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Existence of solutions for subquadratic convex or *B*-concave operator equations and applications to second order Hamiltonian systems

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Abstract. This paper investigates solutions for subquadratic convex or B-concave operator equations. First, some existence results are obtained by the index theory and the critical point theory. Then, some applications to second order Hamiltonian systems satisfying generalized periodic boundary value conditions and Sturm–Liouville boundary value conditions are pointed out. In particular, some well known theorems about periodic solutions for second order Hamiltonian systems are special cases of these results.

Keywords: subquadratic, operator equations, index theory, critical point, second order Hamiltonian systems.

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1 Introduction and main results

Mawhin and Willem [9] investigated the second order Hamiltonian system

$$\begin{cases} -\ddot{x}(t) - m^2\omega^2 x(t) = \nabla_x V(t, x(t)), & \text{a.e. } t \in [0, T], \\ x(0) - x(T) = \dot{x}(0) - \dot{x}(T) = 0, \end{cases} \tag{1.1}$$

where $T > 0, \omega = \frac{2\pi}{T}, m \in \{0, 1, 2, \dots\}, V \in C([0, T] \times \mathbb{R}^n, \mathbb{R}), \nabla_x V$ denotes the gradient of V with respect to $x, \nabla_x V \in C([0, T] \times \mathbb{R}^n, \mathbb{R}^n)$, and for each $x \in \mathbb{R}^n, V(t, x)$ is periodic in t with period T. Using the dual least action principle and the perturbation technique, the Authors, in theirs excellent book [9], proved some existence theorems of solutions for problem (1.1) with subquadratic convex or concave potential. Recently, using the reduction method, the perturbation argument and the least action principle, Tang and Wu [12] proved an abstract critical point theorem without the compactness assumptions which generalizes the results in [7]. As a main application, they successively obtained some existence theorems of problem

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(1.1) with $m = 0$ and subquadratic convex potential or $k(t)$-concave potential, which unify and generalize some earlier results in [9,13,14,16,17]. Later on, applying the abstract critical point theory established in [12], Ye [15] proved some existence theorems of problem (1.1), where $m \geq 1$ and the potential is convex and satisfies conditions which are more general than the subquadratic conditions in [9]. In this paper we reconsider in the framework of the operator equations some theorems proved in [9,12,15].

Let X be a real infinite-dimensional separable Hilbert space with inner product $(\cdot,\cdot)_X$ and the corresponding norm $\|\cdot\|_X$. Let $A : D(A) \subset X \to X$ be an unbounded linear self-adjoint operator with $\sigma(A) = \sigma_d(A)$ bounded from below. Hence, there is an orthonormal basis $\{e_j\}_{j=1}^{\infty}$ of X and $\lambda_1 \leq \lambda_2 \leq \cdots$ such that $Ae_j = \lambda_j e_j, D(A) = \{\sum_{j=1}^{\infty} c_j e_j | \sum_{j=1}^{\infty} \lambda_j^2 c_j^2 < \infty\}$. In addition, let $Z \equiv D(|A|^{\frac{1}{2}}) = \{\sum_{j=1}^{\infty} c_j e_j | \sum_{j=1}^{\infty} |\lambda_j| c_j^2 < \infty\}$ equipped with the norm $\|x\|_Z^2 = \|x\|^2 = \sum_{j=1}^{\infty}(1 + |\lambda_j|)c_j^2$. For any $x = \sum_{j=1}^{\infty} c_j e_j \in Z, y = \sum_{j=1}^{\infty} d_j e_j \in Z$, we can define a bilinear form

$$a(x,y) = \sum_{j=1}^{\infty} \lambda_j c_j d_j.$$

Note that $(Ax,y)_X = a(x,y)$ if $x \in D(A), y \in Z$, this shows that $a(x,y)$ is the extension of $(Ax,y)_X$ on Z. Moreover, let $\mathcal{L}_s(X)$ be the usual space consisting of bounded symmetric operators in X. For given $B \in \mathcal{L}_s(X)$, we define

$$\nu_A(B) = \dim \ker(A - B),$$
$$i_A(B) = \sum_{\lambda<0} \nu_A(B + \lambda Id),$$

as introduced by Dong, see Definition 7.1.1 in [5] or Definition 3.1.1 and Proposition 3.1.4 in [4]. We consider the following operator equation

$$Ax - B_1 x = \nabla\Phi(x), \tag{1.2}$$

where $B_1 \in \mathcal{L}_s(X), \nu_A(B_1) \neq 0$, and Φ satisfies

(Φ_0) $\Phi \in C^1(Z,\mathbb{R})$ is weakly continuous with weakly continuous derivative, that is, $x_n \rightharpoonup x_0$ in Z implies that $\Phi(x_n) \to \Phi(x_0)$ and $\Phi'(x_n) \to \Phi'(x_0)$. Moreover, for every $x \in Z$ there exists $\nabla\Phi(x) \in X$ such that $\Phi'(x)y = (\nabla\Phi(x), y)_X$ for all $y \in Z$.

Let X_1 be a nontrivial subspace of X. For $B_1, B_2 \in \mathcal{L}_s(X)$ we write $B_1 \leq B_2$ with respect to X_1 if and only if $(B_1x,x)_X \leq (B_2x,x)_X$ for all $x \in X_1$; we write $B_1 < B_2$ w.r.t. X_1 if and only if $(B_1x,x)_X < (B_2x,x)_X$ for all $x \in X_1 \backslash \{\theta\}$. If $X_1 = X$, then we just write $B_1 \leq B_2$ or $B_1 < B_2$. In addition, we write $B_1 < B_2$ properly if and only if $B_1 \leq B_2$ and $B_1 < B_2$ w.r.t. $\ker(A - B)$ for all $B \in \mathcal{L}_s(X)$.

Our main results can be stated as follows.

Theorem 1.1. *Assume that Φ satisfies (Φ_0) and*

(Φ_1) *Φ is convex in X;*

(Φ_2) *Φ and Φ' are bounded in Z;*

(Φ_3) *$\Phi(x) \to +\infty$ as $\|x\| \to \infty$ with $x \in \ker(A - B_1)$;*

(Φ_4) *there exist $c > 0$ and $B_2 \in \mathcal{L}_s(X)$ with $B_2 \geq B_1$ and $B_2 > B_1$ w.r.t. $\ker(A - B_1)$, $\nu_A(B_2) \neq 0$ and $i_A(B_2) = i_A(B_1) + \nu_A(B_1)$, such that*

$$\Phi(x) \leq \frac{1}{2}((B_2 - B_1)x,x)_X + c \tag{1.3}$$

for all $x \in X$, *and*

$$\Phi(x) - \frac{1}{2}((B_2 - B_1)x, x)_X \to -\infty \tag{1.4}$$

as $\|\overline{x}\| \to \infty$, *where* $x = \widetilde{x} + \overline{x}$ *with* $\overline{x} \in \ker(A - B_2)$ *and* $\|\tilde{x}\|$ *is bounded.*

Then problem (1.2) *has a solution in* Z.

Theorem 1.2. *The conclusion of Theorem 1.1 still holds if we replace* (Φ_4) *with*

(Φ_4') *there exist* $c > 0$ *and* $B_2 \in \mathcal{L}_s(X)$ *with* $B_2 \geq B_1$ *and* $B_2 > B_1$ *w.r.t.* $\ker(A - B_1)$, $\nu_A(B_2) = 0$ *and* $i_A(B_2) = i_A(B_1) + \nu_A(B_1)$, *such that*

$$\Phi(x) \leq \frac{1}{2}((B_2 - B_1)x, x)_X + c \tag{1.5}$$

for all $x \in X$.

Theorem 1.3. *The conclusion of Theorem 1.1 still holds if we replace* (Φ_1) *and* (Φ_4) *with*

(Φ_1') Φ *is* $(B_2 - B_1)$*-concave, that is,* $-\Phi(x) + \frac{1}{2}((B_2 - B_1)x, x)_X$ *is convex in* X.

(Φ_4'') *there exists* $B_2 \in \mathcal{L}_s(X)$ *with* $B_2 \geq B_1$ *and* $B_2 > B_1$ *w.r.t.* $\ker(A - B_1), i_A(B_1) = 0, \nu_A(B_2) \neq 0$ *and* $i_A(B_2) = i_A(B_1) + \nu_A(B_1)$, *such that*

$$-\Phi(x) + \frac{1}{2}((B_2 - B_1)x, x)_X \to +\infty \tag{1.6}$$

as $\|x\| \to \infty$ *with* $x \in \ker(A - B_2)$, *respectively.*

Theorem 1.4. *The conclusion of Theorem 1.1 still holds if we replace* (Φ_1) *and* (Φ_4) *with* (Φ_1'),

(Φ_4''') *there exists* $B_2 \in \mathcal{L}_s(X)$ *with* $B_2 \geq B_1$ *and* $B_2 > B_1$ *w.r.t.* $\ker(A - B_1), i_A(B_1) = 0, \nu_A(B_2) = 0$, *such that*

$$i_A(B_2) = i_A(B_1) + \nu_A(B_1),$$

respectively.

The paper is organized as follows. In Section 2, we first recall a critical point theorem as given in [12]. Then, following [4, 5], we recall some useful conclusions of index theory for linear self-adjoint operator equations. Finally, we quote a lemma in [3], which shows that (1.2) possesses a variational structure. In Section 3, we prove Theorems 1.1–1.4. In Section 4, we investigate their applications to second order Hamiltonian systems with generalized periodic boundary conditions and Sturm–Liouville boundary conditions. The corresponding results in [9, 12, 15] are special cases of these results.

2 Preliminaries

In order to prove our main results, we recall first two lemmas due to Tang and Wu [12].

