^2)(1+u^2)}\right)(u-u^*)^2\mathrm{d}x - \int_\Omega \frac{1}{1+u^2}(v-v^*)^2\mathrm{d}x \\ &+\int_\Omega \left(\frac{2v^*(u+u^*)}{(1+(u^*)^2)(1+u^2)}-\frac{1}{1+u^2}\right)(u-u^*)(v-v^*)\mathrm{d}x \\ = &\int_\Omega \left(-1+\frac{2b(u+u^*)}{1+u^2}\right)(u-u^*)^2\mathrm{d}x - \int_\Omega \frac{1}{1+u^2}(v-v^*)^2\mathrm{d}x \\ &+\int_\Omega \left(\frac{2b(u+u^*)-1}{1+u^2}\right)(u-u^*)(v-v^*)\mathrm{d}x. \end{aligned}$$

Clearly,

$$\frac{2b(u+u^*)}{1+u^2} < 1 \quad \text{and} \quad \left(1-\frac{2b(u+u^*)}{1+u^2}\right)\frac{4}{1+u^2} > \left(\frac{2b(u+u^*)-1}{1+u^2}\right)^2$$

if $2b(a+c) < 1$. Therefore, we have $I_2 < 0$. It follows from the above arguments that if the conditions of Lemma 5.2 are satisfied, then $L'(t) < 0$ along all trajectories in the first quadrant except (u^*, v^*). Therefore $\mathbf{e} = (u^*, v^*)$ is globally asymptotically stable. □

6 Hopf bifurcation

This section is devoted to the Hopf bifurcation at the nontrivial steady-state solution $\mathbf{e} = (u^*, v^*)^T$ of (1.4) with $a > b$. To be more precise, as a pair of simple complex conjugate eigenvalues of the linearization around $\mathbf{e} = (u^*, v^*)^T$ cross the imaginary axis of the complex plane, the nontrivial steady-state solution $\mathbf{e} = (u^*, v^*)^T$ of (1.4) loses stability and a branch of small-amplitude limit cycles emerges from $\mathbf{e} = (u^*, v^*)^T$. Throughout this section, we always assume that

(H1) $a > b$, λ_i is a simple eigenvalues of the linear operator $-\Delta$ subject to the homogeneous boundary condition $\frac{\partial}{\partial \mathbf{n}} u$ on $\partial\Omega$, φ_i is the eigenvector associated with λ_i satisfying $\int_\Omega \varphi_i^2(x)\mathrm{d}x = 1$.

In what follows, by choosing the cross-diffusion coefficient β as the bifurcation parameter, we shall analyze the occurrence of Hopf bifurcation, the Hopf bifurcation direction and the stability of bifurcating time-periodic solutions. It follows from [11,12] that system (1.4) with $a > b$ undergoes Hopf bifurcation near $\beta = \beta_i$ at the nontrivial steady-state solution $\mathbf{e} = (u^*, v^*)^T$, where $\beta_i \in (0, \infty)$ satisfies

$$T_i(\alpha_1, \beta_i) = 0, \qquad \frac{\partial}{\partial \beta} T_i(\alpha_1, \beta_i) \neq 0, \qquad D_i(\alpha_1, \beta_i) > 0,$$

and

$$T_j(\alpha_1, \beta_i) \neq 0, \qquad D_j(\alpha_1, \beta_i) \neq 0 \quad \text{for all } i \neq j.$$

Note that $T_i(\alpha_1, \beta)$ is monotone decreasing with respect to β, then it is easy to see that $T_i(\alpha_1, \cdot)$ has exactly one zero

$$\beta_i \triangleq \frac{\alpha_1 - b - (d_1 + d_2)\lambda_i}{\theta \lambda_i},$$

which is positive when $\alpha_1 > b + (d_1 + d_2)\lambda_i$. Obviously, $T_j(\alpha_1, \beta_i) \neq 0$ for $j \neq i$. Moreover, $D_i(\alpha_1, \beta_i) = -\alpha_1^2 + 2\alpha_1 d_1 \lambda_i + b^2 + b(d_1 + d_2)\lambda_i + b\theta - d_1^2\lambda_i^2$. Hence, it is easy to see that $D_i(\alpha_1, \beta_i) > 0$ if $b\theta > d_2\lambda_i^2 + b(d_2 - d_1)\lambda_i$ and $\alpha_1 < d_1\lambda_i + \sqrt{b(d_1 + d_2)\lambda_i + b\theta + b^2}$. Next, we only need to verify $D_j(\alpha_1, \beta_i) \neq 0$ for all $j \neq i$. Obviously,

$$D_j(\alpha_1, \beta_i) = -\frac{\lambda_j}{\lambda_i}\alpha_1^2 + \left(\frac{d_1\lambda_j^2}{\lambda_i} + d_1\lambda_j\right)\alpha_1 - \frac{bd_1\lambda_j^2}{\lambda_i} - d_1^2\lambda_j^2 + 2bd_1\lambda_j + bd_2\lambda_j + \frac{b^2\lambda_j}{\lambda_i} + b\theta.$$

Therefore, we have $D_j(\alpha_1, \beta_i) < 0$ for all $j \neq i$ if $\Re < 0$ and $D_j(\alpha_1, \beta_i) \neq 0$ for all $j \neq i$ if $\Re > 0$ and $\alpha_1 \neq \alpha_1^{\pm}$, where

$$\Re = \frac{d_1^2\lambda_j^4}{\lambda_i^2} + d_1^2\lambda_j^2 + \frac{4\lambda_j}{\lambda_i}\left[-\frac{bd_1\lambda_j^2}{\lambda_i} - \frac{1}{2}d_1^2\lambda_j^2 + (2d_1 + d_2)b\lambda_j + \frac{b^2\lambda_j}{\lambda_i} + b\theta\right]$$

and

$$\alpha_1^{\pm} = \frac{d_1(\lambda_j + \lambda_i)\lambda_j \pm \lambda_i\sqrt{\Re}}{2\lambda_j}.$$

Therefore, we shall consider Hopf bifurcation under the following assumptions:

(H2) $b\theta > d_2\lambda_i^2 + b(d_2 - d_1)\lambda_i$ and $b + (d_1 + d_2)\lambda_i < \alpha_1 < d_1\lambda_i + \sqrt{b(d_1 + d_2)\lambda_i + b\theta + b^2}$;

(H3) Either $\Re < 0$ or $\Re > 0$ and $\alpha_1 \neq \alpha_1^{\pm}$.

