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Existence and uniqueness of positive solutions for Kirchhoff type beam equations

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Abstract. This paper is concerned with the existence and uniqueness of positive solutions for the fourth order Kirchhoff type problem

$$\begin{cases} u''''(x) - \left(a + b\int_0^1 (u'(x))^2 dx\right)u''(x) = \lambda f(u(x)), & x \in (0,1), \\ u(0) = u(1) = u''(0) = u''(1) = 0, \end{cases}$$

where $a > 0, b \geq 0$ are constants, $\lambda \in \mathbb{R}$ is a parameter. For the case $f(u) \equiv u$, we use an argument based on the linear eigenvalue problems of fourth order equations and their properties to show that there exists a unique positive solution for all $\lambda > \lambda_{1,a}$, here $\lambda_{1,a}$ is the first eigenvalue of the above problem with $b = 0$; for the case f is sublinear, we prove that there exists a unique positive solution for all $\lambda > 0$ and no positive solution for $\lambda < 0$ by using bifurcation method.

Keywords: fourth order boundary value problem, Kirchhoff type beam equation, global bifurcation, positive solution, uniqueness.

2020 Mathematics Subject Classification: 34B10, 34B18.

1 Introduction

Consider the following nonlinear fourth order Kirchhoff type problem

$$\begin{cases} u''''(x) - \left(a + b\int_0^1 (u'(x))^2 dx\right)u''(x) = \lambda f(u(x)), & x \in (0,1), \\ u(0) = u(1) = u''(0) = u''(1) = 0, \end{cases} \tag{1.1}$$

where $a > 0, b \geq 0$ are constants, $\lambda \in \mathbb{R}$ is a parameter, $f : \mathbb{R} \to \mathbb{R}$ is continuous. Due to the presence of the integral term $(b\int_0^1 (u'(x))^2 dx)u''(x)$, the equation is not a pointwise identity and therefore is a nonlocal integro-differential problem.

Problem (1.1) describes the bending equilibrium of an extensible beam of length 1 which is simply supported at two endpoints $x = 0$ and $x = 1$. The right side term $\lambda f(u)$ in equation

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stands for a force exerted on the beam by the foundation. In fact, (1.1) is related to the stationary problem associated with

$$\frac{\partial^2 u}{\partial t^2}+\frac{EI}{\rho A}\frac{\partial^4 u}{\partial x^4}-\left(\frac{H}{\rho}+\frac{E}{2\rho L}\int_0^L\left|\frac{\partial u}{\partial x}\right|^2dx\right)\frac{\partial^2 u}{\partial x^2}=0, \tag{1.2}$$

which was proposed by Woinowsky-Krieger [29] as a model for the deflection of an extensible beam of length L with hinged ends. In (1.2), $u=u(x,t)$ is the lateral displacement at the space coordinate x and the time t; the letters H, E, ρ, I and A denote, respectively, the tension in the rest position, the Young elasticity modulus, the density, the cross-sectional moment of inertia and the cross-sectional area. The nonlinear term in the brackets is a correction to the classical Euler-Bernoulli equation

$$\frac{\partial^2 u}{\partial t^2}+\frac{EI}{\rho A}\frac{\partial^4 u}{\partial x^4}=0,$$

which does not consider the changes of the tension induced by the variation of the length during the deflection. This kind of correction was proposed by Kirchhoff [9] to generalize D'Alembert's equation with clamped ends. For this reason (1.1) is often called a Kirchhoff type beam equation. Other problems involving fourth-order equations of Kirchhoff type can be found in [7,19].

In the study of problem (1.1) and its generalizations, the nonlocal term under the integral sign causes some mathematical difficulties which make the study of the problem particularly interesting. The existence and multiplicity of solutions for (1.1) and its multi-dimensional case have been studied by several authors, see [13,15–18,27,28,30] and the references there in. Meanwhile, numerical methods of (1.1) have been developed in [3,4,20,21,23,25,26,32].