Lemma 2.1 ([12, Theorem 1.1])**.** *Suppose that* X_1 *and* X_2 *are reflexive Banach spaces,* $I \in C^1(X_1 \times X_2, \mathbb{R})$. $I(x_1, \cdot)$ *is weakly upper semi-continuous for all* $x_1 \in X_1$ *and* $I(\cdot, x_2) : X_1 \to \mathbb{R}$ *is convex for all* $x_2 \in X_2$, *and* I' *is weakly continuous. Assume that*

$$I(\theta, x_2) \to -\infty \tag{2.1}$$

as $\|x_2\| \to +\infty$ *and, for every* $M > 0$

$$I(x_1, x_2) \to +\infty \tag{2.2}$$

as $\|x_1\| \to +\infty$ *uniformly for* $\|x_2\| \leq M$*. Then* I *has at least one critical point.*

Lemma 2.2 ([12, Lemma 5.1])**.** *Suppose that* H *is a real Hilbert space,* $f : H \times H \to \mathbb{R}$ *is a bilinear functional. Then* $g : H \to \mathbb{R}$ *given by*

$$g(x) = f(x, x), \qquad \forall x \in H$$

is convex if and only if

$$g(x) \geq 0, \qquad \forall x \in H.$$

Now we also recall some definitions and propositions in [4,5].

Definition 2.3 ([5, Page 108])**.** For any $B \in \mathcal{L}_s(X)$, we define

$$\psi_{a,B}(x, y) = a(x, y) - (Bx, y)_X, \qquad \forall x, y \in Z.$$

For any $x, y \in Z$ if $\psi_{a,B}(x, y) = 0$ we say that x and y are $\psi_{a,B}$-orthogonal. For any two subspaces Z_1 and Z_2 of Z if $\psi_{a,B}(x, y) = 0$ for any $x \in Z_1, y \in Z_2$ we say that Z_1 and Z_2 are $\psi_{a,B}$-orthogonal.

Proposition 2.4 ([5, Proposition 7.2.1])**.** *For any* $B \in \mathcal{L}_s(X)$*, the space* Z *has a* $\psi_{a,B}$*-orthogonal decomposition*

$$Z = Z_a^+(B) \oplus Z_a^0(B) \oplus Z_a^-(B)$$

such that $\psi_{a,B}$ *is positive definite, null and negative definite on* $Z_a^+(B), Z_a^0(B)$ *and* $Z_a^-(B)$ *respectively. Moreover,* $Z_a^0(B)$ *and* $Z_a^-(B)$ *are finitely dimensional.*

Definition 2.5 ([5, Definition 7.2.1])**.** For any $B \in \mathcal{L}_s(X)$, we define $\nu_a(B) = \dim Z_a^0(B)$, $i_a(B) = \dim Z_a^-(B)$.

Proposition 2.6.

(1) *For any* $B \in \mathcal{L}_s(X)$*, we have*

$$\nu_A(B) = \nu_a(B), \qquad i_A(B) = i_a(B), \qquad \ker(A - B) = Z_a^0(B).$$

([5], Proposition 7.2.2 (i))

(2) *For any* $B_1, B_2 \in \mathcal{L}_s(X)$*, if* $B_1 \leq B_2$ *with respect to* $Z_a^-(B_1)$*, then* $i_a(B_1) \leq i_a(B_2)$*; if* $B_1 \leq B_2$ *with respect to* $Z_a^-(B_1) \oplus Z_a^0(B_1)$*, then* $i_a(B_1) + \nu_a(B_1) \leq i_a(B_2) + \nu_a(B_2)$*; if* $B_1 < B_2$ *with respect to* $Z_a^0(B_1)$ *and* $B_1 \leq B_2$ *with respect to* $Z_a^-(B_1)$*, then* $i_a(B_1) + \nu_a(B_1) \leq i_a(B_2)$*.* ([5], Proposition 7.2.2 (ii))

(3) *For any* $B_1, B_2 \in \mathcal{L}_s(X)$*, if* $B_1(t) \leq B_2(t)$ *and* $B_1(t) < B_2(t)$ *properly, then*

$$i_a(B_2) - i_a(B_1) = \sum_{\lambda \in [0,1)} \nu_a(B_1 + \lambda(B_2 - B_1)).$$

([5], Proposition 7.2.2 (iii))

(4) *(Poincaré inequality.) For any $B \in \mathcal{L}_s(X)$, if $i_a(B) = 0$, then*

$$\psi_{a,B}(x,x) \geq 0, \qquad \forall x \in Z.$$

And the equality holds if and only if $x \in Z_a^0(B)$. ([5], Proposition 7.2.2 (v))

(5) *For any $B_1, B_2 \in \mathcal{L}_s(X)$, if $B_1 \leq B_2$ and $B_1 < B_2$ w.r.t. $\ker(A - B_1)$ and $i_A(B_2) = i_A(B_1) + \nu_A(B_1)$, then $Z = Z_a^-(B_1) \oplus Z_a^0(B_1) \oplus Z_a^0(B_2) \oplus Z_a^+(B_2)$, and $(-\psi_{a,B_1}(x_1,x_1))^{\frac{1}{2}} + (\psi_{a,B_2}(x_2,x_2))^{\frac{1}{2}}$ is an equivalent norm on Z for $x = x_1 + x_2$ with $x_1 \in Z_a^-(B_1)$, $x_2 \in Z_a^+(B_2)$. In particular, for any $B_1 \in \mathcal{L}_s(X)$, then $Z = Z_a^-(B_1) \oplus Z_a^0(B_1) \oplus Z_a^+(B_1)$ and $(-\psi_{a,B_1}(x_1,x_1))^{\frac{1}{2}} + (\psi_{a,B_1}(x_2,x_2))^{\frac{1}{2}}$ is also an equivalent norm on Z for $x = x_1 + x_2$ with $x_1 \in Z_a^-(B_1), x_2 \in Z_a^+(B_1)$.*

Proof. We only prove (5). Let $Z_1 = Z_a^-(B_1) \oplus Z_a^0(B_1), Z_2 = Z_a^0(B_2) \oplus Z_a^+(B_2)$. Noticing that $\psi_{a,B_1}(x,x) \geq \psi_{a,B_2}(x,x)$ for all $x \in Z$, $\psi_{a,B_1}(x,x) \leq 0$ for all $x \in Z_1$ and $\psi_{a,B_2}(x,x) \geq 0$ for all $x \in Z_2$, if $x \in Z_1 \bigcap Z_2$ then $\psi_{a,B_2}(x,x) = 0 = \psi_{a,B_1}(x,x)$, which shows that $x \in Z_a^0(B_2) \bigcap Z_a^0(B_1)$. By $B_1 \leq B_2$ and $B_1 < B_2$ w.r.t. $\ker(A - B_1)$, we have $0 = \psi_{a,B_1}(x,x) > \psi_{a,B_2}(x,x) = 0$ provided $x \in Z_a^0(B_2) \bigcap Z_a^0(B_1) \backslash \{\theta\}$. This is a contradiction, which implies that $Z_1 \bigcap Z_2 = \{\theta\}$. It remains to prove that $Z = Z_1 + Z_2$. By Proposition 2.4, we have $Z = Z_2 \oplus Z_a^-(B_2)$ and for any $x \in Z$ there exists a unique pair $(x_1, x_2) \in Z_2 \times Z_a^-(B_2)$ such that $x = x_1 + x_2$. Let $\{e_j\}_{j=1}^k$ be a basis of $Z_1, e_j = e_j^2 + e_j^-$ with $e_j^2 \in Z_2, e_j^- \in Z_a^-(B_2)$ for $j = 1, 2, \cdots, k = i_A(B_1) + \nu_A(B_1)$. By $i_A(B_2) = i_A(B_1) + \nu_A(B_1) = k$, in order to prove $\{e_j^-\}_{j=1}^k$ is a basis of $Z_a^-(B_2)$ we only need to show that $\{e_j^-\}_{j=1}^k$ is linear independent. In fact, otherwise there exist not all zero constants $c_1, \ldots, c_k$ such that $\sum_{j=1}^k c_j e_j^- = 0$. This leads to $\sum_{j=1}^k c_j e_j \in Z_1 \bigcap Z_2$, a contradiction. The linear independent shows that there exist constants $\{\alpha_j\}_{j=1}^k$ such that $x_2 = \sum_{j=1}^k \alpha_j e_j^-$. And hence $x = x_1 + x_2 = x = x_1 + \sum_{j=1}^k \alpha_j e_j^- = \sum_{j=1}^k \alpha_j e_j + \big(x_1 - \sum_{j=1}^k \alpha_j e_j^2\big)$.

Similar to the proof of Proposition 7.2.2 (iv) in [5], we can prove that $(-\psi_{a,B_1}(x_1,x_1))^{\frac{1}{2}} + (\psi_{a,B_2}(x_2,x_2))^{\frac{1}{2}}$ is an equivalent norm on Z for $x = x_1 + x_2$ with $x_1 \in Z_a^-(B_1), x_2 \in Z_a^+(B_2)$, and $(-\psi_{a,B_1}(x_1,x_1))^{\frac{1}{2}} + (\psi_{a,B_1}(x_2,x_2))^{\frac{1}{2}}$ is also an equivalent norm on Z for $x = x_1 + x_2$ with $x_1 \in Z_a^-(B_1), x_2 \in Z_a^+(B_1)$. □

Finally, let us consider the functional I defined by

$$I(x) = -\frac{1}{2}a(x,x) + \frac{1}{2}(B_1 x, x)_X + \Phi(x), \tag{2.3}$$

for every $x \in Z$. Under assumption (Φ_0), from Theorem 1.2 in [9] it is easy to verify that $I \in C^1(Z, \mathbb{R})$ is weakly upper semi-continuous on Z and I' is weakly continuous with

$$I'(x)y = -a(x,y) + (B_1 x, y)_X + \Phi'(x)y, \tag{2.4}$$

for every $x, y \in Z$.

The following important lemma is an immediate conclusion of Lemma 2.1 in [3].

Lemma 2.7. *Assume that (Φ_0) holds. Then a critical point of $I(x)$ is a solution for problem (1.2).*

3 Proofs of the Theorems

In this section, we present the proof of Theorems 1.1–1.4.

