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The wellposedness and energy estimate for wave equations in domains with a space-like boundary

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Abstract. This paper is concerned with wave equations defined in domains of $\mathbb{R}^2$ with an invariable left boundary and a space-like right boundary which means the right endpoint is moving faster than the characteristic. Different from the case where the endpoint moves slower than the characteristic, this problem with ordinary boundary formulations may cause ill-posedness. In this paper, we propose a new kind of boundary condition to make systems well-posed, based on an idea of transposition. The key is to prove wellposedness and a hidden regularity for the corresponding backward system. Moreover, we establish an exponential decay estimate for the energy of homogeneous systems.

Keywords: wave equation, space-like boundary, wellposedness, energy estimate.

2010 Mathematics Subject Classification: 34K10, 35L10, 35R37.

1 Introduction

Let $T > 0$. Given $t \in [0, T]$, put $\alpha_k(t) = 1 + kt$ and $\Omega(t) = \{(x, t) \in \mathbb{R}^2 \mid 0 < x < \alpha_k(t)\}$. Denote by Q_T^k the non-cylindrical domain in $\mathbb{R}^2 : Q_T^k = \{(x, t) \in \mathbb{R}^2 \mid 0 < x < \alpha_k(t) \text{ and } 0 < t < T\}$. Set $\Gamma_L = \{(0, t) \in \mathbb{R}^2 \mid t \in [0, T]\}$, $\Gamma_R = \{(\alpha_k(t), t) \in \mathbb{R}^2 | t \in [0, T]\}$ and $\Sigma = \Gamma_L \bigcup \Gamma_R$. Let ∂Q_T^k represent the boundary of Q_T^k and $n(p) = (n_x(p), n_t(p))^\top$ denote the unit outward normal at p on ∂Q_T^k, where $n_x(p)$ and $n_t(p)$ are components of $n(p)$ corresponding to space and time, respectively. Q_T^k is named time like, if the inequality $|n_t(p)| < |n_x(p)|$ holds for every point $p \in \Sigma$. If $|n_t(p)| > |n_x(p)|$ holds for every point $p \in \Sigma$, then Q_T^k is named space like. In this article, we assume that $k > 1$. It is easy to see that wave equations are defined in the domain Q_T^k with a space-like boundary Γ_R.

There are many literatures on wave equations in non-cylindrical domains with time-like boundary (see e.g. [2, 3, 5–8, 11–15] and the references cited therein). The systems studied there are well-posed under two boundary conditions. Next we list some works related to wellposedness. To the best of our knowledge, [2] was the first paper, which gave the explicit solutions expressed by series for the one-dimensional wave equation with moving boundary.

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For the N-dimensional case ($N \geq 1$), an idea to convert non-cylindrical domains into cylindrical domains by some invertible transformations was introduced in [12,13]. More precisely, if a function u satisfies the wave equation $u_{tt} - u_{xx} = 0$ in Q_T^k, then by defining the transformation $w(y,t) = u(\alpha_k(t)y, t)$, we can verify that w satisfies

$$w_{tt} - \left[a(y,t)w_y\right]_y + 2b(y,t)w_{ty} = 0 \quad \text{in } Q, \tag{1.1}$$

where $a(y,t) = \frac{1-k^2y^2}{(1+kt)^2}$, $b(y,t) = -\frac{ky}{1+kt}$ and $Q = (0,1) \times (0,T)$. Therefore, the wellposedness problem of u in Q_T^k was transformed to the wellposedness problem of w in (1.1). Clearly, when $0 < k < 1$, $a(y,t)$ is positive definite in Q. Thus the Galerkin method could be applied to deal with the wellposedness problem in this case. Nevertheless, when $k > 1$, $a(y,t)$ changes sign in Q, which brings much trouble. Eventually, we mention that in [5], the authors used the D'Alembert formula to get the wellposedness of the wave equation under two Dirichlet boundary conditions in the case of $k = 1$. Since waves travel at a finite speed, the above methods are not applicable to the case where the motion of the moving endpoint is faster than the wave's motion (i.e. $k > 1$ for Q_T^k). For wave equations in space-like domains, [4] was the one and only one paper we have known providing a condition that solutions and all their first-order derivatives vanish on Σ to make systems well-posed. In the view of controllability, on the one hand, the condition mentioned in [4] is so harsh that we have no chance to impose a boundary control. On the other hand, the solution in [4] has a higher regularity. Relatively, the space in which solutions exist has a poor dual space, which is not good for us to consider a controllability problem.

The aim of this paper is to find a general class of boundary conditions to make systems well-posed in some suitable spaces. From reference [4], we guess that the boundary conditions we are looking for may be different from ordinary formulations. Finally, we consider a system with the following boundary conditions:

$$\begin{cases} u_{tt} - u_{xx} + \alpha u_t + \beta u = 0 & \text{in } Q_T^k, \\ u|_{\Gamma_L} = f_1, u|_{\Gamma_R} = f_2, \frac{\partial u}{\partial l}|_{\Gamma_R} = f_3, & \\ u(x,0) = u^0, u_t(x,0) = u^1 & \text{in } \Omega(0), \end{cases} \tag{1.2}$$

where u is the state variable, (u^0, u^1) is an initial couple, l denotes the direction $\left(\frac{1}{\sqrt{1+k^2}}, \frac{k}{\sqrt{1+k^2}}\right)^\top$, $\frac{\partial u}{\partial l}|_{\Gamma_R}$ is a restriction of the derivative to u along the direction l on Γ_R ($u|_{\Gamma_L}$ and $u|_{\Gamma_R}$ are also restrictions of u on Γ_L and Γ_R, respectively) and α, β are non-negative constants. (u^0, u^1) and $f_i (i = 1,2,3)$ will be given later.

The results we offer will be of significant importance to many related fields such as boundary controllability and qualitative theory of wave equations. We shall give some interpretations.

(1) Our work is a preparation for the study of boundary controllability problems, because in the case of $k > 1$, if we simply exert f_1 or f_2 on the bondary, the system is not controllable (This conclusion follows immediately from the fact that its dual system is not observable).

(2) From (1.2), we claim that such a problem with ordinary Dirichlet boundary conditions may be ill-posed (Since f_3 is given freely in an appropriate function space, we can choose different f_3, but keep f_1 and f_2 the same, and then the system has multiple solutions).

In order to define the transposition solution of (1.2), we introduce the following backward

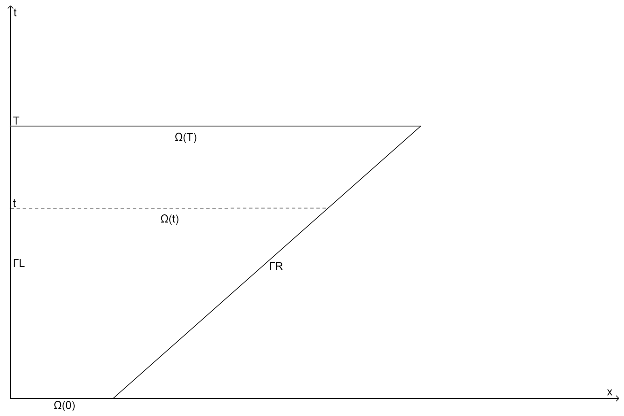

Figure 1.1: The graph of (1.2)

system with zero terminal value.

$$\begin{cases} w_{tt} - w_{xx} - \alpha w_t + \beta w = f & \text{in } Q_T^k, \\ w|_{\Gamma_L} = 0, & \\ w(x,T) = 0, w_t(x,T) = 0 & \text{in } \Omega(T). \end{cases} \tag{1.3}$$

Set $H_L^1(\Omega(t)) = \{w \in H^1(\Omega(t)) \mid \text{the trace of } w \text{ vanish at x=0}\}$, $\forall t \geq 0$. Assume that F is a functional space and F' is its dual space. Let $\langle \cdot, \cdot \rangle_{F,F'}$ denote the dual product between them.

Proposition 1.1. *For any $f \in L^2(Q_T^k)$, (1.3) admits a unique weak solution*

$$w \in L^2(0,T; H_L^1(\Omega(t))) \bigcap H^1(0,T; L^2(\Omega(t))).$$

Definition 1.2. Let $T > 0$. $u \in L^2(0,T;L^2(\Omega(t))) \bigcap H^1(0,T;H^{-1}(\Omega(t)))$ is called a transposition solution of (1.2), if for any $(u^0, u^1) \in L^2(\Omega(0)) \times H^{-1}(\Omega(0))$, $f_1 \in L^2(\Gamma_L)$ and $f_2, f_3 \in L^2(\Gamma_R)$, u satisfies the following equality

$$\begin{aligned} \int_{Q_T^k} fudxdt = & \langle w(0), u^1 \rangle_{H_L^1(\Omega(0)),H^{-1}(\Omega(0))} - \int_{\Omega(0)} [u^0 w_t(0) - \alpha u^0 w(0)] dx \\ & + \int_{\Gamma_L} f_1 w_x ds + \int_{\Gamma_R} \left[w f_3 + f_2 \frac{(-kw_t - w_x + k\alpha w)}{\sqrt{1+k^2}} \right] ds, \qquad \forall f \in L^2(Q_T^k), \end{aligned} \tag{1.4}$$

where w is the solution of (1.3).

The main result of this paper is stated as follows.

Theorem 1.3. *For any given $(u^0, u^1) \in L^2(\Omega(0)) \times H^{-1}(\Omega(0))$, $f_1 \in L^2(\Gamma_L)$ and $f_2, f_3 \in L^2(\Gamma_R)$, (1.2) admits a unique solution*

$$u \in L^2(0,T; L^2(\Omega(t))) \bigcap H^1(0,T; H^{-1}(\Omega(t)))$$

in the sense of transposition.