For convenience, we call a Hopf bifurcation forward if there exist periodic solutions when parameter value $\beta > \beta_i$; and backward if $\beta < \beta_i$. Under assumptions (H1), (H2) and (H3),

$\mathcal{L}_{\alpha_1,\beta_i}$ has exactly one pair of purely imaginary eigenvalues $\pm i\omega_i$ with associated eigenvectors q_i and $\bar{q}_i$, where $\omega_i = \sqrt{D_i(\alpha_1,\beta_i)}$, $q_i = \rho_i\varphi_i$, and the nonzero vector $\rho_i \in \mathbb{C}^2$ satisfies $\mathcal{M}(i\omega_i,\alpha_1,\beta_i,\lambda_i)\rho_i = 0$. It follows that $\rho_i = (\alpha_2, -d_1\lambda_i + \alpha_1 - i\omega_i)^T$. Moreover, there exist a neighborhood $N_1(\beta_i)\times N_2(i\omega_i)$ of $(\beta_i, i\omega_i)$ in $\mathbb{R}_+\times\mathbb{C}$ and a continuously differentiable function σ: $N_1(\beta_i)\to N_2(i\omega_i)$ such that $\sigma(\beta_i) = \pm i\omega_i$ and that the only eigenvalue of $\mathcal{L}_{\alpha_1,\beta}$ in $N_2(i\omega_i)$ is $\sigma(\beta)$. Moreover, as β varies such that $T_i(\alpha_1,\beta)$ decreases and passes through 0, $\sigma(\beta)$ varies from a complex number with a positive real part to a purely imaginary number and then to a complex number with a negative real part. This implies that a codimension one Hopf bifurcation for (1.4) occurs at $\beta=\beta_i$. Namely, in every neighborhood of $(U,\beta)=(\mathbf{e},\beta_i)$ there is a unique branch of time-periodic spatially non-homogeneous solutions $U_\beta(t,x)$, which tends to $\mathbf{e}$ as $\beta\to\beta_i$. The period T_β of $U_\beta(t,x)$ satisfies that $T_\beta\to 2\pi/\omega_i$ as $\beta\to\beta_i$.

Under assumptions (H1), (H2) and (H3), $-i\omega_i$ is also an eigenvalue of $\mathcal{L}^*_{\alpha_1,\beta_i}$ with an associated eigenvector $p_i = \rho_i^*\varphi_i$, where $\rho_i^*\in\mathbb{C}^2\setminus\{0\}$ satisfies

$$\mathcal{M}^T(-i\omega_i,\alpha_1,\beta_i,\lambda_i)\rho_i^* = 0$$

and $\bar{\rho}_i^*\cdot\rho_i = 1$ and $\rho_i^*\cdot\rho_i = 0$. Then, we have $\rho_i^* = \left(\frac{-b-(d_2+\beta_i)\lambda_i+i\omega_i}{2i\alpha_2\omega_i}, \frac{1}{2i\omega_i}\right)^T$. Next, we consider the bifurcation direction and stability of the bifurcating periodic solutions at $\beta=\beta_i$ according to [11,12]. Denote by $\mathfrak{G}^2 = (\mathfrak{G}^2_1,\mathfrak{G}^2_2)^T$, and $\mathfrak{G}^3 = (\mathfrak{G}^3_1,\mathfrak{G}^3_2)^T$ the second- and third-order Fréchet derivatives of $\Delta\Phi(U)+G(U)$ with respect to U at $\mathbf{e}=(u^*,v^*)$, respectively. A straightforward computation yields

$$\begin{aligned}
\mathfrak{G}^2_1(\xi,\zeta) &= 2\left(-1+\frac{3b\theta(1-\theta^2)}{(1+\theta^2)^2}\right)\xi_1\zeta_1 + \frac{\theta^2-1}{(1+\theta^2)^2}(\xi_1\zeta_2+\xi_2\zeta_1),\\
\mathfrak{G}^2_2(\xi,\zeta) &= \Delta[\beta(\xi_1\zeta_2+\xi_2\zeta_1)] - \frac{2}{1+\theta^2}\xi_2\zeta_2 + \frac{4b\theta}{1+\theta^2}(\xi_1\zeta_2+\xi_2\zeta_1) + \frac{2b^2(1-3\theta^2)}{1+\theta^2}\xi_1\zeta_1\\
\mathfrak{G}^3_1(\xi,\zeta,\varsigma) &= \frac{6b(1-6\theta^2+\theta^4)}{(1+\theta^2)^3}\xi_1\zeta_1\varsigma_1 + \frac{2\theta(3-\theta^2)}{(1+\theta^2)^3}(\xi_1\zeta_1\varsigma_2+\xi_2\zeta_1\varsigma_1+\xi_1\zeta_2\varsigma_1),\\
\mathfrak{G}^3_2(\xi,\zeta,\varsigma) &= \frac{4b(1-3\theta^2)}{(1+\theta^2)^2}(\xi_1\zeta_1\varsigma_2+\xi_1\zeta_2\varsigma_1+\xi_2\zeta_1\varsigma_1)\\
&\quad + \frac{4\theta}{(1+\theta^2)^2}(\xi_2\zeta_2\varsigma_1+\xi_1\zeta_2\varsigma_2+\xi_2\zeta_1\varsigma_2) + \frac{24b^2\theta(\theta^2-1)}{(1+\theta^2)^2}\xi_1\zeta_1\varsigma_1
\end{aligned}$$

for all $\xi=(\xi_1,\xi_2)^T$, $(\zeta_1,\zeta_2)^T$ and $\varsigma=(\varsigma_1,\varsigma_2)^T\in\mathbb{X}$. It is well known that the following quantity determines the direction and stability of bifurcating periodic orbits $U_\beta(t,x)$ (see [11, 12])

$$\Upsilon_i = \frac{i}{2\omega_i}\left(\mathbf{g}_{11}\mathbf{g}_{20} - 2|\mathbf{g}_{11}|^2 - \frac{|\mathbf{g}_{02}|^2}{3}\right) + \frac{\mathbf{g}_{21}}{2}, \tag{6.1}$$