Proof of Theorem 1.1. By $\nu_A(B_1) \neq 0$, $B_1 \leq B_2$ and $B_1 < B_2$ w.r.t. $\ker(A - B_1)$ and $i_A(B_2) = i_A(B_1) + \nu_A(B_1)$, we have $Z = Z_a^-(B_1) \oplus Z_a^0(B_1) \oplus Z_a^0(B_2) \oplus Z_a^+(B_2)$ via (5) of Proposition 2.6. Set $X_1 = Z_a^-(B_1) \oplus Z_a^0(B_1), X_2 = Z_a^0(B_2) \oplus Z_a^+(B_2) = Z_a^+(B_1), x \in Z, x = x_1 + x_2$ with $x_1 \in X_1$ and $x_2 \in X_2$. Next, we divide the proof into three steps.

Step 1. We show that $I(\cdot, x_2) : X_1 \to \mathbb{R}$ is convex for all $x_2 \in X_2$. By (Φ_1), it is obvious that $\Phi(x_1 + x_2)$ is convex in $x_1 \in X_1$. From Definition 2.3 and Proposition 2.4 we can see that for every $x_1 \in X_1$,

$$-\frac{1}{2}\psi_{a,B_1}(x_1, x_1) = -\frac{1}{2}a(x_1, x_1) + \frac{1}{2}(B_1 x_1, x_1)_X \geq 0,$$

which implies that $-\frac{1}{2}\psi_{a,B_1}(x_1, x_1)$ is convex in $x_1 \in X_1$ via Lemma 2.2. Hence, for every $x_2 \in X_2$,

$$\begin{aligned} I(x_1 + x_2) &= -\frac{1}{2}a(x_1 + x_2, x_1 + x_2) + \frac{1}{2}(B_1(x_1 + x_2), x_1 + x_2)_X + \Phi(x_1 + x_2) \\ &= -\frac{1}{2}\psi_{a,B_1}(x_1, x_1) + \Phi(x_1 + x_2) - \frac{1}{2}\psi_{a,B_1}(x_2, x_2) \end{aligned}$$

is convex in $x_1 \in X_1$.

Step 2. By contradiction, we prove that (2.2) of Lemma 2.1 holds. Assume that (2.2) of Lemma 2.1 does not hold. Then there exist $M > 0, c_0 > 0$ and two sequences $\{x_{1,n}\} \subset X_1$ and $\{x_{2,n}\} \subset X_2$ with $\|x_{1,n}\| \to +\infty$ as $n \to \infty$ and $\|x_{2,n}\| \leq M$ for all n such that

$$I(x_{1,n} + x_{2,n}) \leq c_0, \ \forall n \in \mathbf{N}. \tag{3.1}$$

For $x_1 \in X_1$, write $x_1 = x_1^- + x_1^0$, where $x_1^- \in Z_a^-(B_1)$ and $x_1^0 \in Z_a^0(B_1)$. We consider the functional $\Phi|_{Z_a^0(B_1)}$. By (Φ_0), we easily see that $\Phi|_{Z_a^0(B_1)}$ is weakly lower semi-continuous on $Z_a^0(B_1)$. Using (Φ_3), by the least action principle (see Theorem 1.1 in [9]), $\Phi|_{Z_a^0(B_1)}$ has a minimum at some $x_{1,0}^0 \in Z_a^0(B_1)$ for which

$$0 = \Phi'(x_{1,0}^0)x_1^0 = (\nabla\Phi(x_{1,0}^0), x_1^0)_X, \ \forall x_1^0 \in Z_a^0(B_1).$$

By assumption (Φ_0) and the convexity of Φ, we have

$$\begin{aligned} \Phi(x_1 + x_2) - \Phi(x_{1,0}^0) &\geq (\nabla\Phi(x_{1,0}^0), x_1^- + x_2 + x_1^0 - x_{1,0}^0)_X \\ &= (\nabla\Phi(x_{1,0}^0), x_1^- + x_2)_X, \end{aligned}$$

and then, from $\|x\|_X \leq \|x\|$ for all $x \in Z$,

$$\begin{aligned} \Phi(x_1 + x_2) &\geq \Phi(x_{1,0}^0) - \|\nabla\Phi(x_{1,0}^0)\|_X \cdot \|x_1^- + x_2\|_X \\ &\geq \Phi(x_{1,0}^0) - \|\nabla\Phi(x_{1,0}^0)\|_X \cdot (\|x_1^-\| + \|x_2\|) \\ &= c_1 - c_2 \cdot (\|x_1^-\| + \|x_2\|) \end{aligned}$$

where $c_1 = \Phi(x_{1,0}^0), c_2 = \|\nabla\Phi(x_{1,0}^0)\|_X \geq 0$. Rewrite $x_{1,n} = x_{1,n}^- + x_{1,n}^0$, where $x_{1,n}^- \in Z_a^-(B_1)$ and $x_{1,n}^0 \in Z_a^0(B_1)$. By (3.1), we have

$$\begin{aligned} c_0 &\geq I(x_{1,n} + x_{2,n}) \\ &= -\frac{1}{2}\psi_{a,B_1}(x_{1,n}^-, x_{1,n}^-) - \frac{1}{2}\psi_{a,B_1}(x_{2,n}, x_{2,n}) + \Phi(x_{1,n} + x_{2,n}) \\ &\geq -\frac{1}{2}\psi_{a,B_1}(x_{1,n}^-, x_{1,n}^-) - \frac{1}{2}\psi_{a,B_1}(x_{2,n}, x_{2,n}) + c_1 - c_2 \cdot (\|x_{1,n}^-\| + \|x_{2,n}\|). \end{aligned}$$

From (Φ_4) and (5) of Proposition 2.6, we know that $(-\psi_{a,B_1}(x_1^-, x_1^-))^{\frac{1}{2}}$ is an equivalent norm on Z for $x_1^- \in Z_a^-(B_1)$ and $(\psi_{a,B_1}(x_2, x_2))^{\frac{1}{2}}$ is an equivalent norm on Z for $x_2 \in Z_a^+(B_1)$. This means that there exist $c_3 > 0$ and $c_4 > 0$ such that

$$
\begin{aligned}
c_0 &\geq I(x_{1,n} + x_{2,n}) \\
&\geq \frac{c_3^2}{2}\|x_{1,n}^-\|^2 - \frac{c_4^2}{2}\|x_{2,n}\|^2 + c_1 - c_2 \cdot (\|x_{1,n}^-\| + \|x_{2,n}\|) \\
&\geq \frac{c_3^2}{2}\|x_{1,n}^-\|^2 - \frac{c_4^2 M^2}{2} + c_1 - c_2 \cdot (\|x_{1,n}^-\| + M)
\end{aligned}
$$

via $\|x_{2,n}\| \leq M$, which shows that $\{\|x_{1,n}^-\|\}$ is bounded. Combining this with assumption (Φ_2) and the convexity of Φ, we see that there exist $c_5 > 0$ and $c_6 = \sup_n \Phi(-x_{1,n}^- - x_{2,n})$ such that

$$
\begin{aligned}
c_0 &\geq I(x_{1,n} + x_{2,n}) \\
&= -\frac{1}{2}\psi_{a,B_1}(x_{1,n}^-, x_{1,n}^-) - \frac{1}{2}\psi_{a,B_1}(x_{2,n}, x_{2,n}) + \Phi(x_{1,n} + x_{2,n}) \\
&\geq \frac{(c_3 c_5)^2}{2} - \frac{c_4^2 M^2}{2} + 2\Phi\left(\frac{1}{2}x_{1,n}^0\right) - \Phi(-x_{1,n}^- - x_{2,n}) \\
&\geq \frac{(c_3 c_5)^2}{2} - \frac{c_4^2 M^2}{2} + 2\Phi\left(\frac{1}{2}x_{1,n}^0\right) - c_6.
\end{aligned}
$$

By (Φ_3), we know that $\{\|x_{1,n}^0\|\}$ is also bounded. This contradicts the fact that $\|x_{1,n}^-\| + \|x_{1,n}^0\| \geq \|x_{1,n}\| \to +\infty$ as $n \to \infty$. Therefore (2.2) of Lemma 2.1 holds.

Step 3. We check that (2.1) of Lemma 2.1 holds. If not, there exist a constant c_7 and a sequence $\{x_{2,n}\}$ in X_2 such that $\|x_{2,n}\| \to +\infty$ as $n \to \infty$ and

$$
I(x_{2,n}) \geq c_7 \tag{3.2}
$$

for all n. For $x_2 \in X_2$, write $x_2 = x_2^0 + x_2^+$, where $x_2^0 \in Z_a^0(B_2)$ and $x_2^+ \in Z_a^+(B_2)$. Notice that $\nu_M^s(B_2) \neq 0$ and $X_2 = Z_a^0(B_2) \oplus Z_a^+(B_2)$. Let $x_{2,n} = x_{2,n}^0 + x_{2,n}^+, x_{2,n}^0 \in Z_a^0(B_2), x_{2,n}^+ \in Z_a^+(B_2)$. Then by (1.3) of (Φ_4), (3.2), Definition 2.3 and Proposition 2.4, we have

$$
\begin{aligned}
c_7 &\leq I(x_{2,n}) \\
&\leq -\frac{1}{2}a(x_{2,n}^0 + x_{2,n}^+, x_{2,n}^0 + x_{2,n}^+) + \frac{1}{2}(B_2(x_{2,n}^0 + x_{2,n}^+), x_{2,n}^0 + x_{2,n}^+)_X + c \\
&= -\frac{1}{2}\psi_{a,B_2}(x_{2,n}^+, x_{2,n}^+) + c
\end{aligned}
$$

which implies that $\{x_{2,n}^+\}$ is bounded since $(-\psi_{a,B_1}(x_1, x_1))^{\frac{1}{2}} + (\psi_{a,B_2}(x_2, x_2))^{\frac{1}{2}}$ is an equivalent norm on Z for $x = x_1 + x_2$ with $x_1 \in Z_a^-(B_1)$ and $x_2 \in Z_a^+(B_2)$, where $x_1 = \theta$. Since $\|x_{2,n}\| \leq \|x_{2,n}^0\| + \|x_{2,n}^+\|$, we have $\|x_{2,n}^0\| \to \infty$ as $n \to +\infty$. By $x_{2,n} \in X_2 = Z_a^0(B_2) \oplus Z_a^+(B_2)$, we have $\psi_{a,B_2}(x_{2,n}, x_{2,n}) \geq 0$ for all n via Proposition 2.4. From $\|x_{2,n}^0\| \to \infty$ as $n \to +\infty$ we have

$$
I(x_{2,n}) \leq \Phi(x_{2,n}) - \frac{1}{2}((B_2 - B_1)x_{2,n}, x_{2,n})_X \to -\infty
$$

via (1.4) of (Φ_4), which contradicts (3.2). Hence (2.1) of Lemma 2.1 holds.