In addition, we study the energy for the homogeneous system of (1.2):

$$\begin{cases} u_{tt} - u_{xx} + \alpha u_t + \beta u = 0 & \text{in } Q_T^k, \\ u|_{\Gamma_L} = 0, u|_{\Gamma_R} = 0, \frac{\partial u}{\partial l}|_{\Gamma_R} = 0, & \\ u(x,0) = u^0, u_t(x,0) = u^1 & \text{in } \Omega(0), \end{cases} \tag{1.5}$$

where (u^0, u^1) is given in (1.2).

Concerning wave equations in domains with variable boundaries, the work on stability and stabilization has been addressed much less in the literature. For the case of time-like domains, we mention [3,11], which provided some first-order polynomial decay results using the multiplier method. As we know, the multiplier method is an efficient way to get a polynomial decay estimate (see e.g. [10]), but it requires that the coefficients of wave equations satisfy certain constraints (e.g. for (1.5), $\alpha^2 = 4\beta$ needed) and depending on the multiplier method, it is hard for us to obtain a better decay estimate. For the sake of getting the desired exponential decay of the energy for (1.5), we borrow an idea introduced in [9]. The difference is the use of an auxiliary functional ρ which will be provided in Section 4. We focus on the case of $\alpha, \beta > 0$ below. On the one hand, when $\alpha = 0$ and $\beta = 0$, it is easy to check the energy $E(t) = \frac{1}{2}\int_{\Omega(t)}[u_t^2(x,t) + u_x^2(x,t)]dx$ for (1.5) is conserved. On the other hand, if either of $\alpha, \beta > 0$ is not true, we have not been able to get the energy estimate for such a problem (we shall provide a further interpretation in Remark 4.1, Section 4).

Define an energy functional:

$$E(u;t) = \frac{1}{2}\int_{\Omega(t)} \left[u_t^2(x,t) + u_x^2(x,t) + \beta u^2(x,t)\right]dx.$$

The energy estimate for (1.5) is as follows.

Theorem 1.4. *There exist constants $\varepsilon_1 > 0$ and $c_1 > 0$ (only depend on α or β), such that the energy functional of (1.5) satisfies*

$$E(u;t) \le \frac{2}{1 - c_1\varepsilon} \exp\left\{-\frac{\varepsilon t}{1 + c_1\varepsilon}\right\} E(u;0), \quad \forall 0 < \varepsilon \le \varepsilon_1,\ \forall t \ge 0. \tag{1.6}$$

Throughout this paper, we let C represent a positive constant which may be different from one line to another. For simplicity of presentation, in what follows we omit the variables of functions sometimes when they are clear in the text.

Remark 1.5. We wish to present an interpretation for the form of (1.4). Without loss of generality, we may assume that functions are sufficiently smooth. Otherwise, we can use the smoothing technique. If u is a solution of (1.2), multiplying the first equation of (1.2) by w and integrating the both sides of the equation on Q_T^k, we have

$$\int_{Q_T^k} (u_{tt} - u_{xx} + \alpha u_t + \beta u)w dxdt = 0,$$

that is,

$$\int_{Q_T^k} [(u_t w - u w_t + \alpha u w)_t - (u_x w - u w_x)_x + u(w_{tt} - w_{xx} - \alpha w_t + \beta w)]dxdt = 0.$$

Using Green's formula, we obtain

$$\int_{\partial Q_T^k} [(u_t w - u w_t + \alpha u w)n_t - (u_x w - u w_x)n_x]ds + \int_{Q_T^k} u(w_{tt} - w_{xx} - \alpha w_t + \beta w)dxdt = 0,$$

where ds is the length of an infinitesimal on boundary ∂Q_T^k. Notice that the unit exterior normal $n(p) = (n_x(p), n_t(p))^\top = (0,1)^\top, (0,-1)^\top, (-1,0)^\top$ and $\left(\frac{1}{\sqrt{1+k^2}}, \frac{-k}{\sqrt{1+k^2}}\right)^\top$ when p lies in

$\Omega(T)$, $\Omega(0)$, Γ_L and Γ_R, respectively. Substituting them into the above equation, we get

$$\int_{\Omega(T)}(u_t w - uw_t + \alpha uw)(x,T)dx - \int_{\Omega(0)}(u_t w - uw_t + \alpha uw)(x,0)dx + \int_{\Gamma_L}(u_x w - uw_x)ds + \int_{\Gamma_R}\left[-w\frac{(ku_t+u_x)}{\sqrt{1+k^2}} + u\frac{(kw_t+w_x-k\alpha w)}{\sqrt{1+k^2}}\right]ds + \int_{Q_T^k} u(w_{tt}-w_{xx}-\alpha w_t+\beta w)dxdt = 0,$$

where we follow the expression $(u+w)(x,T) = u(x,T)+w(x,T)$.

Further, if w is a solution of (1.3), due to the approximation theory of smooth functions, we arrive at

$$\langle w(0),u^1\rangle_{H_L^1(\Omega(0)),H^{-1}(\Omega(0))} - \int_{\Omega(0)}[u^0 w_t(0) - \alpha u^0 w(0)]dx + \int_{\Gamma_L} uw_x ds + \int_{\Gamma_R}\left[w\frac{(ku_t+u_x)}{\sqrt{1+k^2}} + u\frac{(-kw_t-w_x+k\alpha w)}{\sqrt{1+k^2}}\right]ds = \int_{Q_T^k} ufdxdt.$$

Remark 1.6. We would like to show some connections between our results and the existing results. As we know, the definite conditions of the string equation defined in $\mathbb{R}_+$ are the initial displacement and the initial velocity. If we let $k \to +\infty$, then in (1.2), the moving boundary Γ_R goes to the x_+ axis and the direction $l = (\frac{1}{\sqrt{1+k^2}}, \frac{k}{\sqrt{1+k^2}}) \to (0,1)$. At that time, $u|_{\Gamma_R}$ becomes a displacement and $\frac{\partial u}{\partial l}|_{\Gamma_R}$ becomes a velocity. On the other hand, if we let $k \to 1$, then Γ_R turns into a characteristic line of the string equation and $l \to (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$, which is the characteristic direction. According to the compatibility principle, we infer that the boundary conditions in (1.2) become the boundary conditions of the case $k = 1$.

The rest of this paper is organized as follows. In Section 2, we prove Proposition 1.1. In Section 3, we prove Theorem 1.3. In Section 4, we deduce the energy estimate. Section 5 offers an appendix which is a supplement to the proof of Proposition 2.1 in Section 2.

2 The wellposedness of (1.3)

In this section, we start to prove Proposition 1.1. More generally, we consider systems with any given terminal value.

First, we consider the following pure wave system:

$$\begin{cases} w_{tt} - w_{xx} = f & \text{in } Q_T^k, \\ w|_{\Gamma_L} = 0, \\ w(x,T) = w_T^0, w_t(x,T) = w_T^1 & \text{in } \Omega(T). \end{cases} \tag{2.1}$$

Proposition 2.1. *For any given $(w_T^0, w_T^1) \in H_L^1(\Omega(T)) \times L^2(\Omega(T))$ and $f \in L^2(Q_T^k)$, (2.1) admits a unique solution $w \in L^2(0,T;H_L^1(\Omega(t))) \bigcap H^1(0,T;L^2(\Omega(t)))$.*

We postpone the proof of Proposition 2.1 until the appendix for standing out the main part of this paper.

Next consider the system as follows.

$$\begin{cases} w_{tt} - w_{xx} - \alpha w_t + \beta w = f & \text{in } Q_T^k, \\ w|_{\Gamma_L} = 0, \\ w(x,T) = w_T^0, w_t(x,T) = w_T^1 & \text{in } \Omega(T). \end{cases} \tag{2.2}$$

Proposition 2.2. *For any given* $(w_T^0, w_T^1) \in H_L^1(\Omega(T)) \times L^2(\Omega(T))$ *and* $f \in L^2(Q_T^k)$, (2.2) *admits a unique solution* $w \in L^2(0,T;H_L^1(\Omega(t))) \bigcap H^1(0,T;L^2(\Omega(t)))$.

Our idea for the proof of Proposition 2.2 is to transform the backward system into an equivalent forward system. Then we use the contraction mapping principle and an energy method to finish the proof. As a preliminary, some notations are given ahead. Write $\widetilde{\alpha}_k(\tau) = 1 + k(T-\tau)$ for $0 \le \tau \le T$. We let $\widetilde{Q}_T^k$ stand for a non-cylindrical domain in $\mathbb{R}^2$: $\widetilde{Q}_T^k = \{(x,\tau) \in \mathbb{R}^2 \mid 0 < x < \widetilde{\alpha}_k(\tau), 0 < \tau < T\}$. Put $\Gamma_L = \{(0,\tau) \in \mathbb{R}^2 \mid \tau \in [0,T]\}$ and $\widetilde{\Gamma}_R = \{(\widetilde{\alpha}_k(\tau),\tau) \in \mathbb{R}^2 \mid \tau \in [0,T]\}$ (see Figure 2.1).