In [15–17], by using variational methods, Ma considered existence and multiplicity of solutions for (1.1) with $\lambda\equiv 1$ under different nonlinear boundary conditions. In [18], based on the fixed point theorems in cones of ordered Banach spaces, Ma studied existence and multiplicity of positive solutions results for (1.1) with right side term $f(x,u,u')$ in equation.

For multi-dimensional case of (1.1) with $\lambda\equiv 1$, Wang et al. studied the existence and multiplicity of nontrivial solutions by using the mountain pass theorem and the truncation method in [27,28]; for a kind of problem similar to (1.1) in $\mathbb{R}^3$, Xu and Chen [30] established the existence and multiplicity of negative energy solutions based on the genus properties in critical point theory, and very rencently, Mao and Wang [13] studied the existence of nontrivial mountain-pass type of solutions via the Mountain Pass lemma.

It is worth noticing that, in the above mentioned research work, the uniqueness of solutions for the problem has not been discussed. As far as the author knows, there are very few results on the uniqueness of solutions for problem (1.1). In [3], when the right side term $\lambda f(u(x))=g(x)$ is nonpositive, Dang and Luan proved that problem (1.1) has a unique solution by reducing the problem to a nonlinear equation and proposed an iterative method for finding the solution. Very recently, by using contraction mapping principle, Dang and Nguyen [4] obtained a uniqueness result for (1.1) in multi-dimensional case with the right side term $\lambda f(u(x))=g(x,u)$ is bounded. To the best of our knowledge, apart from the two works mentioned above, there is no other result on the uniqueness of solutions for nonlocal integro-differential problem (1.1).

Motivated by the above described works, the object of this paper is to study the existence and uniqueness of positive solutions for (1.1), and our main tool is bifurcation method. It should be emphasized that, global bifurcation phenomena for fourth order problem (1.1) with

$b = 0$ have been investigated in [10, 14, 24], and [1, 5, 8, 11, 12] studied second order Kirchhoff type problem by using the bifurcation theory, but as far as we know, the bifurcation phenomena for fourth order Kirchhoff problem (1.1) has not been discussed.

Concretely, in the present paper we are concerned with problem (1.1) under the two cases: $f(u) \equiv u$ or f is sublinear. For $f(u) \equiv u$, (1.1) can be seen as a nonlinear eigenvalue problem, we use an argument based on the linear eigenvalue problems of fourth order equations and their properties to show that there exists a unique positive solution for all $\lambda > \lambda_{1,a}$, where $\lambda_{1,a}$ is the first eigenvalue of (1.1) with $b = 0$; for the case f is sublinear, such as $f(u) = c_1 u^p + c_2 u^q$ ($c_1, c_2 \geq 0$, $0 < p, q < 1$, see Remark 4.1), we prove that there exists a unique positive solution for all $\lambda > 0$ and no positive solution for $\lambda < 0$ by using bifurcation method.

The rest of the paper is arranged as follows: In Section 2, as preliminaries, we first construct the operator equation corresponding to (1.1). In Section 3, we deal with the case $f(u) \equiv u$ based on the linear eigenvalue problem of fourth order equations and their properties. Finally, for the case f is sublinear, we discuss the existence and uniqueness of positive solutions for (1.1) by using bifurcation method in Section 4.

2 Preliminaries

Let $P := \{u \in C[0, 1] : u(x) \geq 0, \forall\ x \in [0, 1]\}$ be the positive cone in $C[0, 1]$ and let $U := P \cup (-P)$. A solution to problem (1.1) is a function $u \in C^4[0, 1]$ which satisfies the equation and boundary conditions, and moreover, if $u \in C^4[0, 1] \cap P$ we call u a positive solution.

Proposition 2.1. *For each $g \in C[0, 1]$, there exists a solution $u \in C^4[0, 1]$ to the problem*

$$\begin{cases} u''''(x) - (a + b\int_0^1 (u'(x))^2 dx)u''(x) = g(x), & x \in (0, 1), \\ u(0) = u(1) = u''(0) = u''(1) = 0, \end{cases} \tag{2.1}$$

and if $g \in U$, then u is unique. Moreover, the operator $T : U \to U$ defined by

$$T(g) := u$$

is positive and compact.