By Lemma 2.1, I has at least one critical point. Hence problem (1.2) has at least one solution in Z via Lemma 2.7. The proof is complete. □

Proof of Theorem 1.2. By $\nu_A(B_1) \neq 0$, $B_1 \leq B_2$ and $B_1 < B_2$ w.r.t. $\ker(A - B_1)$ and $i_A(B_2) = i_A(B_1) + \nu_A(B_1)$, we have $Z = Z_a^-(B_1) \oplus Z_a^0(B_1) \oplus Z_a^0(B_2) \oplus Z_a^+(B_2)$ via (5) of Proposition 2.6. Note that $\nu_A(B_2) = 0$, we have $Z_a^0(B_2) = \{\theta\}$, which implies that $Z = Z_a^-(B_1) \oplus Z_a^0(B_1) \oplus Z_a^+(B_2)$ and $Z_a^+(B_2) = Z_a^+(B_1)$. Set $X_1 = Z_a^-(B_1) \oplus Z_a^0(B_1), X_2 = Z_a^+(B_2) = Z_a^+(B_1), x \in Z, x = x_1 + x_2$ with $x_1 \in X_1$ and $x_2 \in X_2$.

Let us follow the proof of Theorem 1.1 until (3.2). For $x_{2,n} \in Z_a^+(B_2) = Z_a^+(B_1)$, by (1.5) of (Φ_4'), (3.2), Definition 2.3 and Proposition 2.4, we have

$$\begin{aligned} c_7 &\leq I(x_{2,n}) \\ &\leq -\frac{1}{2}a(x_{2,n}, x_{2,n}) + \frac{1}{2}(B_2 x_{2,n}, x_{2,n})_X + c \\ &= -\frac{1}{2}\psi_{a,B_2}(x_{2,n}, x_{2,n}) + c. \end{aligned}$$

Since $(-\psi_{a,B_1}(x_1, x_1))^{\frac{1}{2}} + (\psi_{a,B_2}(x_2, x_2))^{\frac{1}{2}}$ is an equivalent norm on Z for $x = x_1 + x_2$ with $x_1 \in Z_a^-(B_1)$ and $x_2 \in Z_a^+(B_2)$, where $x_1 = \theta$, we have $\psi_{a,B_2}(x_{2,n}, x_{2,n}) \to +\infty$ via $\|x_{2,n}\| \to \infty$ as $n \to +\infty$. Thus, we have

$$I(x_{2,n}) \leq -\frac{1}{2}\psi_{a,B_2}(x_{2,n}, x_{2,n}) + c \to -\infty$$

as $n \to +\infty$, which contradicts (3.2). Hence (2.1) of Lemma 2.1 holds.

By Lemma 2.1, I has at least one critical point. Hence problem (1.2) has at least one solution in Z via Lemma 2.7. The proof is complete. □

Proof of Theorem 1.3. We apply Lemma 2.1. Consider the functional I_1 defined by

$$I_1(x) = -I(x) = \frac{1}{2}a(x, x) - \frac{1}{2}(B_1 x, x)_X - \Phi(x), \tag{3.3}$$

for every $x \in Z$. Under assumption (Φ_0), it is easy to verify that $I_1 \in C^1(Z, \mathbb{R})$ and I_1' is weakly continuous.

Note that $i_A(B_1) = 0$, we have $Z_a^-(B_1) = \{\theta\}$. By $\nu_A(B_1) \neq 0$, $B_1 \leq B_2$ and $B_1 < B_2$ w.r.t. $\ker(A - B_1)$ and $i_A(B_2) = i_A(B_1) + \nu_A(B_1)$, we have $Z = Z_a^0(B_1) \oplus Z_a^0(B_2) \oplus Z_a^+(B_2)$ via (5) of Proposition 2.6. Set $X_1 = Z_a^0(B_2) \oplus Z_a^+(B_2) = Z_a^+(B_1), X_2 = Z_a^0(B_1), x \in Z, x = x_1 + x_2$ with $x_1 \in X_1$ and $x_2 \in X_2$. From Definition 2.3 and Proposition 2.4, we have

$$I_1(x) = I_1(x_1 + x_2) = \frac{1}{2}a(x_1, x_1) - \frac{1}{2}(B_1 x_1, x_1)_X - \Phi(x_1 + x_2),$$

for every $x \in Z$. Thus, $I_1(x_1, \cdot)$ is weakly upper semi-continuous for all $x_1 \in X_1$ via $\Phi \in C^1(Z, \mathbb{R})$ is weakly continuous.

Next, we still divide the proof into three steps.

Step 1. We show that $I_1(\cdot, x_2) : X_1 \to \mathbb{R}$ is convex for all $x_2 \in X_2$. By (Φ_1'), it is obvious that $-\Phi(x_1 + x_2) + \frac{1}{2}((B_2 - B_1)(x_1 + x_2), x_1 + x_2)_X$ is convex in $x_1 \in X_1$. From Definition 2.3 and Proposition 2.4 we know that for every $x_1 \in X_1$,

$$\frac{1}{2}\psi_{a,B_2}(x_1, x_1) = \frac{1}{2}a(x_1, x_1) - \frac{1}{2}(B_2 x_1, x_1)_X \geq 0,$$

which shows that $\frac{1}{2}\psi_{a,B_2}(x_1, x_1)$ is convex in $x_1 \in X_1$ via Lemma 2.2. Hence, for every $x_2 \in X_2$,

$$\begin{aligned} I_1(x_1 + x_2) &= \frac{1}{2}a(x_1 + x_2, x_1 + x_2) - \frac{1}{2}(B_1(x_1 + x_2), x_1 + x_2)_X - \Phi(x_1 + x_2) \\ &= \frac{1}{2}\psi_{a,B_2}(x_1, x_1) - \Phi(x_1 + x_2) + \frac{1}{2}((B_2 - B_1)(x_1 + x_2), x_1 + x_2)_X + \frac{1}{2}\psi_{a,B_2}(x_2, x_2) \end{aligned}$$

is convex in $x_1 \in X_1$.

Step 2. By contradiction, we verify that (2.2) of Lemma 2.1 holds. If (2.2) of Lemma 2.1 does not hold, there exist $M > 0, c_8 > 0$ and two sequences $\{x_{1,n}\} \subset X_1$ and $\{x_{2,n}\} \subset X_2$ with $\|x_{1,n}\| \to +\infty$ as $n \to \infty$ and $\|x_{2,n}\| \le M$ for all n such that

$$I_1(x_{1,n} + x_{2,n}) \le c_8, \qquad \forall n \in \mathbf{N}. \tag{3.4}$$

For $x_1 \in X_1$, write $x_1 = x_1^0 + x_1^+$, where $x_1^0 \in Z_a^0(B_2)$ and $x_1^+ \in Z_a^+(B_2)$. Let us consider the functional

$$\varphi(x) = -\Phi(x) + \frac{1}{2}((B_2 - B_1)x, x)_X$$

for all $x \in X$. By (Φ_0) and (Φ_1'), we easily see that $\varphi \in C^1(Z, \mathbb{R})$ and φ is weakly lower semi-continuous on $Z_a^0(B_2)$. Using (1.6) of (Φ_4''), by the least action principle (see Theorem 1.1 in [9]), φ has a minimum at some $x_{1,0}^0 \in Z_a^0(B_2)$ for which

$$0 = \varphi'(x_{1,0}^0)x_1^0 = -(\nabla\Phi(x_{1,0}^0), x_1^0)_X + ((B_2 - B_1)x_{1,0}^0, x_1^0)_X, \qquad \forall x_1^0 \in Z_a^0(B_2).$$

By $\varphi \in C^1(Z, \mathbb{R})$ and the $(B_2 - B_1)$-concavity of Φ, we have

$$\begin{aligned}
&\varphi(x_1 + x_2) - \varphi(x_{1,0}^0) \\
&\quad \ge -(\nabla\Phi(x_{1,0}^0), x_1^+ + x_2 + x_1^0 - x_{1,0}^0)_X + ((B_2 - B_1)x_{1,0}^0, x_1^+ + x_2 + x_1^0 - x_{1,0}^0)_X \\
&\quad = -(\nabla\Phi(x_{1,0}^0), x_1^+ + x_2)_X + ((B_2 - B_1)x_{1,0}^0, x_1^+ + x_2)_X,
\end{aligned}$$

and then, from $\|x\|_X \le \|x\|$ for all $x \in Z$,

$$\begin{aligned}
\varphi(x_1 + x_2) &\ge \varphi(x_{1,0}^0) - (\|\nabla\Phi(x_{1,0}^0)\|_X + \|(B_2 - B_1)x_{1,0}^0\|_X) \cdot \|x_1^+ + x_2\|_X \\
&\ge \varphi(x_{1,0}^0) - (\|\nabla\Phi(x_{1,0}^0)\|_X + \|(B_2 - B_1)x_{1,0}^0\|_X) \cdot (\|x_1^+\| + \|x_2\|) \\
&= c_9 - c_{10} \cdot (\|x_1^+\| + \|x_2\|)
\end{aligned}$$

where $c_9 = \varphi(x_{1,0}^0), c_{10} = \|\nabla\Phi(x_{1,0}^0)\|_X + \|(B_2 - B_1)x_{1,0}^0\|_X \ge 0$. Rewrite $x_{1,n} = x_{1,n}^+ + x_{1,n}^0$, where $x_{1,n}^+ \in Z_a^+(B_2)$ and $x_{1,n}^0 \in Z_a^0(B_2)$. By (3.4), we have

$$\begin{aligned}
c_8 \ge I_1(x_{1,n} + x_{2,n}) &= \frac{1}{2}\psi_{a,B_2}(x_{1,n} + x_{2,n}, x_{1,n} + x_{2,n}) \\
&\quad + \frac{1}{2}((B_2 - B_1)(x_{1,n} + x_{2,n}), x_{1,n} + x_{2,n})_X - \Phi(x_{1,n} + x_{2,n}) \\
&= \frac{1}{2}\psi_{a,B_2}(x_{1,n}^+, x_{1,n}^+) + \frac{1}{2}\psi_{a,B_2}(x_{2,n}, x_{2,n}) + \varphi(x_{1,n} + x_{2,n}) \\
&\ge \frac{1}{2}\psi_{a,B_2}(x_{1,n}^+, x_{1,n}^+) + \frac{1}{2}\psi_{a,B_2}(x_{2,n}, x_{2,n}) + c_9 - c_{10} \cdot (\|x_{1,n}^+\| + \|x_{2,n}\|).
\end{aligned}$$