Proof. Step 1. If we take a time transformation $\tau = T - t$ and let $z(x,\tau) = w(x, T-\tau)$, then we see that the wellposedness of (2.2) is equivalent to the wellposedness of the following system:

$$\begin{cases} z_{\tau\tau} - z_{xx} + \alpha z_\tau + \beta z = f & \text{in } \widetilde{Q}_T^k, \\ z|_{\Gamma_L} = 0, \\ z(x,0) = w_T^0, z_\tau(x,0) = w_T^1 & \text{in } \Omega(T). \end{cases} \tag{2.3}$$

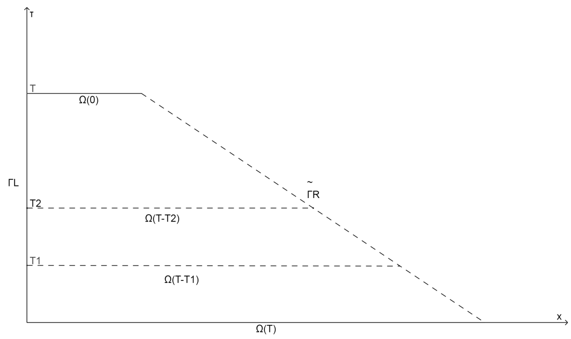

Figure 2.1: The graph of (2.3)

Let $0 < T_1 < T$ and put $\widetilde{Q}_0^{T_1} = \{(x,\tau) \in \mathbb{R}^2 \mid 0 < x < 1 + k(T-\tau), 0 < \tau < T_1\}$. Consider the following system with respect to (2.3) in $\widetilde{Q}_0^{T_1}$:

$$\begin{cases} z_{\tau\tau} - z_{xx} + \alpha \xi_\tau + \beta \xi = f & \text{in } \widetilde{Q}_0^{T_1}, \\ z(0,\tau) = 0 & \text{on } (0,T_1), \\ z(x,0) = w_T^0, z_\tau(x,0) = w_T^1 & \text{in } \Omega(T). \end{cases} \tag{2.4}$$

Set $\mathcal{X} = L^2(0,T_1;H_L^1(\Omega(T-\tau))) \bigcap H^1(0,T_1;L^2(\Omega(T-\tau)))$, where $\Omega(T-\tau) = \{(x,\tau) \in \mathbb{R}^2 | 0 < x < 1 + k(T-\tau)\}$ and put $\|z\|_{\mathcal{X}}^2 = \int_{\widetilde{Q}_0^{T_1}} [z_\tau^2(x,\tau) + z_x^2(x,\tau)] dx d\tau$, $\forall z \in \mathcal{X}$. $\mathcal{X}$ is a Banach space with the norm $\|\cdot\|_{\mathcal{X}}$ which can be found in [1]. Define a mapping $F : \xi \mapsto z$, $\forall \xi \in \mathcal{X}$, where z is the solution of (2.4). By Proposition 2.1, F is well defined. Next we prove F is a contraction mapping on $\mathcal{X}$. Let $z_1 = F(\xi_1)$ and $z_2 = F(\xi_2)$, $\forall \xi_1, \xi_2 \in \mathcal{X}$. Put $\overline{\xi} = \xi_1 - \xi_2$ and $\overline{z} = z_1 - z_2$. From the linearity of (2.4), we know that $\overline{z}$ and $\overline{\xi}$ satisfy

$$\begin{cases} \overline{z}_{\tau\tau} - \overline{z}_{xx} + \alpha \overline{\xi}_\tau + \beta \overline{\xi} = 0 & \text{in } \widetilde{Q}_0^{T_1}, \\ \overline{z}(0,\tau) = 0 & \text{on } (0,T_1), \\ \overline{z}(x,0) = 0, \overline{z}_\tau(x,0) = 0 & \text{in } \Omega(T). \end{cases} \tag{2.5}$$

Multiplying both sides of the first equation of (2.5) by $\bar{z}_\tau$, we get

$$(\bar{z}_{\tau\tau} - \bar{z}_{xx})\bar{z}_\tau = (-\alpha\bar{\xi}_\tau - \beta\bar{\xi})\bar{z}_\tau.$$

Furthermore,

$$\frac{1}{2}(\bar{z}_\tau^2 + \bar{z}_x^2)_\tau - (\bar{z}_\tau\bar{z}_x)_x = (-\alpha\bar{\xi}_\tau - \beta\bar{\xi})\bar{z}_\tau.$$

For any $\tau_1 \in (0, T_1]$, integrating the above equality on $(0, \tau_1) \times \Omega(T-\tau)$ and observing that $n(p) = (n_x(p), n_\tau(p))^\top = \left(\frac{1}{\sqrt{1+k^2}}, \frac{k}{\sqrt{1+k^2}}\right)^\top$, $\forall p \in \widetilde{\Gamma}_R$, we have

$$\begin{aligned}
&\int_{\Omega(T-\tau_1)} \frac{1}{2}\left[\bar{z}_\tau^2(x,\tau_1) + \bar{z}_x^2(x,\tau_1)\right] dx + \int_{\widetilde{\Gamma}_R} \left[\frac{1}{2}(\bar{z}_\tau^2 + \bar{z}_x^2)\frac{k}{\sqrt{1+k^2}} - \frac{1}{\sqrt{1+k^2}}\bar{z}_\tau\bar{z}_x\right] ds \\
&\quad = \int_0^{\tau_1}\int_{\Omega(T-\tau)} (-\alpha\bar{\xi}_\tau - \beta\bar{\xi})\bar{z}_\tau dx d\tau.
\end{aligned}$$

Due to $k > 1$, we get

$$\begin{aligned}
&\int_{\Omega(T-\tau_1)} \frac{1}{2}\left[\bar{z}_\tau^2(x,\tau_1) + \bar{z}_x^2(x,\tau_1)\right] dx \\
&\quad\le \int_0^{\tau_1}\int_{\Omega(T-\tau)} (-\alpha\bar{\xi}_\tau - \beta\bar{\xi})\bar{z}_\tau dx d\tau \\
&\quad\le \int_{\widetilde{Q}_0^{T_1}} (\alpha^2\bar{\xi}_\tau^2 + \beta^2\bar{\xi}^2) dx d\tau + \int_0^{\tau_1}\int_{\Omega(T-\tau)} \frac{1}{2}\left[\bar{z}_\tau^2(x,\tau) + \bar{z}_x^2(x,\tau)\right] dx d\tau.
\end{aligned}$$

Using Gronwall's inequality, we arrive at

$$\int_{\Omega(T-\tau_1)} \frac{1}{2}\left[\bar{z}_\tau^2(x,\tau_1) + \bar{z}_x^2(x,\tau_1)\right] dx \le e^{\tau_1}\int_{\widetilde{Q}_0^{T_1}} (\alpha^2\bar{\xi}_\tau^2 + \beta^2\bar{\xi}^2) dx d\tau. \tag{2.6}$$

We start to estimate $\int_{\widetilde{Q}_0^{T_1}} \bar{\xi}^2 dx d\tau$. Since $\bar{\xi} \in \mathcal{X}$, $\bar{\xi}(0,\tau) = 0$, a.e. $\tau \in (0, T_1)$ and $\bar{\xi}(x,\tau) = \int_0^x \bar{\xi}_y(y,\tau) dy$. Using Hölder's inequality, we have

$$\bar{\xi}^2(x,\tau) = \left[\int_0^x \bar{\xi}_y(y,\tau) dy\right]^2 \le x\int_0^x \bar{\xi}_y^2(y,\tau) dy.$$

Moreover, $0 \le x \le 1 + k(T-\tau)$, $\forall \tau \in [0, T_1]$, so

$$\begin{aligned}
\int_0^{1+k(T-\tau)} \bar{\xi}^2(x,\tau) dx &\le \int_0^{1+k(T-\tau)} x\int_0^x \bar{\xi}_y^2(y,\tau) dy dx \\
&\le \int_0^{1+k(T-\tau)} [1 + k(T-\tau)]\int_0^{1+k(T-\tau)} \bar{\xi}_y^2(y,\tau) dy dx \\
&\le [1 + k(T-\tau)]^2\int_0^{1+k(T-\tau)} \bar{\xi}_y^2(y,\tau) dy.
\end{aligned}$$

Integrating the above inequality on $(0, T_1)$, one has

$$\int_{\widetilde{Q}_0^{T_1}} \bar{\xi}^2(x,\tau) dx d\tau \le (1 + kT)^2\int_{\widetilde{Q}_0^{T_1}} \bar{\xi}_x^2(x,\tau) dx d\tau. \tag{2.7}$$

Using (2.7) in (2.6), one gets

$$\int_{\Omega(T-\tau_1)} \frac{1}{2}\left[\bar{z}_\tau^2(x,\tau_1) + \bar{z}_x^2(x,\tau_1)\right] dx \le e^{\tau_1}\int_{\widetilde{Q}_0^{T_1}} \left[\alpha^2\bar{\xi}_\tau^2 + \beta^2(1 + kT)^2\bar{\xi}_x^2\right] dx d\tau.$$

Since the above inequality holds for any $\tau_1 \in (0, T_1]$, integrating it on $(0, T_1)$, we obtain

$$\int_{\widetilde{Q}_0^{T_1}} \frac{1}{2}\left[\bar{z}_\tau^2(x,\tau) + \bar{z}_x^2(x,\tau)\right] dxd\tau \le T_1 e^{T_1} \int_{\widetilde{Q}_0^{T_1}} \left[\alpha^2 \bar{\xi}_\tau^2 + \beta^2(1+kT)^2 \bar{\xi}_x^2\right] dxd\tau$$
$$\le T_1 e^{T_1} \max\{\alpha^2, \beta^2(1+kT)^2\} \int_{\widetilde{Q}_0^{T_1}} (\bar{\xi}_\tau^2 + \bar{\xi}_x^2) dxd\tau.$$

This implies that

$$\|\bar{z}\|_{\mathcal{X}} \le \left[2T_1 e^{T_1} \max\{\alpha^2, \beta^2(1+kT)^2\}\right]^{\frac{1}{2}} \|\bar{\xi}\|_{\mathcal{X}}.$$