Proof. First, when $g \equiv 0$, we prove that (2.1) has only a unique solution $u \equiv 0$. Assume that u is a solution of (2.1) with $g \equiv 0$, set $w = -u''$, then by (2.1) we have

$$\begin{cases} -w''(x) + (a + b\int_0^1 (u'(x))^2 dx)w(x) = 0, & x \in (0, 1), \\ w(0) = w(1) = 0, \end{cases} \tag{2.2}$$

$$\begin{cases} -u''(x) = w(x), & x \in (0, 1), \\ u(0) = u(1) = 0. \end{cases} \tag{2.3}$$

We claim that the solution of (2.2) is $w \equiv 0$. In fact, suppose on the contrary that $w \not\equiv 0$ is a solution of (2.2), and without loss of generality, $w(\tau) = \max\{w(x) | x \in [0, 1]\} > 0$ for some $\tau \in (0, 1)$, then we have $w''(\tau) \leq 0$, which contradicts with $w''(\tau) = (a + b\int_0^1 (u'(x))^2 dx)w(\tau) > 0$. Substituting $w \equiv 0$ in (2.3), $u \equiv 0$ is an immediate conclusion.

Next, we prove the existence and uniqueness of solutions for (2.1) with $g \neq 0$. For any constant $R \geq 0$, let u_R stands for the unique solution of the linear fourth order problem

$$\begin{cases} u''''(x) - (a + bR)u''(x) = g(x), & x \in (0, 1), \\ u(0) = u(1) = u''(0) = u''(1) = 0, \end{cases} \tag{2.4}$$

then

$$u_R(x) = \int_0^1 \int_0^1 G_1(x,t) G_{2,R}(t,s) g(s) ds dt, \qquad x \in [0,1], \tag{2.5}$$

$$u_R''(x) = -\int_0^1 G_{2,R}(x,t) g(t) dt, \qquad x \in [0,1], \tag{2.6}$$

here

$$G_1(x,t) = \begin{cases} t(1-x), & 0 \le t \le x \le 1, \\ x(1-t), & 0 \le x \le t \le 1, \end{cases} \tag{2.7}$$

and

$$G_{2,R}(t,s) = \begin{cases} \frac{\sinh(\sqrt{a+bR}t)\sinh(\sqrt{a+bR}(1-s))}{\sqrt{a+bR}\sinh\sqrt{a+bR}}, & 0 \le t \le s \le 1, \\ \frac{\sinh(\sqrt{a+bR}s)\sinh(\sqrt{a+bR}(1-t))}{\sqrt{a+bR}\sinh\sqrt{a+bR}}, & 0 \le s \le t \le 1, \end{cases} \tag{2.8}$$

are Green functions of

$$\begin{cases} -u''(x) = 0, & x \in (0,1), \\ u(0) = u(1) = 0, \end{cases} \tag{2.9}$$

and

$$\begin{cases} -w''(t) + (a+bR)w(t) = 0, & t \in (0,1), \\ w(0) = w(1) = 0, \end{cases} \tag{2.10}$$

respectively. Since $0 \le G_1(x,t) \le G_1(x,x)$ and $0 \le G_{2,R}(t,s) \le G_{2,R}(t,t) \le \frac{(\sinh\frac{\sqrt{a}}{2})^2}{\sqrt{a}\sinh\sqrt{a}}$, then by (2.5)–(2.8) we have that there exist two positive constants C_1 and C_1 such that

$$\|u_R\|_\infty \le C_1 \|g\|_\infty, \qquad \|u_R''\|_\infty \le C_2 \|g\|_\infty. \tag{2.11}$$

Multiplying the equation in (2.4) by u_R and integrating it over $[0,1]$, based on boundary conditions and integration by parts we obtain

$$\int_0^1 (u_R'(x))^2 dx = \frac{\int_0^1 g(x) u_R(x) dx - \int_0^1 (u_R''(x))^2 dx}{a+bR}. \tag{2.12}$$