From (Φ_4'') and (5) of Proposition 2.6, we know that $(\psi_{a,B_2}(x,x))^{\frac{1}{2}}$ is an equivalent norm on Z for $x \in Z_a^+(B_2)$. Noticing that $-\psi_{a,B_2}(x,x) > 0$ for all $x \in Z_a^-(B_2)\backslash\{\theta\}$, so $(-\psi_{a,B_2}(x,x))^{\frac{1}{2}}$ is a norm on $Z_a^-(B_2)$, which is equivalent to $\|\cdot\|_Z = \|\cdot\|$ because of the finiteness of the subspace $Z_a^-(B_2)$. This means that there exist $c_{11} > 0$ and $c_{12} > 0$ such that

$$\begin{aligned}
c_8 &\ge I_1(x_{1,n} + x_{2,n}) \\
&\ge \frac{c_{11}^2}{2}\|x_{1,n}^+\|^2 - \frac{c_{12}^2}{2}\|x_{2,n}\|^2 + c_9 - c_{10} \cdot (\|x_{1,n}^+\| + \|x_{2,n}\|) \\
&\ge \frac{c_{11}^2}{2}\|x_{1,n}^+\|^2 - \frac{c_{12}^2 M^2}{2} + c_9 - c_{10} \cdot (\|x_{1,n}^+\| + M)
\end{aligned}$$

via $\|x_{2,n}\| \le M$, which shows that $\{\|x_{1,n}^+\|\}$ is bounded. Combining this with assumption (Φ_2) and the $(B_2 - B_1)$-concavity of Φ, we see that there exist $c_{13} > 0$ and $c_{14} = \sup_n \varphi(-x_{1,n}^+ - x_{2,n})$ such that

$$\begin{aligned} c_8 &\ge I_1(x_{1,n} + x_{2,n}) \\ &= \frac{1}{2}\psi_{a,B_2}(x_{1,n}^+, x_{1,n}^+) + \frac{1}{2}\psi_{a,B_2}(x_{2,n}, x_{2,n}) + \varphi(x_{1,n} + x_{2,n}) \\ &\ge \frac{(c_{11}c_{13})^2}{2} - \frac{c_{12}^2 M^2}{2} + 2\varphi\left(\frac{1}{2}x_{1,n}^0\right) - \varphi(-x_{1,n}^+ - x_{2,n}) \\ &\ge \frac{(c_{11}c_{13})^2}{2} - \frac{c_{12}^2 M^2}{2} + 2\varphi\left(\frac{1}{2}x_{1,n}^0\right) - c_{14}. \end{aligned}$$

By (1.6) of (Φ_4''), we know that $\{\|x_{1,n}^0\|\}$ is also bounded. This contradicts the fact that $\|x_{1,n}^+\| + \|x_{1,n}^0\| \ge \|x_{1,n}\| \to +\infty$ as $n \to \infty$. Therefore (2.2) of Lemma 2.1 holds.

Step 3. By $X_2 = Z_a^0(B_1)$, we have $I_1(x_2) = -\Phi(x_2)$ for all $x_2 \in X_2$. Thus, (2.1) of Lemma 2.1 holds via (Φ_3).

By Lemma 2.1, I_1 has at least one critical point. Hence problem (1.2) has at least one solution in Z via Lemma 2.7. The proof is complete. □

Proof of Theorem 1.4. we still consider the functional I_1 defined by (3.3). Under assumption (Φ_0), it is easy to verify that $I_1 \in C^1(Z, \mathbb{R})$ and I_1' is weakly continuous.

By $\nu_A(B_1) \ne 0$, $B_1 \le B_2$ and $B_1 < B_2$ w.r.t. $\ker(A - B_1)$ and $i_A(B_2) = i_A(B_1) + \nu_A(B_1)$, we have $Z = Z_a^-(B_1) \oplus Z_a^0(B_1) \oplus Z_a^0(B_2) \oplus Z_a^+(B_2)$ via (5) of Proposition 2.6. Note that $i_A(B_1) = 0$ and $\nu_A(B_2) = 0$, we have $Z_a^-(B_1) = Z_a^0(B_2) = \{\theta\}$, which implies that $Z = Z_a^0(B_1) \oplus Z_a^+(B_2)$, $Z_a^-(B_2) = Z_a^0(B_1)$ and $Z_a^+(B_2) = Z_a^+(B_1)$. Set $X_1 = Z_a^+(B_2) = Z_a^+(B_1), X_2 = Z_a^0(B_1), x \in Z, x = x_1 + x_2$ with $x_1 \in X_1$ and $x_2 \in X_2$.

From the proof of Theorem 1.3, it is not difficult to see that we only need to verify the validity of (2.2) in Lemma 2.1. If (2.2) of Lemma 2.1 does not hold, there exist $M > 0, c_{15} > 0$ and two sequences $\{x_{1,n}\} \subset X_1$ and $\{x_{2,n}\} \subset X_2$ with $\|x_{1,n}\| \to +\infty$ as $n \to \infty$ and $\|x_{2,n}\| \le M$ for all n such that

$$I_1(x_{1,n} + x_{2,n}) \le c_{15}, \qquad \forall n \in \mathbf{N}. \tag{3.5}$$

We consider the functional

$$\varphi(x) = -\Phi(x) + \frac{1}{2}((B_2 - B_1)x, x)_X$$

for all $x \in X$. By (Φ_0) and (Φ_1'), we easily see that $\varphi \in C^1(Z, \mathbb{R})$. From the $(B_2 - B_1)$-concavity of Φ, we have

$$\begin{aligned} \varphi(x_1 + x_2) - \varphi(\theta) &\ge -(\nabla\Phi(\theta), x_1 + x_2)_X + ((B_2 - B_1)\theta, x_1 + x_2)_X \\ &= -(\nabla\Phi(\theta), x_1 + x_2)_X, \end{aligned}$$

and then, from $\|x\|_X \le \|x\|$ for all $x \in Z$,

$$\begin{aligned} \varphi(x_1 + x_2) &\ge \varphi(\theta) - \|\nabla\Phi(\theta)\|_X \cdot \|x_1 + x_2\|_X \\ &\ge \varphi(\theta) - \|\nabla\Phi(\theta)\|_X(\|x_1\| + \|x_2\|). \end{aligned}$$

From (Φ_4''') and (5) of Proposition 2.6, we know that $(\psi_{a,B_2}(x,x))^{\frac{1}{2}}$ is an equivalent norm on Z for $x \in Z_a^+(B_2)$. Noticing that $-\psi_{a,B_2}(x,x) > 0$ for all $x \in Z_a^-(B_2)\backslash\{\theta\}$, so $(-\psi_{a,B_2}(x,x))^{\frac{1}{2}}$ is a norm on $Z_a^-(B_2)$, which is equivalent to $\|\cdot\|_Z = \|\cdot\|$ because of the finiteness of the subspace $Z_a^-(B_2)$. Combining (3.5), we know that there exist $c_{16} > 0$ and $c_{17} > 0$ such that

$$\begin{aligned} c_{15} &\geq I_1(x_{1,n} + x_{2,n}) \\ &= \frac{1}{2}\psi_{a,B_2}(x_{1,n}, x_{1,n}) + \frac{1}{2}\psi_{a,B_2}(x_{2,n}, x_{2,n}) + \varphi(x_{1,n} + x_{2,n}) \\ &\geq \frac{c_{16}^2}{2}\|x_{1,n}\|^2 - \frac{c_{17}^2 M^2}{2} + \varphi(\theta) - \|\nabla\Phi(\theta)\|_X(\|x_{1,n}\| + M), \end{aligned}$$

which shows that $\{\|x_{1,n}\|\}$ is bounded. This contradicts the fact that $\|x_{1,n}\| \to +\infty$ as $n \to \infty$. Therefore (2.2) of Lemma 2.1 holds. The proof is complete. $\square$

4 Applications to the second order Hamiltonian systems

In this section, we consider the applications of the main results to the second order Hamiltonian systems satisfying two boundary value conditions including generalized periodic boundary value conditions and Sturm-Liouville boundary value conditions. For more details about Hamiltonian systems, we refer to the excellent books [6,8,9,11] and the papers [1,2,10].

4.1 Second order Hamiltonian systems satisfying generalized periodic boundary value conditions

As a first example, we consider a generalized periodic boundary value problem

$$-\ddot{x} - \bar{B}_1(t)x = \nabla_x V(t,x) \quad \text{a.e. } t \in [0,1], \tag{4.1}$$

$$x(1) = Mx(0), \qquad \dot{x}(1) = N\dot{x}(0), \tag{4.2}$$

where $\bar{B}_1(t) \in L^\infty([0,1], \mathcal{L}_s(\mathbb{R}^n)) = \{B(t) = (b_{jk})_{n\times n} | b_{jk}(t) = b_{kj}(t),\ t \in [0,1],\ b_{jk}(t) \in L^\infty([0,1])\}$, $M, N \in GL(n) = \{A = (a_{jk})_{n\times n}\ | a_{jk} \in \mathbb{R} \text{ and } \det(A) \neq 0\}$, and $MN^T = I_n$, where I_n is the unit matrix of order n, and $\nabla_x V(t,x)$ denotes the gradient of $V(t,x)$ for $x \in \mathbb{R}^n$. We suppose that $V : [0,1] \times \mathbb{R}^n \to \mathbb{R}$ satisfies the following condition:

(H_0) $V(t,x)$ is measurable in t for every $x \in \mathbb{R}^n$ and continuously differentiable in x for a.e. $t \in [0,1]$.