We can choose T_1 to be small such that $\|\bar{z}\|_{\mathcal{X}} \le \frac{1}{2}\|\bar{\xi}\|_{\mathcal{X}}$ holds, i.e., $\|F(\xi_1) - F(\xi_2)\|_{\mathcal{X}} \le \frac{1}{2}\|\xi_1 - \xi_2\|_{\mathcal{X}}$. For this T_1, F is a contraction mapping on $\mathcal{X}$. According to the contraction mapping principle, we know that F has a fixed point in $\mathcal{X}$ which is a solution of (2.3) in $\widetilde{Q}_0^{T_1}$. Let $T_0 = 0$ and $T_1 = T_1$. For any $T_i (i \ge 0, i \in \mathbb{Z})$, we put $\widetilde{Q}_{T_i}^{T_{i+1}} = \{(x,\tau) \in \mathbb{R}^2 \mid 0 < x < 1 + k(T-\tau), T_i < \tau < T_{i+1}\}$ and $\mathcal{X}_i = L^2(T_i, T_{i+1}; H_L^1(\Omega(T-\tau))) \bigcap H^1(T_i, T_{i+1}; L^2(\Omega(T-\tau)))$. Going through the same process we used in $\widetilde{Q}_0^{T_1}$ for $\widetilde{Q}_{T_i}^{T_{i+1}}$, we can get $\|\bar{z}\|_{\mathcal{X}_i} \le \left[2(T_{i+1} - T_i)e^{(T_{i+1}-T_i)} \max\{\alpha^2, \beta^2(1+k(T-T_i))^2\}\right]^{\frac{1}{2}} \|\bar{\xi}\|_{\mathcal{X}_i}$. Let $T_{i+1} - T_i \le T_1$, then we have $\left[2(T_{i+1} - T_i)e^{(T_{i+1}-T_i)} \max\{\alpha^2, \beta^2(1+k(T-T_i))^2\}\right]^{\frac{1}{2}} \le \left[2T_1 e^{T_1} \max\{\alpha^2, \beta^2(1+kT)^2\}\right]^{\frac{1}{2}}$, which implies that F is a contraction mapping on every $\mathcal{X}_i$. Continuing this process until $T_{i+1} \ge T$ for some i, we deduce that (2.3) admits a solution in $\widetilde{Q}_T^k$.

Step 2. We shall use an energy method for proving the uniqueness of the solution. Multiplying both sides of the first equation of (2.3) by z_τ, we have

$$(z_{\tau\tau} - z_{xx} + \alpha z_\tau + \beta z)z_\tau = f z_\tau.$$

Further,

$$\frac{1}{2}(z_\tau^2 + z_x^2 + \beta z^2)_\tau - (z_x z_\tau)_x + \alpha z_\tau^2 = f z_\tau.$$

For any T_1 in $(0, T]$, integrating the above equality on $\widetilde{Q}_0^{T_1}$ and using the boundary condition again, we get

$$\int_{\Omega(T-T_1)} \frac{1}{2}\left[z_\tau^2(x,T_1) + z_x^2(x,T_1) + \beta z^2(x,T_1)\right] dx$$
$$\le \int_0^{T_1}\int_{\Omega(T-\tau)} (f z_\tau - \alpha z_\tau^2) dxd\tau + \int_{\Omega(T)} \frac{1}{2}\left[(w_T^1)^2 + \left(\frac{\partial w_T^0}{\partial x}\right)^2 + \beta (w_T^0)^2\right] dx$$
$$\le \int_0^{T_1}\int_{\Omega(T-\tau)} \left[\frac{1}{2\delta} f^2 + \left(\frac{\delta}{2} - \alpha\right) z_\tau^2\right] dxd\tau + \int_{\Omega(T)} \frac{1}{2}\left[(w_T^1)^2 + \left(\frac{\partial w_T^0}{\partial x}\right)^2 + \beta (w_T^0)^2\right] dx.$$

If $\alpha > 0$, we can choose $\frac{\delta}{2} < \alpha$; if $\alpha = 0$, we use Gronwall's inequality again. Hence

$$\int_{\Omega(T-T_1)} \frac{1}{2}\left[z_\tau^2(x,T_1) + z_x^2(x,T_1) + \beta z^2(x,T_1)\right] dx$$
$$\le C(T)\left[\int_{\widetilde{Q}_T^k} f^2 dxd\tau + \int_{\Omega(T)} \frac{1}{2}\left[(w_T^1)^2 + \left(\frac{\partial w_T^0}{\partial x}\right)^2 + \beta (w_T^0)^2\right] dx\right]. \tag{2.8}$$

Because (2.8) holds for every $T_1 \in [0, T]$, it is shown that the solution of (2.3) is unique. The conclusion of Proposition 2.2 follows from the equivalence of wellposedness between (2.2) and (2.3). □

In Proposition 2.2, letting $(w_T^0, w_T^1) = (0, 0)$, we have Proposition 1.1.

3 The proof of Theorem 1.3

This section is devoted to establishing a hidden regularity result for (1.3) by the multiplier technique and confirming the conclusion of Theorem 1.3. We divide our proof in three steps.

Proof. Step 1. Define a functional $F : \forall f \in L^2(Q_T^k)$,

$$\begin{aligned} F(f) = \langle w(0), u^1 \rangle_{H_L^1(\Omega(0)), H^{-1}(\Omega(0))} - \int_{\Omega(0)} (u^0 w_t(0) - \alpha u^0 w(0)) dx \\ \int_{\Gamma_L} f_1 w_x ds + \int_{\Gamma_R} \left[w f_3 + f_2 \frac{(-k w_t - w_x + k\alpha w)}{\sqrt{1+k^2}} \right] ds, \end{aligned} \tag{3.1}$$

where w is the solution of (1.3) with f. In (3.1), u^0, u^1, f_1, f_2 and f_3 are known. It is clear that F is a linear functional. Next, we are going to prove it is bounded.

Step 2. We multiply both sides of the first equation of (1.3) by w_t and integrate it on Q_T^k. Observing that $n(p) = (n_x(p), n_t(p))^\top = (\frac{1}{\sqrt{1+k^2}}, \frac{-k}{\sqrt{1+k^2}})^\top$, $\forall p \in \Gamma_R$, we have

$$\begin{aligned} \int_{Q_T^k} f w_t dx dt &= \int_{Q_T^k} (w_{tt} - w_{xx} - \alpha w_t + \beta w) w_t dx \\ &= \int_{Q_T^k} \left[\frac{1}{2}(w_t^2 + w_x^2 + \beta w^2)_t - (w_t w_x)_x - \alpha w_t^2 \right] dx dt \\ &= -E(0) + \int_{\Gamma_R} \left[\frac{1}{2}(w_t^2 + w_x^2 + \beta w^2) \frac{-k}{\sqrt{1+k^2}} - (w_t w_x) \frac{1}{\sqrt{1+k^2}} \right] ds \\ &\quad - \alpha \int_{Q_T^k} w_t^2 dx dt, \end{aligned} \tag{3.2}$$

where $E(t) = \int_{\Omega(t)} \frac{1}{2} [w_t^2(x,t) + w_x^2(x,t) + \beta w^2(x,t)] dx$.

From (2.8), when $(w_T^0, w_T^1) = (0, 0)$, we know

$$E(t) \le C(T) \int_{Q_T^k} f^2 dx dt, \quad \forall t \in [0, T]. \tag{3.3}$$

Step 3. Multiplying both sides of the first equation of (1.3) by w_x and integrating it on Q_T^k, we get

$$\begin{aligned} \int_{Q_T^k} f w_x dx dt &= \int_{Q_T^k} (w_{tt} - w_{xx} - \alpha w_t + \beta w) w_x dx \\ &= \int_{Q_T^k} \left[(w_t w_x)_t - \frac{1}{2}(w_t^2 + w_x^2 - \beta w^2)_x - \alpha w_t w_x \right] dx dt \\ &= -\int_{\Omega(0)} w_t(x,0) w_x(x,0) dx + \int_{\Gamma_L} \frac{1}{2} w_x^2 ds \\ &\quad + \int_{\Gamma_R} \left[w_t w_x \frac{-k}{\sqrt{1+k^2}} - \frac{1}{2}(w_t^2 + w_x^2 - \beta w^2) \frac{1}{\sqrt{1+k^2}} \right] ds - \alpha \int_{Q_T^k} w_t w_x dx dt. \end{aligned} \tag{3.4}$$

(3.4) $- k \times$ (3.2), yields

$$\begin{aligned} \int_{Q_T^k} (f w_x - k f w_t) dx dt = &- \int_{\Omega(0)} w_t(x,0) w_x(x,0) dx + \int_{\Gamma_L} \frac{1}{2} w_x^2 ds \\ &+ \int_{\Gamma_R} \left[-\frac{1}{2}(w_t^2 + w_x^2 - \beta w^2) \frac{1}{\sqrt{1+k^2}} + \frac{1}{2}(w_t^2 + w_x^2 + \beta w^2) \frac{k^2}{\sqrt{1+k^2}} \right] ds \\ &+ \int_{Q_T^k} (-\alpha w_t w_x + k\alpha w_t^2) dx dt + kE(0). \end{aligned}$$

Rearranging the above equality, we obtain

$$\begin{aligned}&\int_{\Gamma_L}\frac{1}{2}w_x^2ds+\int_{\Gamma_R}\left[\frac{(k^2-1)}{2\sqrt{1+k^2}}(w_t^2+w_x^2)+\frac{(k^2+1)\beta}{2\sqrt{1+k^2}}w^2\right]ds\\&=\int_{Q_T^k}\left[f(w_x-kw_t)+\alpha(w_tw_x-kw_t^2)\right]dxdt+\int_{\Omega(0)}w_t(x,0)w_x(x,0)dx-kE(0)\\&\leq C(\alpha,\beta,k,T)\int_{Q_T^k}f^2dxdt,\end{aligned}\tag{3.5}$$

where the last inequality in (3.5) is derived using (3.3). From (3.5), it follows that $w_x|_{\Gamma_L}\in L^2(\Gamma_L)$, $w|_{\Gamma_R}$, $w_t|_{\Gamma_R}$ and $w_x|_{\Gamma_R}\in L^2(\Gamma_R)$. We shall make the assumptions:$(u^0,u^1)\in L^2(\Omega(0))\times H^{-1}(\Omega(0))$, $f_1\in L^2(\Gamma_L)$, $f_2\in L^2(\Gamma_R)$ and $f_3\in L^2(\Gamma_R)$. The definition for $F(f)$ in (3.1), together with (3.5), indicates that there exists a positive constant C (only depending on α,β,k,T) such that

$$|F(f)|\leq C(\alpha,\beta,k,T)\left[\|(u^0,u^1)\|_{L^2(\Omega(0))\times H^{-1}(\Omega(0))}+\|f_1\|_{L^2(\Gamma_L)}+\|f_2\|_{L^2(\Gamma_R)}+\|f_3\|_{L^2(\Gamma_R)}\right]\|f\|_{L^2(Q_T^k)}.$$

According to the Riesz's theorem, one can find a unique $u\in L^2(0,T;L^2(\Omega(t)))$ such that (1.4) holds. We claim that (1.2) has a unique solution $u\in L^2(0,T;L^2(\Omega(t)))$ in the sense of transposition. Without loss of generality, let $g\in C_0^\infty(0,T;H_0^1(\Omega(t)))$ and replace f with g_t at the right end of the first equation in (1.3). In the same manner we can get $\frac{\partial u}{\partial t}\in L^2(0,T;H^{-1}(\Omega(t)))$. □

4 The proof of Theorem 1.4

In this section, we prove Theorem 1.4 for the system (1.5) in Section 1.