where

$$\begin{aligned}
\mathbf{g}_{20} &= \langle p_i, \mathfrak{G}^2(q_i,q_i)\rangle,\\
\mathbf{g}_{11} &= \langle p_i, \mathfrak{G}^2(q_i,\bar{q}_i)\rangle,\\
\mathbf{g}_{02} &= \langle p_i, \mathfrak{G}^2(\bar{q}_i,\bar{q}_i)\rangle,\\
\mathbf{g}_{21} &= \langle p_i, \mathfrak{G}^3(q_i,q_i,\bar{q}_i)\rangle + 2\langle p_i, \mathfrak{G}^2(W_{11},q_i)\rangle + \langle p_i, \mathfrak{G}^2(W_{20},\bar{q}_i)\rangle,
\end{aligned}$$

and

$$\begin{aligned}
W_{20} &= [2i\omega_i - \mathcal{L}_{\alpha_1,\beta_i}]^{-1}\left[\mathfrak{G}^2(q_i,q_i) - \langle p_i,\mathfrak{G}^2(q_i,q_i)\rangle q_i - \langle\bar{p}_i,\mathfrak{G}^2(q_i,q_i)\rangle\bar{q}_i\right],\\
W_{11} &= -[\mathcal{L}_{\alpha_1,\beta_i}]^{-1}\left[\mathfrak{G}^2(q_i,\bar{q}_i) - \langle p_i,\mathfrak{G}^2(q_i,\bar{q}_i)\rangle q_i - \langle\bar{p}_i,\mathfrak{G}^2(q_i,\bar{q}_i)\rangle\bar{q}_i\right].
\end{aligned}$$

Therefore, we obtain the following result.

Theorem 6.1. *In addition to assumptions (H1), (H2) and (H3), a Hopf bifurcation for* (1.4) *occurs at* $\beta = \beta_i$ *if* $a > b$*. Namely, when* $a > b$*, in a neighborhood of* $(U, \beta) = (\mathbf{e}, \beta_i)$ *there is a branch of periodic solutions* $U_\beta(x,t)$ *satisfying* $U_\beta(x,t) \to \mathbf{e}$ *as* $\beta \to \beta_i$*. The period* T_β *of* $U_\beta(x,t)$ *satisfies that* $T_\beta \to 2\pi/\omega_*$ *as* $\beta \to \beta_i$*. Moreover, the bifurcation is backward (respectively, forward) if* $\mathrm{Re}(\Upsilon_i) < 0$ $(respectively, > 0)$*.*

Obviously, in Theorem 6.1, if λ_i is not the principal eigenvalue of the linear operator $-\Delta$ subject to the homogeneous boundary condition $\frac{\partial}{\partial \mathbf{n}} u = 0$ on $\partial\Omega$, then the Hopf bifurcating periodic solutions $U_\beta(x,t)$ is spatially nonhomogeneous and unstable. However, if λ_i is the principal eigenvalue $\lambda_0 = 0$, then the associated eigenvector φ_0 can be a positive constant function on Ω. In this case, assumption (H1) is obviously satisfied and $\alpha_1 - b$ is sufficiently close to zero. Hence, we can regard b as a bifurcation parameter. Obviously, we have $T_0(\alpha_1, \beta) = 0$, $T_j(\alpha_1, \beta) < 0$ and $D_0(\alpha_1, \beta) = b\theta > 0$, $D_j(\alpha_1, \beta) > 0$ for all $j \in \mathbb{N}$ if $b = b_* \triangleq \frac{\theta(1+\theta^2)}{\theta^2 - 1}$ and one of the following conditions is satisfied

(A1) $\theta > 1$ and $d_1 - d_2 - 2\beta\theta \geq 0$;

(A2) $\theta > 1$, $d_1 > d_2$ and $d_1 - d_2 - 2\beta\theta < 0$ and $b(d_1 - d_2 - 2\beta\theta)^2 < 4d_1(d_2 + \beta\theta)\theta$;

(A3) $\theta > 1$ and $d_1 < d_2$ and $b(d_1 - d_2 - 2\beta\theta)^2 < 4d_1(d_2 + \beta\theta)\theta$.

It is easy to evaluate $\sigma(b)$ at $b = b_*$ to get $\mathrm{Re}\sigma'(b_*) = \frac{\theta^2 - 1}{2(1+\theta^2)} > 0$. Thus, it remains to calculate the direction of Hopf bifurcation and the stability of bifurcating periodic orbits bifurcating from $(U, b) = (\mathbf{e}, b^*)$. In virtue of (6.1), we have

$$\mathrm{Re}(\Upsilon_0) = \frac{3\theta^3(2 - 3\theta^2 + 6\theta^4 - \theta^6)}{2(1+\theta^2)^4(\theta^2 - 1)}.$$

Corollary 6.2. *Under one of conditions (A1)-(A3), if* $a > b$ *then in every neighborhood of* $(U, b) = (\mathbf{e}, b_*)$ *there is a branch of spatially homogeneous periodic solutions* $U_b(x,t)$ *satisfying* $U_b(x,t) \to \mathbf{e}$ *as* $b \to b_*$ *and the Hopf bifurcation is forward (respectively, backward) and the bifurcation periodic solutions are orbitally asymptotically stable (respectively, unstable) if* $2 - 3\theta^2 + 6\theta^4 - \theta^6 < 0$ *(respectively,* > 0*), where* $b_* \triangleq \frac{\theta(1+\theta^2)}{\theta^2 - 1}$*.*

7 Bogdanov–Takens bifurcation

Apart from the occurrence of Hopf bifurcation discussed so far, codimension 2 bifurcation such as Bogdanov–Takens bifurcation is also possible in system (1.4). In order to discuss codimension 2 bifurcation, in addition to taking β as a bifurcation parameter, we need another parameter. It is easy to see that $T_i(\alpha_1, \beta)$ depends on (β, α_1 θ, b) and $D_i(\alpha_1, \beta)$ on (β, b, α_1, α_2 θ). More precisely, α_1 depends on θ and b, α_2 on θ. For convenience, we choose α_1 and β as bifurcation parameters. In this section, we investigate the Bogdanov–Takens bifurcation at the nontrivial steady-state solution $\mathbf{e} = (u^*, v^*)^T$ of (1.4) under the condition (H1) and the following assumption

(H4) $T_i(\alpha_1, \beta) = 0$, $D_i(\alpha_1, \beta) = 0$, $T_j(\alpha_1, \beta) \neq 0$, $D_j(\alpha_1, \beta) \neq 0$ for $j \neq i$.