Moreover, there exist $a(\cdot) \in C(\mathbb{R}^+, \mathbb{R}^+)$ and $b(t) \in L^1([0,1], \mathbb{R}^+)$ such that

$$|V(t,x)| \leq a(|x|)b(t) \quad \text{and} \quad |\nabla_x V(t,x)| \leq a(|x|)b(t)$$

for all $x \in \mathbb{R}^n$ and a.e. $t \in [0,1]$, where $\mathbb{R}^+ = [0, +\infty)$.

Let $X = L^2([0,1], \mathbb{R}^n)$. Define $A_1 : D(A_1) \to X$ by $(A_1 x)(t) = -\ddot{x}(t)$ where $D(A_1) = \{x \in H^2([0,1], \mathbb{R}^n) | x \text{ satisfies } (4.2)\}$. Set $(B_1 x)(t) = \bar{B}_1(t)x(t)$ with $D(B_1) = X$. From Corollary 1.21 in [3], we know that A_1 is self-adjoint in X and $\sigma(A_1) = \sigma_d(A_1) \subset [0, +\infty)$. Define $i_M(\bar{B}_1) = i_{A_1}(B_1), \nu_M(\bar{B}_1) = \nu_{A_1}(B_1)$, that is, $\nu_M(\bar{B}_1)$ is the dimension of the solution subspace of (4.1)–(4.2) with $V(t,x) \equiv 0$ and $i_M(\bar{B}_1) = \sum_{\lambda<0} \nu_M(\bar{B}_1 + \lambda I_n)$.

Assume that $\nu_M(\bar{B}_1) \neq 0$. Meanwhile, set $Z_1 = \{x \in H^1([0,1], \mathbb{R}^n) | x(1) = Mx(0)\}$. Then, from Corollary 1.21 in [3] again, we have $Z_1 = D(|A_1|^{\frac{1}{2}})$.

Remark 4.1 ([5, Remark 7.1.3], [4, Example 2.4.3]). Let $\alpha_1 \le \alpha_2 \le \cdots \le \alpha_n$ be the eigenvalues of a constant $n \times n$ symmetric matrix B. Then

$$i_{I_n}(B) = {}^{\#}\{k : \alpha_k > 0\} + 2\sum_{k=1}^{n} {}^{\#}\{j \in \mathbf{N} : 4(j\pi)^2 < \alpha_k\}, \tag{4.3}$$

$$\nu_{I_n}(B) = {}^{\#}\{k : \alpha_k = 0\} + 2\sum_{k=1}^{n} {}^{\#}\{j \in \mathbf{N} : 4(j\pi)^2 = \alpha_k\}, \tag{4.4}$$

$$i_{-I_n}(B) = 2\sum_{k=1}^{n} {}^{\#}\{j \in \mathbf{N} : ((2j-1)\pi)^2 < \alpha_k\},$$

$$\nu_{-I_n}(B) = 2\sum_{k=1}^{n} {}^{\#}\{j \in \mathbf{N} : ((2j-1)\pi)^2 = \alpha_k\},$$

where ${}^{\#}E$ denotes the number of elements in a set E. For $\eta \in \mathbb{R}\backslash\{\pm 1, 0\}$ with $\lambda_0 = \arccos \frac{2}{\eta^{-1}+\eta}$, we have

$$i_{\eta I_n}(B) = \sum_{k=1}^{n} {}^{\#}\{j \in \mathbf{N} : (2j\pi + \lambda_0)^2 < \alpha_k\} + \sum_{k=1}^{n} {}^{\#}\{j \in \mathbf{N} : (2\pi - \lambda_0 + 2j\pi)^2 < \alpha_k\},$$

$$\nu_{\eta I_n}(B) = \sum_{k=1}^{n} {}^{\#}\{j \in \mathbf{N} : (2j\pi + \lambda_0)^2 = \alpha_k\} + \sum_{k=1}^{n} {}^{\#}\{j \in \mathbf{N} : (2\pi - \lambda_0 + 2j\pi)^2 = \alpha_k\}.$$

In particular, formulae (4.3) and (4.4) were given first by Mawhin and Willem in [9].

In addition, set

$$\Phi(x) = \int_0^1 V(t,x)dt, \qquad \forall x \in Z_1.$$

Then, $\Phi \in C^1(Z_1, \mathbb{R})$ is weakly continuous with weakly continuous derivative and for every $x \in Z_1$,

$$\Phi'(x)y = \int_0^1 (\nabla_x V(t,x), y)dt, \qquad \forall y \in Z_1$$

because of (H_0). Hence, (Φ_0) holds. Moreover, for each $x \in Z_1$, we can write the norm

$$\|x\|^2 = \int_0^1 [|\dot{x}(t)|^2 + |x(t)|^2]dt.$$

Let $\|\cdot\|_\infty$ be the norm of $C([0,1], \mathbb{R}^n)$. Then, there is a constant $\delta_0 > 0$ such that

$$|x| \le \|x\|_\infty \le \delta_0 \|x\| \tag{4.5}$$

for any $x \in Z_1$. By (4.5) and (H_0), we can verify that (Φ_2) holds.

For any $\bar{B}_1(t), \bar{B}_2(t) \in L^\infty([0,1], \mathcal{L}_s(\mathbb{R}^n))$, we write $\bar{B}_1 \le \bar{B}_2$ if $\bar{B}_1(t) \le \bar{B}_2(t)$ for a.e. $t \in [0,1]$ and define $\bar{B}_1 < \bar{B}_2$ if $\bar{B}_1 \le \bar{B}_2$ and $\bar{B}_1(t) < \bar{B}_2(t)$ on a subset of $(0,1)$ with positive measure.

Now, the following four results hold.

Theorem 4.2. *Assume that $V(t,x)$ satisfies (H_0) and*

(H_1) *$V(t,x)$ is convex in x for a.e. $t \in [0,1]$;*

(H_2) *$\int_0^1 V(t,x)dt$ as $\|x\| \to \infty$ with $x \in \ker(A_1 - \bar{B}_1)$;*

(H_3) *there exist* $\gamma(t) \in L^1([0,1],\mathbb{R}^+)$ *and* $\bar{B}_2(t) \in L^\infty([0,1],\mathcal{L}_s(\mathbb{R}^n))$ *with* $\bar{B}_2 > \bar{B}_1$, $\nu_M(\bar{B}_2) \neq 0$ *and* $i_M(\bar{B}_2) = i_M(\bar{B}_1) + \nu_M(\bar{B}_1)$, *such that*

$$V(t,x) \le \frac{1}{2}((\bar{B}_2(t) - \bar{B}_1(t))x, x) + \gamma(t) \tag{4.6}$$

for all $x \in \mathbb{R}^n$ *and a.e.* $t \in [0,1]$, *and*

$$\text{meas}\left\{ t \in [0,1] \middle| V(t,x) - \frac{1}{2}((\bar{B}_2(t) - \bar{B}_1(t))x, x) \to -\infty \text{ as } \|\bar{x}\| \to \infty \right\} > 0, \tag{4.7}$$

where $x = \tilde{x} + \bar{x} \in Z_1$ *with* $\bar{x} \in \ker(A_1 - \bar{B}_2)$ *and* $\|\tilde{x}\|$ *is bounded.*

Then problem (4.1)–(4.2) *has a solution in* Z_1.

Proof. Clearly, (H_0) implies that (Φ_0) and (Φ_2) hold, (H_1) implies that (Φ_1) holds, and (H_2) implies that (Φ_3). We need only to show that (Φ_4) follows from (H_3). First, since $\bar{B}_2 > \bar{B}_1$, then exists $E_0 \subset [0,1]$ with meas$E_0 > 0$ such that $\bar{B}_2(t) > \bar{B}_1(t)$ for all $t \in E_0$ and $\bar{B}_2(t) \ge \bar{B}_1(t)$ for all $t \in [0,1]\backslash E_0$. Hence

$$\begin{aligned} ((\bar{B}_2 - \bar{B}_1)x, x)_X &= \int_0^1 ((\bar{B}_2(t) - \bar{B}_1(t))x(t), x(t))dt \\ &\ge \int_{E_0} ((\bar{B}_2(t) - \bar{B}_1(t))x(t), x(t))dt > 0 \end{aligned}$$

for all $x \in \ker(A_1 - \bar{B}_1)$, because $x(t) \in \ker(A_1 - \bar{B}_1)$ only has finite zeros. This implies that $\bar{B}_2 \ge \bar{B}_1$ and $\bar{B}_2 > \bar{B}_1$ w.r.t. $\ker(A_1 - \bar{B}_1)$. Next, by (4.6), we have

$$\begin{aligned} \Phi(x) = \int_0^1 V(t,x)dt &\le \frac{1}{2}\int_0^1 ((\bar{B}_2(t) - \bar{B}_1(t))x(t), x(t))dt + \int_0^1 \gamma(t)dt \\ &= \frac{1}{2}((\bar{B}_2 - \bar{B}_1)x, x)_X + c \end{aligned}$$

for all $x \in X$, where $c = \int_0^1 \gamma(t)dt$, which shows that (1.3) of (Φ_4) holds. Finally, set $E_1 = \{t \in [0,1] \mid V(t,x) - \frac{1}{2}((\bar{B}_2(t) - \bar{B}_1(t))x, x) \to -\infty \text{ as } \|\bar{x}\| \to \infty\}$, where $x = \tilde{x} + \bar{x} \in Z_1$ with $\bar{x} \in \ker(A_1 - \bar{B}_2)$ and $\|\tilde{x}\|$ is bounded. Thus, by (4.7) and meas$E_1 > 0$, we have

$$\begin{aligned} &\Phi(x) - \frac{1}{2}((\bar{B}_2 - \bar{B}_1)x, x)_X \\ &= \int_0^1 [V(t,x) - \frac{1}{2}((\bar{B}_2(t) - \bar{B}_1(t))x, x)]dt \\ &\le \int_{E_1} [V(t,x) - \frac{1}{2}((\bar{B}_2(t) - \bar{B}_1(t))x, x)]dt + \int_0^1 \gamma(t)dt \to -\infty \end{aligned}$$

as $\|\bar{x}\| \to \infty$ with $x = \tilde{x} + \bar{x}$, $\bar{x} \in \ker(A_1 - \bar{B}_2)$ and $\|\tilde{x}\|$ is bounded, which implies that (1.4) of (Φ_4) holds. Now, we can apply Theorem 1.1 to conclude that the system (4.1) – (4.2) has a solution in Z_1. □