Noticing that $u|_{\Gamma_R}=0$ and $\frac{\partial u}{\partial l}|_{\Gamma_R}=0$ in (1.5), we can deduce that

$$u_x|_{\Gamma_R}=u_t|_{\Gamma_R}=0.$$

Thus the boundary conditions in (1.5) are equivalent to

$$u|_{\Gamma_L}=0,\ u|_{\Gamma_R}=u_x|_{\Gamma_R}=u_t|_{\Gamma_R}=0.\tag{4.1}$$

Inspired by [9], we introduce an auxiliary functional. For any $\varepsilon>0$, let

$$E_\varepsilon(u;t)=E(u;t)+\varepsilon\rho(u;t),\quad\forall t\geq 0,$$

where

$$\rho(u;t)=\int_0^{\alpha_k(t)}u_t(x,t)u(x,t)dx.$$

Proof. First, suppose that the initial value (u^0,u^1) and the solution u are sufficiently smooth.

On account of

$$|\rho(u;t)|\leq\int_0^{\alpha_k(t)}|u_tu|dx\leq\frac{1}{2}\int_0^{\alpha_k(t)}(\frac{1}{\beta}u_t^2+\beta u^2+u_x^2)dx,$$

when $\beta\geq 1$, put $c_1=1$; when $\beta<1$, put $c_1=\frac{1}{\beta}$. Hence,

$$\varepsilon^{-1}|E_\varepsilon(u;t)-E(u;t)|=|\rho(u;t)|\leq c_1E(u;t).\tag{4.2}$$

Using the first equation in (1.5), we have

$$
\begin{aligned}
E_\varepsilon'(u;t) &= E'(u;t) + \varepsilon\rho'(u;t) \\
&= \int_0^{\alpha_k(t)} (u_t u_{tt} + u_x u_{xt} + \beta u u_t)dx + \varepsilon \int_0^{\alpha_k(t)} (u_{tt}u + u_t^2)dx \\
&= \int_0^{\alpha_k(t)} \left[u_t(u_{xx} - \alpha u_t - \beta u) + u_x u_{xt} + \beta u u_t\right]dx \\
&\quad + \varepsilon \int_0^{\alpha_k(t)} \left[(u_{xx} - \alpha u_t - \beta u)u + u_t^2\right]dx \\
&= \int_0^{\alpha_k(t)} \left[(u_t u_x)_x - \alpha u_t^2\right]dx \\
&\quad + \varepsilon \int_0^{\alpha_k(t)} \left[(u_x u)_x - u_x^2 - \alpha u_t u - \beta u^2 + u_t^2\right]dx,
\end{aligned}
$$

where $E_\varepsilon'(u;t)$ represents the derivative of $E_\varepsilon(u;t)$ with respect to time.

Using (4.1), we arrive at

$$
E_\varepsilon'(u;t) = -\alpha \int_0^{\alpha_k(t)} u_t^2 dx + \varepsilon \int_0^{\alpha_k(t)} (-u_x^2 - \alpha u_t u - \beta u^2 + u_t^2)dx.
$$

Furthermore,

$$
\begin{aligned}
&\varepsilon \int_0^{\alpha_k(t)} (-u_x^2 - \alpha u_t u - \beta u^2 + u_t^2)dx \\
&\quad \le \varepsilon \int_0^{\alpha_k(t)} \left[-\frac{1}{2}u_x^2 + \left(\frac{\beta}{2}u^2 + \frac{\alpha^2}{2\beta}u_t^2\right) - \beta u^2 + u_t^2\right] dx \\
&\quad = \varepsilon \int_0^{\alpha_k(t)} \left[-\frac{1}{2}\left(u_x^2 + \beta u^2 + u_t^2\right) + \left(\frac{3}{2} + \frac{\alpha^2}{2\beta}\right) u_t^2\right] dx \\
&\quad = -\varepsilon E(u;t) + \varepsilon \left(\frac{3}{2} + \frac{\alpha^2}{2\beta}\right) \int_0^{\alpha_k(t)} u_t^2 dx.
\end{aligned}
$$

Letting $\varepsilon \le \dfrac{\alpha}{\frac{3}{2} + \frac{\alpha^2}{2\beta}} = \varepsilon_0$, we have

$$
E_\varepsilon'(u;t) \le -\varepsilon E(u;t).
$$

On the other hand, when $\varepsilon < \frac{1}{c_1}$, (4.2) implies that

$$
(1 - c_1\varepsilon)E(u;t) \le E_\varepsilon(u;t) \le (1 + c_1\varepsilon)E(u;t), \quad \forall t \ge 0.
$$

So

$$
E_\varepsilon'(u;t) \le -\varepsilon E(u;t) \le -\frac{\varepsilon}{1 + c_1\varepsilon} E_\varepsilon(u;t). \tag{4.3}
$$

Set $\varepsilon_1 = \min\left\{\frac{1}{c_1}, \varepsilon_0\right\}$. Using (4.2) combined with (4.3), we get

$$
\begin{aligned}
(1 - c_1\varepsilon)E(u;t) \le E_\varepsilon(u;t) &\le \exp\left\{-\frac{\varepsilon t}{1 + c_1\varepsilon}\right\} E_\varepsilon(u;0) \\
&\le \exp\left\{-\frac{\varepsilon t}{1 + c_1\varepsilon}\right\} (1 + c_1\varepsilon)E(u;0),
\end{aligned}
$$

$\forall 0 < \varepsilon \le \varepsilon_1$, $\forall t \ge 0$.

It means that

$$\begin{aligned} E(u;t) &\le \exp\left\{-\frac{\varepsilon t}{1+c_1\varepsilon}\right\}\frac{1+c_1\varepsilon}{1-c_1\varepsilon}E(u;0) \\ &\le \frac{2}{1-c_1\varepsilon}\exp\left\{-\frac{\varepsilon t}{1+c_1\varepsilon}\right\}E(u;0), \quad \forall t \ge 0. \end{aligned} \tag{4.4}$$

Finally, using the approximation theory of smooth functions in (4.4), we obtain Theorem 1.4. □

Remark 4.1. From the process of the proof of Theorem 1.4, we can see that if $\alpha \le 0$, the energy of (1.5) does not decrease; if $\beta = 0$, we are not able to get the desired decay estimate in the same manner (because the Poincaré inequality does not hold for some uniform constant in such an increasing domain).

Remark 4.2. In the case of $0 < k \le 1$, by [5], we have that the wave equation is well-posed with two Dirichlet boundary conditions. We can also put a damping αu_t ($\alpha > 0$) and a compensation βu ($\beta > 0$) into the homogeneous system to consider the energy estimate. Although there will be some extra terms coming from boundary, we can deal with them as well depending on the method mentioned above, and then get the exponential decay results. Nevertheless, if the moving boundaries are general, things may be complicated and it may be not easy for us to deal with those boundary terms and get the desired energy estimate based on the method above.

5 Appendix. The proof of Proposition 2.1

We follow the notations in Section 2.

Proof. Taking a time transformation $\tau = T - t$ and letting $z(x,\tau) = w(x, T-\tau)$, we change (2.1) to the following system of z :

$$\begin{cases} z_{\tau\tau} - z_{xx} = f & \text{in } \widetilde{Q}_T^k, \\ z|_{\Gamma_L} = 0, \\ z(x,0) = w_T^0, z_\tau(x,0) = w_T^1 & \text{in } \Omega(T). \end{cases} \tag{5.1}$$

We will denote by $\overline{\widetilde{Q}_T^k}$ the closure of $\widetilde{Q}_T^k$ and by $D(\overline{\widetilde{Q}_T^k})$ the space of infinitely differentiable functions defined on $\overline{\widetilde{Q}_T^k}$. Let $(D(\overline{\widetilde{Q}_T^k}))^2$ be the quadratic space of $D(\overline{\widetilde{Q}_T^k})$ and $\mathcal{H} = (L^2(\widetilde{Q}_T^k))^2$. $\forall Y, W \in \mathcal{H}$, with $Y = (y_1, y_2)^\top$ and $W = (w_1, w_2)^\top$, an inner product in $\mathcal{H}$ is defined to be

$$(Y, W)_{\mathcal{H}} = \int_{\widetilde{Q}_T^k} \left[y_1(x,t)w_1(x,t) + y_2(x,t)w_2(x,t)\right] dxdt.$$