That is, the Bogdanov–Takens bifurcation is a bifurcation in a two-parameter family of system (1.4) at which $\mathbf{e} = (u^*, v^*)^T$ has a zero eigenvalue of geometric multiplicity one and algebraic multiplicity two. Assume that $\alpha_1 > b + (d_1 + d_2)\lambda_i$ and $b\theta > d_2\lambda_i^2 + b(d_2 - d_1)\lambda_i$, then the only choice of (α_1, β) satisfying assumption (H4) is (α_1^*, β^*), where

$$\alpha_1^* = d_1\lambda_i + \sqrt{b(d_1 + d_2)\lambda_i + b\theta + b^2}, \qquad \beta^* = \frac{\alpha_1 - b - (d_1 + d_2)\lambda_i}{\theta\lambda_i}.$$

Clearly, if $j \neq i$, we have $T_j(\alpha_1^*, \beta^*) \neq 0$. Furthermore,

$$D_j(\alpha_1^*, \beta^*) = \left(1 - \frac{\lambda_j}{\lambda_i}\right)\left(bd_1\lambda_j + b\theta - d_1\lambda_j\sqrt{b(d_1 + d_2)\lambda_i + b\theta + b^2}\right)$$

Hence, it is easy to see that $D_j(\alpha_1^*, \beta^*) \neq 0$ for all $j \neq i$ if $\sqrt{b(d_1 + d_2)\lambda_i + b\theta + b^2} \neq b + \frac{b\theta}{d_1\lambda_j}$. Therefore, we have the following result:

Theorem 7.1. *Under the assumption (H1), if $a > b$ and $\sqrt{b(d_1 + d_2)\lambda_i + b\theta + b^2} \neq b + \frac{b\theta}{d_1\lambda_j}$ and $b\theta > d_2\lambda_i^2 + b(d_2 - d_1)\lambda_i$, then near $(\alpha_1, \beta) = (\alpha_1^*, \beta^*)$ system (1.4) has a Bogdanov–Takens singularity at the positive constant steady-state solution $\mathbf{e} = (u^*, v^*)^T$.*

Under assumptions (H1) and (H4), $\mathcal{L}_{\alpha_1^*,\beta^*}$ has exactly a zero eigenvalue of geometric multiplicity one and algebraic multiplicity two. Let $\mathbb{P}$ be the subspace of $\mathcal{L}_{\alpha_1^*,\beta^*}$ associated with zero eigenvalues. Let $\Phi = (\phi_1, \phi_2) = (\mathbf{c}_1\varphi_i, \mathbf{c}_2\varphi_i)$ be a basis for $\mathbb{P}$, and $\Psi = (\psi_1, \psi_2)^T = (\mathbf{d}_1\varphi_i, \mathbf{d}_2\varphi_i)^T$ be the basis for the dual space $\mathbb{P}^*$ in $\mathbb{X}$, such that $\langle\psi_j, \phi_s\rangle = \delta_{js}$, where δ_{js} is the Kronecker delta. Obviously,

$$\mathcal{M}(0, \alpha_1^*, \beta^*, \lambda_i)\mathbf{c}_1 = 0 \qquad \text{and} \qquad \mathcal{M}(0, \alpha_1^*, \beta^*, \lambda_i)\mathbf{c}_2 = \mathbf{c}_1,$$
$$\mathcal{M}^T(0, \alpha_1^*, \beta^*, \lambda_i)\mathbf{d}_2 = 0 \qquad \text{and} \qquad \mathcal{M}^T(0, \alpha_1^*, \beta^*, \lambda_i)\mathbf{d}_2 = \mathbf{d}_1,$$

where

$$\mathbf{c}_1 = \left(1, \frac{-d_1\lambda_i + \alpha_1^*}{\alpha_2}\right)^T, \qquad \mathbf{c}_2 = \left(0, -\frac{1}{\alpha_2}\right)^T,$$

and

$$\mathbf{d}_1 = (1, 0)^T, \qquad \mathbf{d}_2 = ((d_2 + \beta^*\theta)\lambda_i + b, -\alpha_2)^T.$$

Thus, $\langle\Psi, \Phi\rangle = \text{Id}_2$ and $\dot{\Phi} = B\Phi$, where

$$B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

We adopt the framework of [11], we rewrite system (1.4) as

$$\frac{dU}{dt} = \mathcal{L}(\alpha_1, \beta, U)U + F(\alpha_1, \beta, U), \tag{7.1}$$

where

$$F(\alpha_1, \beta, U) = \begin{pmatrix} u\left(a - u - \dfrac{v}{1 + u^2}\right) - d_1\Delta u - \alpha_1 u - \alpha_2 v \\ v\left(b - \dfrac{v}{1 + u^2}\right) - \beta b(1 + \theta^2)\Delta u - \alpha_3 u - (d_2 + \beta\theta)\Delta v + bv \end{pmatrix}.$$

We decompose $\mathbb{X} = \mathbb{X}^c + \mathbb{X}^s$, with $\mathbb{X}^c \triangleq \{z\Phi | z \in \mathbb{R}^2\}$, $\mathbb{X}^s \triangleq \{U \in \mathbb{X} | \langle\Psi, U\rangle = 0\}$. For any $U = (u, v)^T \in \mathbb{X}$, there exist $z \in \mathbb{R}$ and $y = (y_1, y_2) \in \mathbb{X}^s$ such that $U = \mathbf{e} + \Phi z + y$. Then system (7.1) is reduced to the following system in (z, y) coordinates:

$$\begin{cases} \frac{dz}{dt} = Bz + \langle\Psi, F(\mathbf{e} + \Phi z + y, \alpha_1, \beta)\rangle, \\ \frac{dy}{dt} = \mathcal{L}(\alpha_1, \beta)y + H(z_1, z_2, y), \end{cases} \tag{7.2}$$

where

$$H(z_1, z_2, y) = F(\mathbf{e} + \Phi z + y, \alpha_1, \beta) - \langle \Psi, F(\mathbf{e} + \Phi z + y, \alpha_1, \beta) \rangle.$$