Remark 4.3. In particular, set $\bar{B}_1(t) \equiv m^2(2\pi)^2, \bar{B}_2(t) = (m+1)^2(2\pi)^2, m \in \{0,1,2,\dots\}$ and $M = I_n$. Then, $Z_1 = \{x \in H^1([0,1],\mathbb{R}^n) | x(1) = x(0)\}$, $\sigma(A_1) = \{(2m\pi)^2 | m \in \mathbf{N}\}$ and $\ker(A_1 - \bar{B}_1) = \{a\cos(2mt\pi) + b\sin(2mt\pi) | a, b \in \mathbb{R}^n\}$. Hence, the following problem

$$-\ddot{x}(t) - m^2(2k\pi)^2 x(t) = \nabla_x V(t, x(t)), \qquad x(0) - x(1) = \dot{x}(0) - \dot{x}(1) = 0$$

has a solution via Theorems 4.2. In addition, for the interval $[0, T]$ considered in second order Hamiltonian systems satisfying periodic boundary value conditions, if $T = 1$, in Theorem 3.1 ($m = 0$) of [12] and Theorem 1.1 ($m \neq 0$) of [15], assume that $V(t, x)$ satisfies (H_0), (H_1), (H_2), and

$(\mathrm{H}_{3,1})$ there exists $\gamma(t) \in L^1([0,1], \mathbb{R}^+)$ such that

$$V(t,x) \leq \frac{2m+1}{2}(2\pi)^2|x|^2 + \gamma(t) \tag{4.8}$$

for all $x \in \mathbb{R}^n$ and a.e. $t \in [0,1]$, and

$$\operatorname{meas}\left\{t \in [0,1] \middle| V(t,x) - \frac{2m+1}{2}(2\pi)^2|x|^2 \to -\infty \text{ as } |x| \to \infty\right\} > 0, \tag{4.9}$$

then the conclusion of Theorem 4.2 is also true. In fact, set $\bar{B}_2(t) = (m+1)^2(2\pi)^2$, $x = \tilde{x} + \bar{x} \in Z_1$ with $\bar{x} \in \ker(A_1 - (m+1)^2(2\pi)^2)$. If $\|\bar{x}\| \to \infty$ and $\|\tilde{x}\|$ is bounded, we can obtain that $|x| \to \infty$ via the proof of Theorem 1.1 in [15]. From (4.9), we know that (4.7) holds. Noticing that $i_{I_n}((m+1)^2(2\pi)^2) = \nu_{I_n}(m^2(2\pi)^2) + i_{I_n}(m^2(2\pi)^2)$, we have (H_3) holds via $(\mathrm{H}_{3,1})$. So Theorem 4.2 generalizes in Theorem 3.1 ($m = 0$) of [12] and Theorem 1.1 ($m \neq 0$) of [15]. By the remarks in [12] and [15] we can see that Theorem 4.2 also generalizes the corresponding theorems in [9] as $T = 1$.

Theorem 4.4. *The conclusion of Theorem 4.2 still holds if we replace* (H_3) *with*

(H_3') *there exist* $\alpha(t) \in L^\infty([0,1], \mathbb{R}^+)$, $\gamma(t) \in L^1([0,1], \mathbb{R}^+)$ *and* $\bar{B}_3(t) \in L^\infty([0,1], \mathcal{L}_s(\mathbb{R}^n))$ *with* $\bar{B}_3 > \bar{B}_1$, $\nu_M(\bar{B}_3) \neq 0$ *and* $i_M(\bar{B}_3) = i_M(\bar{B}_1) + \nu_M(\bar{B}_1)$, *such that* $\alpha(t) I_n \leq \bar{B}_3(t) - \bar{B}_1(t)$ *for a.e.* $t \in [0,1]$ *with*

$$\operatorname{meas}\left\{t \in [0,1] \middle| 0 < \alpha(t) I_n < \bar{B}_3(t) - \bar{B}_1(t)\right\} > 0, \tag{4.10}$$

and

$$V(t,x) \leq \frac{1}{2}\alpha(t)|x|^2 + \gamma(t) \tag{4.11}$$

for a.e. $t \in [0,1]$ *and for all* $x \in \mathbb{R}^n$.

Proof. Similarly to the proof of Theorem 4.2, We need only to show that (Φ_4') follows from (H_3'). Set $\bar{B}_2(t) = \bar{B}_1(t) + \alpha(t) I_n$, we have $\bar{B}_2(t) \in L^\infty([0,1], \mathcal{L}_s(\mathbb{R}^n))$ via $\alpha(t) \in L^\infty([0,1], \mathbb{R}^+)$ and $\bar{B}_2(t) \geq \bar{B}_1(t)$. By (4.10), we have $\bar{B}_2 \geq \bar{B}_1$ and $\bar{B}_2 > \bar{B}_1$ w.r.t. $\ker(A_1 - \bar{B}_1)$ and $\bar{B}_3 \geq \bar{B}_2$ and $\bar{B}_3 > \bar{B}_2$ w.r.t. $\ker(A_1 - \bar{B}_2)$ via the similar proof in Theorem 4.2. From (2) of Proposition 2.6, we can find that

$$i_M(\bar{B}_1) + \nu_M(\bar{B}_1) = i_M(\bar{B}_3) \geq i_M(\bar{B}_2) + \nu_M(\bar{B}_2) \geq i_M(\bar{B}_2) \geq i_M(\bar{B}_1) + \nu_M(\bar{B}_1),$$

which implies that $i_M(\bar{B}_2) = i_M(\bar{B}_1) + \nu_M(\bar{B}_1)$ and $\nu_M(\bar{B}_2) = 0$. Again by (4.11), we have

$$\begin{aligned}\Phi(x) = \int_0^1 V(t,x)dt &\leq \frac{1}{2}\int_0^1 ((\bar{B}_2(t) - \bar{B}_1(t))x(t), x(t))dt + \int_0^1 \gamma(t)dt \\ &= \frac{1}{2}((\bar{B}_2 - \bar{B}_1)x, x)_X + c\end{aligned}$$

for all $x \in X$, where $c = \int_0^1 \gamma(t)dt$. This shows that (Φ_4') holds. Next, we can apply Theorem 1.2 to conclude that the system (4.1)–(4.2) has a solution in Z_1. □

Remark 4.5. In particular, set $\bar{B}_1(t) \equiv m^2(2\pi)^2, \bar{B}_2(t) = (m+1)^2(2\pi)^2, m \in \{0,1,2,\dots\}$ and $M = I_n$. Then, the following problem

$$-\ddot{x}(t) - m^2(2k\pi)^2 x(t) = \nabla_x V(t,x(t)), \qquad x(0)-x(1) = \dot{x}(0)-\dot{x}(1) = 0$$

has a solution via Theorems 4.4. In addition, as $T = 1$, in Theorem 3.3 ($m = 0$) of [12] and Theorem 1.10 ($m \neq 0$) of [15], assume that $V(t,x)$ satisfies (H_0), (H_1), (H_2), and

$(\mathrm{H}'_{3,1})$ there exist $\gamma(t), \alpha(t) \in L^1([0,1],\mathbb{R}^+)$ with $\int_0^1 \alpha(t)dt < \frac{12(2m+1)}{(m+1)^2}$, such that (4.11) holds.

Then the conclusion of Theorem 4.4 is also true.

Obviously, $\alpha(t) \in L^\infty([0,1],\mathbb{R}^+) \subset L^1([0,1],\mathbb{R}^+)$. But, for $\alpha(t) \in L^\infty([0,1],\mathbb{R}^+)$, we have $\int_0^1 \alpha(t)dt < \frac{12(2m+1)}{(m+1)^2} \not\Rightarrow 0 < \alpha(t) < (2m+1)(2\pi)^2$ and $0 < \alpha(t) < (2m+1)(2\pi)^2 \not\Rightarrow \int_0^1 \alpha(t)dt < \frac{12(2m+1)}{(m+1)^2}$. Indeed, if

$$\alpha(t) = \begin{cases} (2m+1)(2\pi)^2, & x \in [0, \frac{1}{(2m+1)(2\pi)^2}], \\ 0, & x \in (\frac{1}{(2m+1)(2\pi)^2}, 1], \end{cases}$$

then $\int_0^1 \alpha(t)dt = 1 \leq \frac{12(2m+1)}{(m+1)^2}$ as $m \leq 22$ and $\alpha(t) \geq (2m+1)(2\pi)^2$ for $x \in [0, \frac{1}{(2m+1)(2\pi)^2}]$; if $\frac{12(2m+1)}{(m+1)^2} < \alpha(t) < (2m+1)(2\pi)^2$, then $\int_0^1 \alpha(t)dt > \frac{12(2m+1)}{(m+1)^2}$. So Theorem 4.4 is a new result and, in some sence, it represent a development of Theorem 3.3 ($m = 0$) of [12] and Theorem 1.10 ($m \neq 0$) of [15].