We use the symbol $\langle \cdot, \cdot \rangle_{\mathbb{R}^2}$ to denote the scalar product in $\mathbb{R}^2$. The notation $|\cdot|_{\mathbb{R}^2}$ means the canonical norm induced by $\langle \cdot, \cdot \rangle_{\mathbb{R}^2}$. Define an operator A :

$$D(A) = \{W \in \mathcal{H} \mid AW \in \mathcal{H}\},$$

$$AW = \begin{pmatrix} \frac{\partial w_1}{\partial t} - \frac{\partial w_2}{\partial x} \\ \frac{\partial w_2}{\partial t} - \frac{\partial w_1}{\partial x} \end{pmatrix}, \quad \forall W = (w_1, w_2)^\top \in D(A).$$

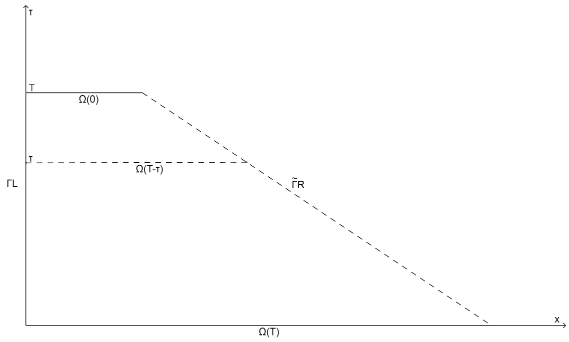

Figure 5.1: The graph of (5.1)

Let $\|W\|^2_{D(A)} = \|W\|^2_{\mathcal{H}} + \|AW\|^2_{\mathcal{H}}$, $\forall W \in D(A)$. It is easy to check that $D(A)$ is a Hilbert space with the norm $\|\cdot\|_{D(A)}$.

Using an argument similar to that in [4], we shall prove three lemmas. Notice that Q^k_T is different from the domain described in [4] and the distinction is Γ_L. Consequently, the focus of our proofs is on estimating the value of functions discussed below on Γ_L.

Lemma 5.1. *The following two propositions are equivalent.*

(i) For every triplet $(w^0_T, w^1_T, f) \in H^1_L(\Omega(T)) \times L^2(\Omega(T)) \times L^2(\widetilde{Q}^k_T)$*, find* $z \in L^2(0, T; H^1_L(\Omega(T-\tau)))$ *such that* $\frac{\partial z}{\partial \tau} \in L^2(0, T; L^2(\Omega(T-\tau)))$*, and z satisfies:*

$$\begin{aligned} z_{\tau\tau} - z_{xx} &= f, \\ z(\cdot, 0) = w^0_T, &\quad z_\tau(\cdot, 0) = w^1_T, \end{aligned} \tag{5.2}$$

with the additional boundary conditions $z\big|_{\Gamma_L} = \frac{\partial z}{\partial \tau}\big|_{\Gamma_L} = 0$.

(ii) For every triplet $(w^0_T, w^1_T, f) \in H^1_L(\Omega(T)) \times L^2(\Omega(T)) \times L^2(\widetilde{Q}^k_T)$*, find* $Z = (z_1, z_2)^\top \in D(A)$*, such that*

$$AZ = F, \quad Z(\cdot, 0) = Z_0 \quad \textit{and} \quad z_2|_{\Gamma_L} = 0, \tag{5.3}$$

where F and Z_0 *are given as follows*

$$F = \begin{pmatrix} 0 \\ f \end{pmatrix}, \quad Z_0 = \begin{pmatrix} \frac{\partial w^0_T}{\partial x} \\ w^1_T \end{pmatrix}.$$

Proof. $(i) \Rightarrow (ii)$ If z satisfies (i), then let

$$Z := \begin{pmatrix} \frac{\partial z}{\partial x} \\ \frac{\partial z}{\partial \tau} \end{pmatrix}.$$

According to the definition of a distributional derivative, we have

$$\left(\frac{\partial}{\partial x}\right)\left(\frac{\partial z}{\partial \tau}\right) = \left(\frac{\partial}{\partial \tau}\right)\left(\frac{\partial z}{\partial x}\right). \tag{5.4}$$

Using (5.2) combined with (5.4), we obtain $AZ = F$ and $Z \in D(A)$. From the initial condition and boundary condition, we get

$$Z_0 = Z(\cdot, 0) = \begin{pmatrix} \frac{\partial z}{\partial x}(\cdot, 0) \\ \frac{\partial z}{\partial \tau}(\cdot, 0) \end{pmatrix} = \begin{pmatrix} \frac{\partial w^0_T}{\partial x} \\ w^1_T \end{pmatrix} \quad \text{and} \quad z_2|_{\Gamma_L} = \frac{\partial z}{\partial \tau}\bigg|_{\Gamma_L} = 0.$$

$(ii) \Rightarrow (i)$ For any $Z = (z_1, z_2)^\top \in D(A)$, let

$$z(x,\tau) := \int_0^\tau z_2(x,s)ds + w_T^0(x). \tag{5.5}$$

Since $D(A)\bigcap(D(\overline{\widetilde{Q}_T^k}))^2$ is dense in $D(A)$, one can choose a $Z^\varepsilon = (z_1^\varepsilon, z_2^\varepsilon)^\top \in D(A)\bigcap(D(\overline{\widetilde{Q}_T^k}))^2$ such that

$$Z^\varepsilon \to Z(\varepsilon \to 0+) \quad \text{in } D(A),$$

that is, $Z^\varepsilon \to Z$ in $\mathcal{H}$, $AZ^\varepsilon \to AZ$ in $\mathcal{H}$.

Put

$$h^\varepsilon := \frac{\partial z_1^\varepsilon}{\partial \tau} - \frac{\partial z_2^\varepsilon}{\partial x}.$$

Since $\frac{\partial z_1}{\partial \tau} - \frac{\partial z_2}{\partial x} = 0$, $h^\varepsilon \to 0(\varepsilon \to 0+)$ in $L^2(\widetilde{Q}_T^k)$. Thus

$$\begin{aligned}
\frac{\partial}{\partial x}\int_0^\tau z_2^\varepsilon(x,s)ds &= \int_0^\tau \frac{\partial}{\partial x} z_2^\varepsilon(x,s)ds \\
&= \int_0^\tau [\frac{\partial z_1^\varepsilon}{\partial s}(x,s) - h^\varepsilon(x,s)]ds \\
&= z_1^\varepsilon(x,\tau) - z_1^\varepsilon(x,0) - \int_0^\tau h^\varepsilon(x,s)ds.
\end{aligned}$$

It follows that

$$\lim_{\varepsilon\to 0+} \frac{\partial}{\partial x}\int_0^\tau z_2^\varepsilon(x,s)ds = z_1(x,\tau) - \frac{\partial w_T^0}{\partial x}(x). \tag{5.6}$$

Using (5.5) and (5.6), we have

$$\begin{aligned}
\frac{\partial z(x,\tau)}{\partial x} &= \frac{\partial}{\partial x}\int_0^\tau z_2(x,s)ds + \frac{\partial w_T^0}{\partial x}(x) \\
&= \lim_{\varepsilon\to 0+}\frac{\partial}{\partial x}\int_0^\tau z_2^\varepsilon(x,s)ds + \frac{\partial w_T^0}{\partial x}(x) \\
&= z_1(x,\tau) - \frac{\partial w_T^0}{\partial x}(x) + \frac{\partial w_T^0}{\partial x}(x) \\
&= z_1(x,\tau).
\end{aligned}$$

From (5.5), it is easy to check that $\frac{\partial z(x,\tau)}{\partial \tau} = z_2(x,\tau)$. Therefore,

$$z_{\tau\tau} - z_{xx} = \frac{\partial z_2}{\partial \tau} - \frac{\partial z_1}{\partial x} = f,$$

where the last equality is obtained using the first equation in (5.3). Because of $z_2|_{\Gamma_L} = 0$, by (5.5), we deduce

$$z|_{\Gamma_L} = \frac{\partial z}{\partial \tau}\bigg|_{\Gamma_L} = 0,$$

and it is easy to see that

$$z(\cdot,0) = w_T^0 \quad \text{and} \quad z_\tau(\cdot,0) = w_T^1. \qquad \square$$

Let us consider a special case of Lemma 5.1, that is, $Z(\cdot,0) = 0$. We put $D_0 = \{Z \in D(A) \mid Z(\cdot,0) = 0\}$. Furthermore, write $D_0^{\Gamma_L} = \{Z \in D_0 \mid z_2|_{\Gamma_L} = 0\}$.

Lemma 5.2. *For any $F \in \mathcal{H}$, there exists one and only one $Z \in D_0^{\Gamma_L}$, such that*

$$AZ = F.$$

Proof. We go through two steps to prove Lemma 5.2.