According to [11], the normal form of Bogdanov–Takens bifurcation under conditions (H1) and (H4) is given by

$$\begin{cases} \dfrac{dz_1}{dt} = z_2, \\ \dfrac{dz_2}{dt} = C_{10}(\alpha_1, \beta) z_1 + C_{01}(\alpha_1, \beta) z_2 + C_{20}(\alpha_1^*, \beta^*) z_1^2 + C_{11}(\alpha_1^*, \beta^*) z_1 z_2, \end{cases} \tag{7.3}$$

where

$$\begin{aligned} C_{10}(\alpha_1, \beta) &= \langle \psi_2, \mathcal{L}^1_{\alpha_1} \phi_1 \alpha_1 + \mathcal{L}^1_{\beta} \phi_1 \beta \rangle \\ &= [(d_2 + \beta^* \theta)\lambda_i + b](\alpha_1 - \alpha_1^*) + [b(1+\theta^2)\lambda_i \alpha_2 + \theta(\alpha_1^* - d_1 \lambda_i)\lambda_i](\beta - \beta^*), \\ C_{01}(\alpha_1, \beta) &= \langle \psi_2, \mathcal{L}^1_{\alpha_1} \phi_2 \alpha_1 + \mathcal{L}^1_{\beta} \phi_2 \beta \rangle + \langle \psi_1, \mathcal{L}^1_{\alpha_1} \phi_1 \alpha_1 + \mathcal{L}^1_{\beta}(\phi_1)\beta \rangle = \alpha_1 - \alpha_1^* - \theta\lambda_i(\beta - \beta^*), \\ C_{20}(\alpha_1^*, \beta^*) &= \langle \psi_2, \mathfrak{G}^2(\phi_1, \phi_1) \rangle, \\ C_{11}(\alpha_1^*, \beta^*) &= \langle \psi_1, \mathfrak{G}^2(\phi_1, \phi_1) \rangle + \langle \psi_2, \mathfrak{G}^2(\phi_1, \phi_2) \rangle. \end{aligned}$$

For convenience, we denote

$$\tilde{G} = \begin{bmatrix} (d_2 + \beta^* \theta)\lambda_i + b & b(1+\theta^2)\alpha_2 \lambda_i + \theta(\alpha_1^* - d_1 \lambda_i)\lambda_i \\ 1 & -\theta\lambda_i \end{bmatrix}.$$

It is easy to see that $\det\tilde{G} > 0$. Therefore, we have the following conclusion.

Theorem 7.2. *Under assumptions (H1) and (H4), if $a > b$ and $C_{20}C_{11} \neq 0$, then system (7.1) undergoes a Bogdanov–Takens bifurcation. More precisely, if $C_{20}C_{11} < 0$, then, in the (C_{10}, C_{01}) bifurcation diagram, the Hopf bifurcation curve Γ_1 and the homoclinic bifurcation curve Γ_2 lie in the region $\mathfrak{W}$. Both the homoclinic loop and the periodic orbit are unstable, where*

$$\begin{aligned} \mathfrak{W} &= \{(C_{10}, C_{01}) \mid C_{10} > 0 \text{ and } C_{01} < 0\} \\ \Gamma_1 &= \left\{ (\alpha_1, \beta) \,\middle|\, C_{01}(\alpha_1, \beta) = \frac{C_{11}}{C_{20}} C_{10}(\alpha_1, \beta) + h.o.t,\ C_{10}(\alpha_1, \beta) > 0 \right\} \\ \Gamma_2 &= \left\{ (\alpha_1, \beta) \,\middle|\, C_{01}(\alpha_1, \beta) = \delta\sqrt{C_{10}(\alpha_1, \beta)} C_{10}(\alpha_1, \beta) + h.o.t,\ C_{10}(\alpha_1, \beta) > 0 \right\} \end{aligned}$$

and δ is a continuous and differentiable function satisfying $\delta(0) = \frac{6C_{11}}{7C_{20}}$.

8 Conclusions and numerical simulations

In this paper, we have shown that all solutions of system (1.4) exist globally and are uniformly bounded if β satisfies (2.4). But we don't know whether the solution of system (1.4) can blow up in a finite time or exists globally if (2.4) does not hold. This is a problem filled with challenge. Next, this paper presents the existence of the non-constant positive steady states of system (1.4). In view of Theorems 4.5 and 4.6, we see that system (1.4) has a non-constant positive steady state if either the diffusion coefficient d_1 is small or the cross-diffusion coefficient β is large. This implies the predator and prey species may coexist in the interacting habit nonuniformly if the predator disperses quickly from a high density of prey to a low density one, or the prey move slowly from a higher to a lower concentration region. Our theoretical analysis shows that the cross-diffusion phenomenon has the potential to play an important role in the coexistence information. From the biological point of view, our analysis gives a theoretical support for studying coexistence phenomena of reaction-diffusion systems with cross-diffusion.

Sections 6 and 7 show that system (1.4) is capable of producing much more abundant dynamics than the corresponding ODEs. For example, system (1.4) may have multiple bifurcation under certain conditions, and both Hopf bifurcation and homoclinic bifurcation are possible. According to Section 7, we know that the ODEs associated with (1.4) (i.e., with $d_1 = d_2 = \beta = 0$) cannot show Bogdanov–Takens singularity, but system (1.4) can show Bogdanov–Takens singularity (see Theorem 7.1); this indicates that diffusion plays a fundamental role in producing a rich dynamics and even Bogdanov–Takens bifurcation phenomena. Meanwhile, the existence and properties of the spatially nonhomogeneous Hopf bifurcation of system (1.4) (i.e., $\lambda_i \neq \lambda_0$) are established in Theorem 6.1, and Corollary 6.2 is devoted to spatially homogeneous Hopf bifurcation of system (1.4) (i.e., $\lambda_i = \lambda_0$).