Theorem 4.6. *The conclusion of Theorem 4.2 still holds if we replace* (H_1) *and* (H_3) *with*

(H'_1) *$V(t,\cdot)$ is $(\bar{B}_2(t) - \bar{B}_1(t))$-concave, that is, $-V(t,x) + \frac{1}{2}((\bar{B}_2(t) - \bar{B}_1(t))x, x)$ is convex in x for a.e. $t \in [0,1]$.*

(H''_3) *there exists $\bar{B}_2(t) \in L^\infty([0,1], \mathcal{L}_s(\mathbb{R}^n))$ with $\bar{B}_2 > \bar{B}_1$, $i_M(\bar{B}_1) = 0$, $\nu_M(\bar{B}_2) \neq 0$ and $i_M(\bar{B}_2) = i_M(\bar{B}_1) + \nu_M(\bar{B}_1)$, such that*

$$\int_0^1 \left(-V(t,x) + \frac{1}{2}((\bar{B}_2(t) - \bar{B}_1(t))x, x) \right) dt \to +\infty \tag{4.12}$$

as $\|x\| \to \infty$ with $x \in \ker(A_1 - \bar{B}_2)$,

respectively.

The proof Theorem 4.6 is similar to that of Theorem 4.2. Here we omit it.

Remark 4.7. In particular, set $\bar{B}_1(t) \equiv 0, \bar{B}_2(t) = (2\pi)^2$ and $M = I_n$. Then, the following problem

$$-\ddot{x}(t) = \nabla_x V(t,x(t)), \qquad x(0)-x(1) = \dot{x}(0)-\dot{x}(1) = 0$$

has a solution via Theorems 4.6. In addition, as $T = 1$, then Theorem 4.6 reduces to Theorem 5.2 in [12].

Theorem 4.8. *The conclusion of Theorem 4.2 still holds if we replace* (H_1) *and* (H_3) *with*

(H''_1) *$V(t,\cdot)$ is $\beta(t)$-concave, that is, $-V(t,x) + \frac{1}{2}\beta(t)|x|^2$ is convex in x for a.e. $t \in [0,1]$.*

(H_3''') *there exist* $\beta(t) \in L^\infty([0,1],\mathbb{R}^+)$ *and* $\bar{B}_3(t) \in L^\infty([0,1],\mathcal{L}_s(\mathbb{R}^n))$ *with* $\bar{B}_3 > \bar{B}_1$, $i_M(\bar{B}_1) = 0$, $\nu_M(\bar{B}_3) \neq 0$ *and* $i_M(\bar{B}_3) = i_M(\bar{B}_1) + \nu_M(\bar{B}_1)$, *such that* $\beta(t) \leq \bar{B}_3(t) - \bar{B}_1(t)$ *for a.e.* $t \in [0,1]$ *with*

$$\text{meas}\left\{t \in [0,1] \;\middle|\; 0 < \beta(t) < \bar{B}_3(t) - \bar{B}_1(t)\right\} > 0, \tag{4.13}$$

respectively.

The proof Theorem 4.8 is similar to that of Theorem 4.4. Here we omit it.

Remark 4.9. In particular, set $\bar{B}_1(t) \equiv 0$, $\bar{B}_2(t) = (2\pi)^2$ and $M = I_n$. Then, the following problem

$$-\ddot{x}(t) = \nabla_x V(t, x(t)), \qquad x(0) - x(1) = \dot{x}(0) - \dot{x}(1) = 0$$

has a solution via Theorems 4.8. Moreover, as $T = 1$, then Theorem 4.8 reduces to Theorem 5.1 of [12] as $k(t) \in L^\infty([0,1],\mathbb{R}^+)$.

In addition, as $T = 1$, in Theorem 1.4 of [12], assume that $V(t,x)$ satisfies (H_0), (H_2), and

($H''_{1,1}$) there exist $\beta(t) \in L^1([0,1],\mathbb{R}^+)$ with $\int_0^1 \beta(t)dt < 12$, such that $V(t,\cdot)$ is $\beta(t)$-concave.

Then the conclusion of Theorem 4.8 is also true.

Obviously, $\beta(t) \in L^\infty([0,1],\mathbb{R}^+) \subset L^1([0,1],\mathbb{R}^+)$. But, for $\beta(t) \in L^\infty([0,1],\mathbb{R}^+)$, we have $\int_0^1 \beta(t)dt < 12 \not\Rightarrow 0 < \beta(t) < (2\pi)^2$ and $0 < \beta(t) < (2\pi)^2 \not\Rightarrow \int_0^1 \beta(t)dt < 12$. Indeed, if

$$\beta(t) = \begin{cases} (2\pi)^2, & x \in [0, \frac{1}{(2\pi)^2}], \\ 0, & x \in (\frac{1}{(2\pi)^2}, 1], \end{cases}$$

then $\int_0^1 \beta(t)dt = 1$ and $\beta(t) \geq (2\pi)^2$ for $x \in [0, \frac{1}{(2\pi)^2}]$; if $12 < \beta(t) < (2\pi)^2$, then $\int_0^1 \beta(t)dt > 12$. So Theorem 4.8 is a new result and, in some sence, it represent a development of Theorem 1.4 of [12]. By the remarks in [12] we can see that Theorem 4.8 also generalizes the corresponding theorems in [9,14,16,17] as $T = 1$.

4.2 Second order Hamiltonian systems satisfying Sturm–Liouville boundary value conditions

As a second example, we consider Sturm–Liouville boundary value problem

$$-\ddot{x} - \tilde{B}_1(t)x = \nabla_x V(t,x), \tag{4.14}$$

$$x(0)\cos\alpha - \dot{x}(0)\sin\alpha = 0, \tag{4.15}$$

$$x(1)\cos\beta - \dot{x}(1)\sin\beta = 0, \tag{4.16}$$

where $\tilde{B}_1 \in L^\infty([0,1],\mathcal{L}_s(\mathbb{R}^n))$, $\nabla_x V(t,x)$ denotes the gradient of $V(t,x)$ for $x \in \mathbb{R}^n$ and $0 \leq \alpha < \pi, 0 < \beta \leq \pi$. We suppose that $V : [0,1] \times \mathbb{R}^n \to \mathbb{R}$ satisfies (H_0).

Let $X = L^2([0,1],\mathbb{R}^n)$. Define $A_2 : D(A_2) \to X$ by $(A_2x)(t) = -\ddot{x}(t)$ with $D(A_2) = \{x \in H^2([0,1],\mathbb{R}^n) | x$ satisfies (4.15) and (4.16)$\}$. Set $(B_1x)(t) = \tilde{B}_1(t)x(t)$ with $D(B_1) = X$. From Proposition 1.17 in [3], we know that A_2 is self-adjoint in X and $\sigma(A_2) = \sigma_d(A_2)$ is bounded from below. Define $i_{\alpha,\beta}(\tilde{B}_1) = i_{A_1}(B_1), \nu_{\alpha,\beta}(\tilde{B}_1) = \nu_{A_1}(B_1)$, that is, $\nu_{\alpha,\beta}(\tilde{B}_1)$ is the dimension of the solution subspace of (4.14)–(4.16) with $V(t,x) \equiv 0$.

Assume that $\nu_{\alpha,\beta}(\tilde{B}_1) \neq 0$. Meanwhile, set

$$Z_2 = \begin{cases} \{x \in H^1([0,1],\mathbb{R}^n) | x(1) = 0\}, & \alpha = 0, \beta \in (0,\pi); \\ \{x \in H^1([0,1],\mathbb{R}^n) | x(0) = 0\}, & \alpha \in (0,\pi), \beta = \pi; \\ \{x \in H^1([0,1],\mathbb{R}^n) | x(1) = x(0) = 0\}, & \alpha = 0, \beta = \pi; \\ H^1([0,1],\mathbb{R}^n), & \alpha,\beta \in (0,\pi). \end{cases}$$

Then, from Proposition 1.17 in [3] again, we have $Z_2 = D(|A_1|^{\frac{1}{2}})$. Moreover, set

$$\Phi(x) = \int_0^1 V(t,x)dt, \qquad \forall x \in Z_2.$$

Then, $\Phi \in C^1(Z_2, \mathbb{R})$ is weakly continuous with weakly continuous derivative and for every $x \in Z_2$,

$$\Phi'(x)y = \int_0^1 (\nabla_x V(t,x), y)dt, \qquad \forall y \in Z_2$$

because of (H_0). Hence, (Φ_0) holds. Further, for each $x \in Z_2$, we can write the norm

$$\|x\|^2 = \int_0^1 [|\dot{x}(t)|^2 + |x(t)|^2]dt.$$

By (4.5) and (H_0), we can verify that (Φ_2) holds. Then, the following four results hold. Since their proofs are similar to Theorems 4.2–4.8, and we omit them here.

Theorem 4.10. *Assume that $V(t,x)$ satisfies (H_0), (H_1), (H_2) and (H_3) with $\bar{B}_1, \bar{B}_2$ and A_1 replaced with $\tilde{B}_1, \tilde{B}_2$ and A_2 respectively, then problem (4.14)–(4.16) has a solution in Z_2.*

Theorem 4.11. *The conclusion of Theorem 4.10 still holds if we replace (H_3) and $\bar{B}_3$ with (H_3') and $\tilde{B}_3$, respectively.*

Theorem 4.12. *The conclusion of Theorem 4.10 still holds if we replace (H_1) and (H_3) with (H_1') and (H_3''), respectively.*

Theorem 4.13. *The conclusion of Theorem 4.10 still holds if we replace $\bar{B}_3$, (H_1) and (H_3) with $\tilde{B}_3$, (H_1'') and (H_3'''), respectively.*

Remark 4.14. In particular, set $\tilde{B}_1(t) \equiv \pi^2 I_n$ and $\alpha = 0, \beta = \pi$. Then, $Z_2 = H_0^1$, $\sigma(A_2) = \{k^2\pi^2 | k \in \mathbf{N}\backslash\{0\}\}$ and $\ker(A_2 - \tilde{B}_1) = \{a \sin t\pi \mid a \in \mathbb{R}^n\}$. Hence, the following problem

$$-\ddot{x}(t) = \nabla_x V(t,x(t)), \qquad x(0) = x(1) = 0$$

has a solution via Theorems 4.10–4.13 respectively, where $\tilde{B}_2(t) \equiv 4\pi^2 I_n = \tilde{B}_3(t)$.

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