Step 1. Let $Z = (z_1, z_2)^\top \in D_0^{\Gamma_L}$. Using Green's formula, we have

$$\begin{aligned}(AZ, Z)_{\mathcal{H}} &= \int_{\widetilde{Q}_T^k} \left\langle \begin{pmatrix} \frac{\partial z_1}{\partial \tau} - \frac{\partial z_2}{\partial x} \\ \frac{\partial z_2}{\partial \tau} - \frac{\partial z_1}{\partial x} \end{pmatrix}, \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \right\rangle_{\mathbb{R}^2} dx d\tau \\ &= \int_{\widetilde{Q}_T^k} \left(\frac{\partial z_1}{\partial \tau} z_1 - \frac{\partial z_2}{\partial x} z_1 + \frac{\partial z_2}{\partial \tau} z_2 - \frac{\partial z_1}{\partial x} z_2 \right) dx d\tau \\ &= \int_{\widetilde{Q}_T^k} \left[\frac{1}{2} \left(z_1^2 + z_2^2 \right)_\tau - \left(z_1 z_2 \right)_x \right] dx d\tau \\ &= \int_{\Omega(0)} \frac{1}{2} \left| Z(x, T) \right|_{\mathbb{R}^2}^2 dx + \int_{\widetilde{\Gamma}_R} \left[\frac{1}{2} (z_1^2 + z_2^2) n_\tau - z_1 z_2 n_x \right] ds.\end{aligned} \tag{5.7}$$

Notice that $n(p) = (n_x(p), n_\tau(p))^\top = \left(\frac{1}{\sqrt{1+k^2}}, \frac{k}{\sqrt{1+k^2}} \right)^\top$, $\forall p \in \widetilde{\Gamma}_R$, when $k \geq 1$,

$$\int_{\widetilde{\Gamma}_R} \left[\frac{1}{2} (z_1^2 + z_2^2) \frac{k}{\sqrt{1+k^2}} - z_1 z_2 \frac{1}{\sqrt{1+k^2}} \right] ds \geq 0.$$

Replacing the upper limit T of the integral with τ in (5.7), we have

$$\int_{\Omega(T-\tau)} \left| Z(x, \tau) \right|_{\mathbb{R}^2}^2 dx \leq 2 \int_0^\tau \int_{\Omega(T-s)} \langle AZ, Z \rangle_{\mathbb{R}^2} dx ds.$$

According to the Cauchy–Schwartz inequality, it follows that

$$\left| \langle AZ, Z \rangle_{\mathbb{R}^2} \right| \leq \left| AZ \right|_{\mathbb{R}^2} \left| Z \right|_{\mathbb{R}^2} \leq \frac{1}{2} (\left| AZ \right|_{\mathbb{R}^2}^2 + \left| Z \right|_{\mathbb{R}^2}^2),$$

and moreover,

$$\begin{aligned}\int_{\Omega(T-\tau)} \left| Z(x, \tau) \right|_{\mathbb{R}^2}^2 dx &\leq \int_0^\tau \int_{\Omega(T-s)} \left| AZ \right|_{\mathbb{R}^2}^2 dx ds + \int_0^\tau \int_{\Omega(T-s)} \left| Z \right|_{\mathbb{R}^2}^2 dx ds \\ &\leq \int_0^T \int_{\Omega(T-s)} \left| AZ \right|_{\mathbb{R}^2}^2 dx ds + \int_0^\tau \int_{\Omega(T-s)} \left| Z \right|_{\mathbb{R}^2}^2 dx ds.\end{aligned}$$

Using Gronwall's inequality, we obtain

$$\int_{\Omega(T-\tau)} \left| Z(x, \tau) \right|_{\mathbb{R}^2}^2 dx \leq C(T) \int_{\widetilde{Q}_T^k} \left| AZ \right|_{\mathbb{R}^2}^2 dx d\tau.$$

Integrating above inequality on $(0, T)$, one get

$$\|Z\|_{L^2(\widetilde{Q}_T^k)}^2 \leq C(T) \|AZ\|_{L^2(\widetilde{Q}_T^k)}^2, \quad \forall Z \in D_0^{\Gamma_L}. \tag{5.8}$$

We now see that (5.8) implies A is an injective mapping and

$$A : D_0^{\Gamma_L} \to \operatorname{Im}(A)$$

is an isomorphism. It is easy to check that A^{-1} is a bounded linear operator and continuous. $D_0^{\Gamma_L}$ is a closed subspace of $D(A)$, thus $\operatorname{Im}(A)$ is closed in $\mathcal{H}$ from the continuity of A^{-1}.

Step 2. Suppose that $V=(v_1,v_2)^\top\in\mathcal{H}$. $\mathrm{Im}(A)$ is dense in $\mathcal{H}$ if

$$(AZ,V)_{\mathcal{H}}=0,\quad \forall Z\in D_0^{\Gamma_L}\Rightarrow V=0. \tag{5.9}$$

Next let us prove (5.9). Let $Z\in(C_0^\infty(\widetilde{Q}_T^k))^2$. Then

$$\begin{aligned}
(AV,Z)_{\mathcal{H}}&=\int_{\widetilde{Q}_T^k}\left\langle\begin{pmatrix}\frac{\partial v_1}{\partial\tau}-\frac{\partial v_2}{\partial x}\\ \frac{\partial v_2}{\partial\tau}-\frac{\partial v_1}{\partial x}\end{pmatrix},\begin{pmatrix}z_1\\ z_2\end{pmatrix}\right\rangle_{\mathbb{R}^2}dxd\tau\\
&=\int_{\widetilde{Q}_T^k}\left(\frac{\partial v_1}{\partial\tau}z_1-\frac{\partial v_2}{\partial x}z_1+\frac{\partial v_2}{\partial\tau}z_2-\frac{\partial v_1}{\partial x}z_2\right)dxd\tau\\
&=\int_{\widetilde{Q}_T^k}-\left(v_1\frac{\partial z_1}{\partial\tau}-v_2\frac{\partial z_1}{\partial x}+v_2\frac{\partial z_2}{\partial\tau}-v_1\frac{\partial z_2}{\partial x}\right)dxd\tau\\
&=-\int_{\widetilde{Q}_T^k}\left\langle\begin{pmatrix}\frac{\partial z_1}{\partial\tau}-\frac{\partial z_2}{\partial x}\\ \frac{\partial z_2}{\partial\tau}-\frac{\partial z_1}{\partial x}\end{pmatrix},\begin{pmatrix}v_1\\ v_2\end{pmatrix}\right\rangle_{\mathbb{R}^2}dxd\tau\\
&=-(AZ,V)_{\mathcal{H}}=0.
\end{aligned}$$

Hence, $AV=0$ and $V\in D(A)$.

$\forall Z,V\in(D(\overline{\widetilde{Q}_T^k}))^2$,

$$\begin{aligned}
&(AZ,V)_{\mathcal{H}}+(Z,AV)_{\mathcal{H}}\\
&=\int_{\widetilde{Q}_T^k}\left\langle\begin{pmatrix}\frac{\partial z_1}{\partial\tau}-\frac{\partial z_2}{\partial x}\\ \frac{\partial z_2}{\partial\tau}-\frac{\partial z_1}{\partial x}\end{pmatrix},\begin{pmatrix}v_1\\ v_2\end{pmatrix}\right\rangle_{\mathbb{R}^2}dxd\tau+\int_{\widetilde{Q}_T^k}\left\langle\begin{pmatrix}z_1\\ z_2\end{pmatrix},\begin{pmatrix}\frac{\partial v_1}{\partial\tau}-\frac{\partial v_2}{\partial x}\\ \frac{\partial v_2}{\partial\tau}-\frac{\partial v_1}{\partial x}\end{pmatrix}\right\rangle_{\mathbb{R}^2}dxd\tau\\
&=\int_{\widetilde{Q}_T^k}\left(\frac{\partial z_1}{\partial\tau}v_1-\frac{\partial z_2}{\partial x}v_1+\frac{\partial z_2}{\partial\tau}v_2-\frac{\partial z_1}{\partial x}v_2+\frac{\partial v_1}{\partial\tau}z_1-\frac{\partial v_2}{\partial x}z_1+\frac{\partial v_2}{\partial\tau}z_2-\frac{\partial v_1}{\partial x}z_2\right)dxd\tau\\
&=\int_{\widetilde{Q}_T^k}\left[\left(z_1v_1+z_2v_2\right)_\tau-\left(z_2v_1+z_1v_2\right)_x\right]dxd\tau\\
&=\int_{\Omega(0)}\langle Z(x,T),V(x,T)\rangle_{\mathbb{R}^2}\,dx-\int_{\Omega(T)}\langle Z(x,0),V(x,0)\rangle_{\mathbb{R}^2}\,dx\\
&\quad+\int_{\Gamma_L}\left\langle\frac{1}{\sqrt{1+k^2}}\begin{pmatrix}z_2\\ z_1\end{pmatrix},\begin{pmatrix}v_1\\ v_2\end{pmatrix}\right\rangle_{\mathbb{R}^2}ds\\
&\quad+\int_{\widetilde{\Gamma}_R}\left\langle\frac{1}{\sqrt{1+k^2}}\begin{pmatrix}k&-1\\ -1&k\end{pmatrix}\begin{pmatrix}z_1\\ z_2\end{pmatrix},\begin{pmatrix}v_1\\ v_2\end{pmatrix}\right\rangle_{\mathbb{R}^2}ds.
\end{aligned}\tag{5.10}$$

Using the density of $(D(\overline{\widetilde{Q}_T^k}))^2\bigcap D(A)$ in $D(A)$, one can choose a set of functions $\{V_\varepsilon\}_{\varepsilon>0}\subset(D(\overline{\widetilde{Q}_T^k}))^2$, such that $V_\varepsilon\to V$ in $\mathcal{H}$, $AV_\varepsilon\to AV$ in $\mathcal{H}$. Put $Z\in(D(\overline{\widetilde{Q}_T^k}))^2\bigcap D_0^{\Gamma_L}$, with the property S : $\operatorname{supp}Z\cap\Omega(0)$ is contained in a compact subset of $\Omega(0)$, and $\operatorname{supp}Z\cap\Sigma$ is contained in a compact subset of Σ. Letting $\varepsilon\to0+$ in (5.10), from $AV=0$ and the hypothesis of (5.9), we have

$$\begin{aligned}
0=&\int_{\Omega(0)}\langle Z(x,T),V(x,T)\rangle\,dx+\int_{\Gamma_L}\left\langle\frac{1}{\sqrt{1+k^2}}\begin{pmatrix}z_2\\ z_1\end{pmatrix},\begin{pmatrix}v_1\\ v_2\end{pmatrix}\right\rangle ds\\
&+\int_{\widetilde{\Gamma}_R}\left\langle\frac{1}{\sqrt{1+k^2}}\begin{pmatrix}k&-1\\ -1&k\end{pmatrix}\begin{pmatrix}z_1\\ z_2\end{pmatrix},\begin{pmatrix}v_1\\ v_2\end{pmatrix}\right\rangle ds,
\end{aligned}\tag{5.11}$$

where $\langle\cdot,\cdot\rangle$ stands for a dual relation in form. Since $\begin{pmatrix}k&-1\\ -1&k\end{pmatrix}$ $(k>1)$ is positive definite and (5.11) holds for every $Z\in(D(\overline{\widetilde{Q}_T^k}))^2\bigcap D_0^{\Gamma_L}$ with the property S, we can deduce that

$$V|_{\Omega(0)}=0,\quad v_2|_{\Gamma_L}=0\quad\text{and}\quad V|_{\widetilde{\Gamma}_R}=0.$$