Finally, we present some numerical simulations to support and supplement our analytic results given in the previous sections. For the spatially homogeneous model (1.4), it follows from Corollary 6.2 that **e** is locally asymptotically stable if $a > b$, $b < \frac{\theta(1+\theta^2)}{\theta^2-1}$ and one of conditions (A1)–(A3) is satisfied, and is unstable if $\theta > 1$ and $b > \frac{\theta(1+\theta^2)}{\theta^2-1}$. In addition, when b passes through $\frac{\theta(1+\theta^2)}{\theta^2-1}$ from the left of $\frac{\theta(1+\theta^2)}{\theta^2-1}$, **e** will lose its stability and a family of periodic solutions bifurcate from the interior equilibrium **e**. It also follows from Corollary 6.2 that the direction of Hopf bifurcation is forward and the bifurcating periodic solutions are asymptotically stable if $2 - 3\theta^2 + 6\theta^4 - \theta^6 < 0$. For system (1.4) with $\Omega = (0, 2\pi)$ and initial values $u(x,t) = \cos^2 \frac{x}{\pi}$ and $v(x,t) = \cos^2 \frac{x}{\pi}$, if we fix $\theta = 2.4$, then the critical point $\frac{\theta(1+\theta^2)}{\theta^2-1} = 3.4084$ and $2 - 3\theta^2 + 6\theta^4 - \theta^6 = -7.3174 < 0$. Next, we can choose the following three sets of parameter values, which satisfy conditions (A1)–(A3) respectively:

(P1) $d_1 = 3$, $d_2 = 1$, $\beta = 0.3$;

(P2) $d_1 = 3$, $d_2 = 1$, $\beta = 1$;

(P3) $d_1 = 3$, $d_2 = 5$, $\beta = 0.5$.

Obviously, if the values of a and b are fixed, the mathematical phenomena described by the above three sets of parameters are quite similar. Without loss of generality, we illustrate our analytical results by numerical simulations only under the condition (P1). If $a = 5.55$ and $b = 3.15 < 3.4084$, then the positive constant solution **e** is locally asymptotically stable (see Figure 8.1). Choose $a = 5.81$, $b = 3.41 > 3.4084$, then we see that a limit cycle arises out of Hopf bifurcation around **e** (see Figure 8.2). Lemma 5.2 tells us the positive constant solution **e** is globally asymptotically stable if

$$a > b,\, 2b(a+c) < 1 \quad \text{and} \quad \beta \leq \frac{2}{c}\sqrt{\frac{d_1 d_2 \theta}{b(1+\theta^2)}}.$$

Here, for system (1.4) with $\Omega = (0, 2\pi)$, we choose $d_1 = d_2 = 1$, $a = 1.5$, $b = 0.15$ $\beta = 2.38$, and initial values $u(x,t) = \cos^2 \frac{x}{\pi}$, $v(x,t) = \cos^2 \frac{x}{\pi}$, then $c = a = 1.5$, $\beta < \frac{2}{c}\sqrt{\frac{d_1 d_2 \theta}{b(1+\theta^2)}} = 2.3809$ and $2b(a+c) = 0.9$. Thus, as depicted in Figure 8.3, the positive constant solution **e** is globally asymptotically stable. Nevertheless, we do not know whether the conclusion of Lemma 5.2 holds true if the value b does not satisfy $2b(a+c) < 1$. Therefore, we just find a sufficient condition ensuring the global asymptotical stability of the positive steady-state solution **e**. However, when $a > b$, we can get the critical value b_* of the parameter b by fixing the value of θ. It follows from Corollary 6.2 that the positive steady-state solution **e** is locally asymptotically stable if $b < b_*$ and will lose its stability when b passes b_* from the left of b_*.

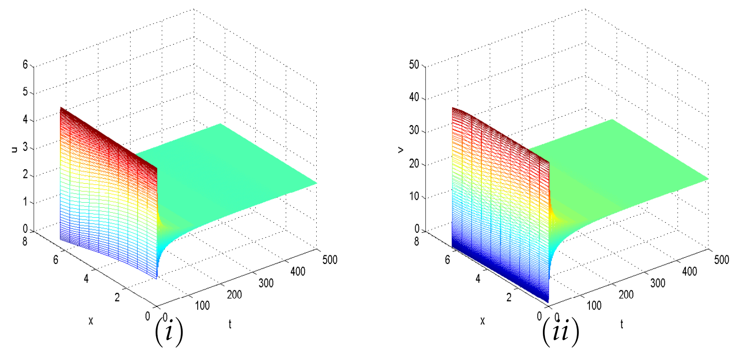

Figure 8.1: The solutions of model (1.4) tends to a positive steady state with parameters $b = 3.15 < 3.4084$ and $a = 5.55$.

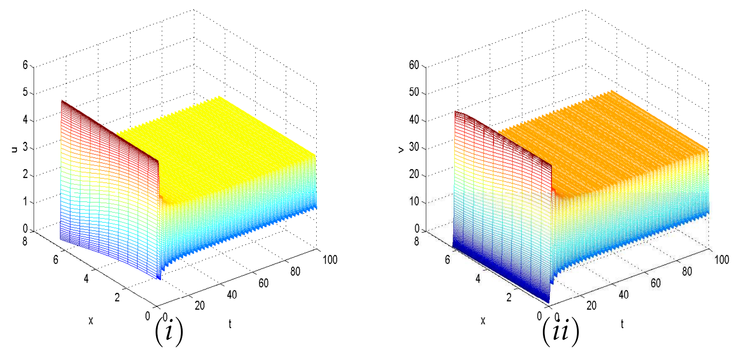

Figure 8.2: The solutions of model (1.4) with $b = 3.41 > 3.4084$ and $a = 5.81$ tends to a positive spatially homogeneous time-periodic orbit.

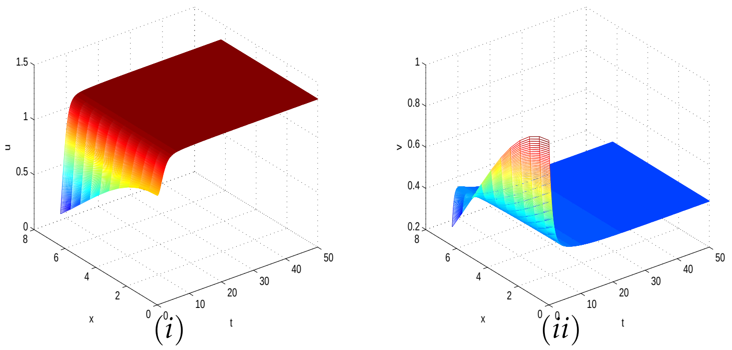

Figure 8.3: The positive steady-state solution $\mathbf{e} = (1.35, 0.4234)$ of model (1.4) with parameters $a = 1.5$ $b = 0.15$ and $\beta = 2.38$ is globally asymptotically stable
.

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References

[1] H. Amann, Dynamic theory of quasilinear parabolic equations. II. Reaction-diffusion systems, *Differential Integral Equations* **3**(1990), No. 1, 13–75. MR1014726.

[2] H. Amann, Dynamic theory of quasilinear parabolic systems. III. Global existence, *Math. Z.* **202**(1989), No. 2, 219–250. MR1013086.

[3] J. F. Andrews, A mathematical model for the continuous culture of microorganisms utilizing inhibitory substrates, *Biotechnol. Bioeng.* **10**(1968), No. 6, 707–723. https://doi.org/10.1002/bit.260100602.

[4] M. A. Aziz-Alaoui, M. D. Okiye, Boundedness and global stability for a predator-prey model with modified Leslie–Gower and Holling-type II schemes, *Appl. Math. Lett.* **16**(2003), No. 7, 1069–1075. https://doi.org/10.1016/S0893-9659(03)90096-6; MR2013074.

[5] W. Chen, R. Peng, Stationary patterns created by cross-diffusion for the competitor-competitor-mutualist model, *J. Math. Anal. Appl.* **291**(2004), No. 2, 550–564. https://doi.org/10.1016/j.jmaa.2003.11.015; MR2039069.

[6] V. H. Edwards, The influence of high substrate concentrations on microbial kinetics, *Biotechnol. Bioeng.* **12**(1970), No. 5, 679–712. https://doi.org/10.1002/bit.260120504

[7] G. Gambino, M. C. Lombardo, M. Sammartino, Pattern formation driven by cross-diffusion in a 2D domain, *Nonlinear Anal. Real World Appl.* **14**(2013), No. 3, 1755–1779. https://doi.org/10.1016/j.nonrwa.2012.11.009; MR3004535.

[8] J. P. Gao, S. J. Guo, Patterns in a modifed Leslie–Gower model with Beddington–DeAngelis functional response and nonlocal prey competition, *Internat. J. Bifur. Chaos Appl. Sci. Engrg.* **30**(2020), No. 5, 2050074. https://doi.org/10.1142/S0218127420500741; MR4095965.

[9] S. J. Guo, Bifurcation and spatio-temporal patterns in a diffusive predator-prey system, *Nonlinear Anal. Real World Appl.* **42**(2018), 448–477. https://doi.org/10.1016/j.nonrwa.2018.01.011; MR3773369.

[10] S. J. Guo, Patterns in a nonlocal time-delayed reaction-diffusion equation, *Z. Angew. Math. Phys.* **69**(2018), No. 1, Art. 10. https://doi.org/10.1007/s00033-017-0904-7; MR3737363.

[11] S. J. Guo, J. H. Wu, *Bifurcation theory of functional differential equations*, Applied Mathematical Sciences, Vol. 184, Springer, New York, 2013. https://doi.org/10.1007/978-1-4614-6992-6; MR3098815.

[12] B. D. Hassard, N. D. Kazarinoff, Y. H. Wan, *Theory and applications of Hopf bifurcation*, CUP Archive, 1981.

[13] C. S. Holling, Some characteristics of simple types of predation and parasitism, *Can. Entomol.* **91**(1959), No. 7, 385–398. https://doi.org/10.4039/Ent91385-7.

[14] D. Horstmann, M. Winkler, Boundedness vs. blow-up in a chemotaxis system, *J. Differential Equations.* **215**(2005), No. 1, 52–107. https://doi.org/10.1016/j.jde.2004.10.022; MR2146345.

[15] J. Y. Jin, J. P. Shi, J. J. Wei, F. Q. Yi, Bifurcations of patterned solutions in the diffusive Lengyel–Epstein system of CIMA chemical reactions, *Rocky Mountain J. Math.* **43**(2013), No. 5, 1637–1674. https://doi.org/10.1216/RMJ-2013-43-5-1637; MR3127841; Zbl 1288.35051.

[16] O. A. Ladyzhenskaja, V. A. Solonnikov, N. N. Ural'tseva, *Linear and quasi-linear equations of parabolic type.* American Mathematical Soc., 1988.

[17] P. H. Leslie, J. C. Gower, The properties of a stochastic model for the predator-prey type of interaction between two species, *Biometrika* **47**(1960), No. (3/4), 219–234. https://doi.org/10.1093/biomet/47.3-4.219; MR0122603.

[18] S. Z. Li, S. J. Guo, Stability and Hopf bifurcation in a Hutchinson model, *Appl. Math. Lett.* **101**(2020), 106066. https://doi.org/10.1016/j.aml.2019.106066; MR4015747.

[19] S. Z. Li, S. J. Guo, Hopf bifurcation for semilinear FDEs in general Banach spaces, *Internat. J. Bifur. Chaos Appl. Sci. Engrg.* **30**(2020), No. 9, 2050130; https://doi.org/10.1142/S0218127420501308; MR4132011.

[20] S. B. Li, J. H. Wu, H. Nie, Steady-state bifurcation and Hopf bifurcation for a diffusive Leslie–Gower predator–prey model, *Comput. Math. Appl.* **70**(2015), No. 12, 3043–3056. https://doi.org/10.1016/j.camwa.2015.10.017; MR3427905.

[21] C. S. Lin, W. M. Ni, I. Takagi, Large amplitude stationary solutions to a chemotaxis system, *J. Differential Equations* **72**(1988), No. 1, 1–27. https://doi.org/10.1016/0022-0396(88)90147-7; MR0929196.

[22] Y. Y. Liu, Y. S. Tao, Dynamics in a parabolic-elliptic two-species population competition model with cross-diffusion for one species, *J. Math. Anal. Appl.* **456**(2017), No. 1, 1–15. https://doi.org/10.1016/j.jmaa.2017.05.058; MR3680953.

[23] Z. H. Liu, R. Yuan, The effect of diffusion for a predator-prey system with nonmonotonic functional response, *Internat. J. Bifur. Chaos Appl. Sci. Engrg.* **14**(2004), No. 12, 4309–4316. https://doi.org/10.1142/S0218127404011867; MR2118653.