Set

$$\widetilde{D} = \{V \in D(A) \mid V|_{\Omega(0)} = 0,\ v_2|_{\Gamma_L} = 0 \quad \text{and} \quad V|_{\widetilde{\Gamma}_R} = 0\}.$$

For every $V \in (D(\overline{\widetilde{Q}_T^k}))^2 \bigcap \widetilde{D}$, replacing Z with V in (5.10) and using Hölder's inequality, we get

$$\|V\|_{\mathcal{H}} \le C(T)\|AV\|_{\mathcal{H}}.$$

Using the density of $(D(\overline{\widetilde{Q}_T^k}))^2 \bigcap \widetilde{D}$ in $\widetilde{D}$, we see that the above inequality remains true for every $V \in \widetilde{D}$. From $AV = 0$, we deduce $V = 0$. Since $\mathrm{Im}(A)$ is dense and closed in $\mathcal{H}$, $\mathrm{Im}(A) = \mathcal{H}$. Furthermore, A is an isomorphism, so the proof is complete. □

The following Lemma 5.3 tells us that (ii) of Lemma 5.1 holds.

Lemma 5.3. *For every couple $(Z_0, F) \in (L^2(\Omega(T)))^2 \times \mathcal{H}$, there exists one and only one $Z \in D(A)$, such that*

$$AZ = F, \quad Z(\cdot, 0) = Z_0 \quad \textit{and} \quad z_2|_{\Gamma_L} = 0.$$

Proof. As $Z_0 = (z_{01}, z_{02})^\top \in (L^2(\Omega(T)))^2$, there exists a sequence of functions $\{Z^\eta\}_{\eta>0} \subset (D(\overline{\widetilde{Q}_T^k}))^2$ with $Z^\eta = (z_1^\eta, z_2^\eta)^\top$ such that

$$(1) \quad \lim_{\eta\to+0} \|Z^\eta(\cdot,0) - Z_0\|_{(L^2(\Omega(T)))^2} = 0,$$
$$(2) \quad z_2^\eta|_{\Gamma_L} = 0.$$

For any $\eta > 0$, using Lemma 5.2, one can find a $Z_\eta = (z_{\eta 1}, z_{\eta 2})^\top \in D_0^{\Gamma_L}$ satisfying

$$AZ_\eta = F - AZ^\eta.$$

Thus $A(Z_\eta + Z^\eta) = F$.

Let $Z_\eta^\varepsilon = (D(\overline{\widetilde{Q}_T^k}))^2 \bigcap D_0^{\Gamma_L}$ such that $Z_\eta^\varepsilon \to Z_\eta$ in $D(A)$. Replacing Z and V with $Z_\eta^\varepsilon + Z^\eta$ in (5.10) and using conditions $z_2^\eta|_{\Gamma_L} = 0$ and $z_{\eta 2}^\varepsilon|_{\Gamma_L} = 0$, we have

$$\begin{aligned}
&2\big(A(Z_\eta^\varepsilon + Z^\eta), Z_\eta^\varepsilon + Z^\eta\big)_{\mathcal{H}} \\
&= \int_{\Omega(0)} \big|(Z_\eta^\varepsilon + Z^\eta)(x,T)\big|_{\mathbb{R}^2}^2 dx - \int_{\Omega(T)} \big|(Z_\eta^\varepsilon + Z^\eta)(x,0)\big|_{\mathbb{R}^2}^2 dx \\
&\quad + \int_{\widetilde{\Gamma}_R} \left\langle \frac{1}{\sqrt{1+k^2}} \begin{pmatrix} k & -1 \\ -1 & k \end{pmatrix} \begin{pmatrix} z_{\eta 1}^\varepsilon + z_1^\eta \\ z_{\eta 2}^\varepsilon + z_2^\eta \end{pmatrix}, \begin{pmatrix} z_{\eta 1}^\varepsilon + z_1^\eta \\ z_{\eta 2}^\varepsilon + z_2^\eta \end{pmatrix} \right\rangle_{\mathbb{R}^2} ds.
\end{aligned}$$

Since $\begin{pmatrix} k & -1 \\ -1 & k \end{pmatrix}$ $(k > 1)$ is positive definite, we have

$$2\big(A(Z_\eta^\varepsilon + Z^\eta), Z_\eta^\varepsilon + Z^\eta\big)_{\mathcal{H}} \ge \int_{\Omega(0)} \big|(Z_\eta^\varepsilon + Z^\eta)(x,T)\big|_{\mathbb{R}^2}^2 dx - \int_{\Omega(T)} \big|(Z_\eta^\varepsilon + Z^\eta)(x,0)\big|_{\mathbb{R}^2}^2 dx. \quad (5.12)$$

Changing upper limit T of the integral to τ in (5.12), we have

$$\begin{aligned}
&2\int_0^\tau \int_{\Omega(T-s)} \left\langle A(Z_\eta^\varepsilon + Z^\eta), Z_\eta^\varepsilon + Z^\eta \right\rangle_{\mathbb{R}^2} dxds \\
&\quad \ge \int_{\Omega(T-\tau)} \big|(Z_\eta^\varepsilon + Z^\eta)(x,\tau)\big|_{\mathbb{R}^2}^2 dx - \int_{\Omega(T)} \big|(Z_\eta^\varepsilon + Z^\eta)(x,0)\big|_{\mathbb{R}^2}^2 dx.
\end{aligned}$$

Moreover,

$$\begin{aligned}&\int_{\Omega(T-\tau)} \left|(Z^\varepsilon_\eta + Z^\eta)(x,\tau)\right|^2_{\mathbb{R}^2} dx \\ &\quad \le \int_{\Omega(T)} \left|(Z^\varepsilon_\eta + Z^\eta)(x,0)\right|^2_{\mathbb{R}^2} dx + \int_0^T \int_{\Omega(T-\tau)} \left|A(Z^\varepsilon_\eta + Z^\eta)\right|^2_{\mathbb{R}^2} dx d\tau \\ &\qquad + \int_0^\tau \int_{\Omega(T-s)} \left|(Z^\varepsilon_\eta + Z^\eta)\right|^2_{\mathbb{R}^2} dx ds.\end{aligned}$$

Using Gronwall's inequality, we have

$$\int_{\Omega(T-\tau)} \left|(Z^\varepsilon_\eta + Z^\eta)(x,\tau)\right|^2_{\mathbb{R}^2} dx \le C(T)\left(\int_{\Omega(T)} \left|(Z^\varepsilon_\eta + Z^\eta)(x,0)\right|^2_{\mathbb{R}^2} dx + \|A(Z^\varepsilon_\eta + Z^\eta)\|_{\mathcal{H}}\right).$$

Integrating the above inequality on $(0,T)$, we get

$$\|(Z^\varepsilon_\eta + Z^\eta)\|^2_{\mathcal{H}} \le C(T)\left(\int_{\Omega(T)} \left|(Z^\varepsilon_\eta + Z^\eta)(x,0)\right|^2_{\mathbb{R}^2} dx + \|A(Z^\varepsilon_\eta + Z^\eta)\|_{\mathcal{H}}\right).$$

Let $\varepsilon \to 0+$, it holds that

$$\|Z_\eta + Z^\eta\|^2_{\mathcal{H}} \le C(T)\left(\int_{\Omega(T)} \left|Z^\eta(x,0)\right|^2_{\mathbb{R}^2} dx + ||F||^2_{\mathcal{H}}\right).$$

When η is small enough, $\{Z_\eta + Z^\eta\}_\eta$ remains bounded in $D(A)$. Since $D(A)$ is a Hilbert space, there exists a subsequence of $\{Z_\eta + Z^\eta\}_\eta$, still remember $\{Z_\eta + Z^\eta\}_\eta$, and $Z \in D(A)$, such that

$$Z_\eta + Z^\eta \rightharpoonup Z \quad \text{in } D(A).$$

Noticing that $A(Z_\eta + Z^\eta) = F$, we deduce

$$AZ = F, \quad z_2|_{\Gamma_L} = 0 \quad \text{and} \quad Z(\cdot,0) = Z_0.$$

The uniqueness of the solution is easily obtained from Lemma 5.2. □

Combining the results of Lemma 5.1 and Lemma 5.3, we claim that for every triplet $(w^0_T, w^1_T, f) \in H^1_L(\Omega(T)) \times L^2(\Omega(T)) \times L^2(\widetilde{Q}^k_T)$, system (5.1) admits a unique solution $z \in L^2(0,T;H^1_L(\Omega(T-\tau)))$ and $\frac{\partial z}{\partial \tau} \in L^2(0,T;L^2(\Omega(T-\tau)))$. Because of $w(x,t) = w(x,T-\tau) = z(x,\tau)$, it implies that under the same assumptions, system (2.1) admits a unique solution $w \in L^2(0,T;H^1_L(\Omega(t)))$. Moreover, $\frac{\partial z}{\partial \tau}(x,\tau) = -\frac{\partial w}{\partial t}(x,T-\tau) = -\frac{\partial w}{\partial t}(x,t)$, so $\frac{\partial w}{\partial t} \in L^2(0,T;L^2(\Omega(t)))$. Until now, we finish the proof of Proposition 2.1. □

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