[24] Y. Lou, W. M. Ni, Diffusion, self-diffusion and cross-diffusion, *J. Differential Equations* **131**(1996), No. 1, 79–131. https://doi.org/10.1006/jdeq.1996.0157; MR1415047.

[25] L. Ma, S. J. Guo, Bifurcation and stability of a two-species diffusive Lotka–Volterra model, *Commun. Pure Appl. Anal.* **19**(2020), No. 3, 1205–1232. https://doi.org/10.3934/cpaa.2020056; MR4064028.

[26] C. L. Mu, L. C. Wang, P. Zheng, Q. N. Zhang, Global existence and boundedness of classical solutions to a parabolic-parabolic chemotaxis system, *Nonlinear Anal. Real World Appl.* **14**(2013), No. 3, 1634–1642. https://doi.org/10.1016/j.nonrwa.2012.10.022; MR3004526.

[27] L. Nirenberg, An extended interpolation inequality, *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (3)* **20**(1966), No. 4, 733–737. MR0208360; Zbl 0163.29905.

[28] L. Nirenberg, *Topics in nonlinear functional analysis*, American Mathematical Soc., 1974.

[29] P. Y. H. Pang, M. X. Wang, Strategy and stationary pattern in a three-species predator–prey model, *J. Differential Equations* **200**(2004), No. 2, 245–273. https://doi.org/10.1016/j.jde.2004.01.004; MR2052615.

[30] H. H. Qiu, S. J. Guo, S. Z. Li, Stability and bifurcation in a predator-prey system with prey-taxis, *Internat. J. Bifur. Chaos Appl. Sci. Engrg.* **30**(2020), No. 2, 2050022. https://doi.org/10.1142/S0218127420500224; MR4074214.

[31] W. Sokol, J. A. Howell, Kinetics of phenol oxidation by washed cells, *Biotechnol. Bioeng.* **23**(1981), No. 9: 2039–2049. https://doi.org/10.1002/bit.260230909.

[32] Y. S. Tao, Boundedness in a chemotaxis model with oxygen consumption by bacteria, *J. Math. Anal. Appl.* **381**(2011), No. 2, 521–529. https://doi.org/10.1016/j.jmaa.2011.02.041; MR2802089.

[33] Y. S. Tao, Global existence of classical solutions to a predator–prey model with nonlinear prey-taxis, *Nonlinear Anal. Real World Appl.* **11**(2010), No. 3, 2056–2064. https://doi.org/10.1016/j.nonrwa.2009.05.005; MR2646615.

[34] V. Volterra, Variazioni e fluttuazioni del numero d'individui in specie animali conviventi, *Memorie Della Reale Accademia Nazionale Dei Lincei* **2**(1926), 5–112.

[35] H. Y. Wang, S. J. Guo, S. Z. Li, Stationary solutions of advective Lotka-Volterra models with a weak Allee effect and large diffusion, *Nonlinear Anal. Real World Appl.* **56**(2020), 103171. https://doi.org/10.1016/j.nonrwa.2020.103171; MR4115566.

[36] J. F. Wang, J. P. Shi, J. J. Wei, Dynamics and pattern formation in a diffusive predator–prey system with strong Allee effect in prey, *J. Differential Equations* **251**(2011), No. 4–5, 1276–1304. https://doi.org/10.1016/j.jde.2011.03.004; MR2812590.

[37] D. Wei, S. J. Guo, Qualitative analysis of a Lotka–Volterra competition-diffusion-advection system, *Discrete Contin. Dyn. Syst. Ser. B* https://doi.org/10.3934/dcdsb.2020197.

[38] M. Winkler, Absence of collapse in a parabolic chemotaxis system with signal-dependent sensitivity, *Math. Nachr.* **283**(2010), No. 11, 1664–1673. https://doi.org/10.1002/mana.200810838; MR2759803.

[39] M. Winkler, K. C. Djie, Boundedness and finite-time collapse in a chemotaxis system with volume-filling effect, *Nonlinear Anal.* **72**(2010), No. 2, 1044–1064. https://doi.org/10.1007/978-1-4612-4050-1; MR1415838.

[40] J. Wu, *Theory and applications of partial functional differential equations*, Springer Science & Business Media, 2012. https://doi.org/10.1007/978-1-4612-4050-1

[41] S. N. Wu, J. P. Shi, B. Y. Wu, Global existence of solutions and uniform persistence of a diffusive predator-prey model with prey-taxis, *J. Differential Equations* **260**(2016), No. 7, 5847–5874. https://doi.org/10.1016/j.jde.2015.12.024; MR3456817.

[42] F. Q. YI, J. J. WEI, J. P. SHI, Bifurcation and spatiotemporal patterns in a homogeneous diffusive predator–prey system, *J. Differential Equations* **246**(2009), No. 5, 1944–1977. https://doi.org/10.1016/j.jde.2008.10.024; MR2494694.

[43] X. Z. ZENG, Z. H. LIU, Nonconstant positive steady states for a ratio-dependent predator-prey system with cross-diffusion, *Nonlinear Anal. Real World Appl.* **11**(2010), No. 1, 372–390. https://doi.org/10.1016/j.nonrwa.2008.11.010; MR2570557.

[44] J. F. ZHANG, W. T. LI, Y. X. WANG, Turing patterns of a strongly coupled predator–prey system with diffusion effects, *Nonlinear Anal.* **74**(2011), No. 3, 847–858. https://doi.org/10.1016/j.na.2010.09.035; MR2738636.

[45] J. F. ZHANG, W. T. LI, X. P. YAN, Multiple bifurcations in a delayed predator–prey diffusion system with a functional response, *Nonlinear Anal. Real World Appl.* **11**(2010), No. 4, 2708–2725. https://doi.org/10.1016/j.nonrwa.2009.09.019; MR2661938.

[46] J. ZHOU, Bifurcation analysis of a diffusive predator–prey model with ratio-dependent Holling type III functional response, *Nonlinear Dynam.* **81**(2015), No. 3, 1535–1552. https://doi.org/10.1007/s11071-015-2088-z; MR3367172.

[47] R. ZOU, S. J. GUO, Dynamics of a diffusive Leslie–Gower predator–prey model in spatially heterogeneous environment, *Discrete Contin. Dyn. Syst. Ser. B*. **25**(2020), No. 11, 4189–4210. https://doi.org/10.3934/dcdsb.2020093.