Now to get a solution of (2.1), we only need to find R such that

$$R = y(R) := \frac{\int_0^1 g(x) u_R(x) dx - \int_0^1 (u_R''(x))^2 dx}{a+bR} = \int_0^1 (u_R'(x))^2 dx, \tag{2.13}$$

that is, find a fixed point of $R = y(R)$. Obviously, $y(0) > 0$. On the other hand, by (2.11) we have

$$|y(R)| = \frac{\left|\int_0^1 g(x) u_R(x) dx - \int_0^1 (u_R''(x))^2 dx\right|}{a+bR} \le \frac{C_1\|g\|_\infty^2 + C_2^2\|g\|_\infty^2}{a} \le C. \tag{2.14}$$

This concludes the existence of fixed point for $R = y(R)$ which yields a solution u of (2.1) in $C^4[0,1]$.

Now, we show that if $g \in U$, the solution of (2.1) is unique. Without loss of generality, we assume on the contrary that for some $g \in P$, there exist two solutions $u \ne v$. By (2.5) and (2.6), we have

$$u \ge 0, \quad u'' \le 0; \qquad v \ge 0, \quad v'' \le 0. \tag{2.15}$$

Since u and v satisfy the equation in (2.1), we have

$$\begin{aligned} u''''(x) - v''''(x) - \left[a + b\int_0^1 (u'(x))^2 dx\right] (u''(x) - v''(x)) \\ - b\left[\int_0^1 (u'(x))^2 dx - \int_0^1 (v'(x))^2 dx\right] v''(x) = 0. \end{aligned} \tag{2.16}$$

Set $w = -(u'' - v'')$. If $\int_0^1 (u'(x))^2 dx = \int_0^1 (v'(x))^2 dx$, then (2.2) holds for $w = -(u'' - v'')$ and consequently we can obtain $u \equiv v$ arguing as above. If we assume that $\int_0^1 (u'(x))^2 dx > \int_0^1 (v'(x))^2 dx$, then by (2.16) and (2.15) we have

$$u''''(x) - v''''(x) - \left[a + b\int_0^1 (u'(x))^2 dx\right] (u''(x) - v''(x)) \le 0, \tag{2.17}$$

that is

$$-w''(x) + \left[a + b\int_0^1 (u'(x))^2 dx\right] w(x) \le 0. \tag{2.18}$$

We claim that (2.18) implies $w \le 0$. In fact, suppose on the contrary that $w(\tau) = \max\{w(x) | x \in [0,1]\} > 0$ for some $\tau \in (0,1)$, then $w''(\tau) \le 0$. This contradicts with (2.18) with $x = \tau$. On the other hand, based on boundary conditions and integration by parts, from the assumption

$$\begin{aligned} \int_0^1 (u'(x))^2 dx - \int_0^1 (v'(x))^2 dx &= \int_0^1 [u'(x) + v'(x)][u'(x) - v'(x)] dx \\ &= -\int_0^1 (u(x) + v(x))(u''(x) - v''(x)) dx \\ &= \int_0^1 (u(x) + v(x)) w(x) dx > 0. \end{aligned} \tag{2.19}$$

Since (2.15) guarantees that $u(x) + v(x) \ge 0$, then (2.19) contradicts with $w \le 0$. The uniqueness of solutions for (2.1) is proved.

At the end, let $T : U \to C[0,1]$ be the operator defined by $Tg = u$, where u is the solution of (2.1). By (2.5) and the positiveness of Green functions $G_1(x,t)$, $G_{2,R}(t,s)$ in (2.7) and (2.8), we conclude that T is a positive operator, that is $T : U \to U$. Now, we show that T is compact. Without loss of generality, let $B \subseteq P$ be any bounded set. Combining (2.5) with (2.11) we can see that TB is a bounded set in P; On the other hand, (2.6) with (2.11) imply that TB is bounded in $C^2[0,1]$ and then we can deduce that TB is equicontinuous. Consequently, by Arzelà–Ascoli theorem we have that $T : P \to P$ is a completely continuous operator. Therefore $T : U \to U$ is a compact operator and the proof is completed. □

Remark 2.2. When $g(x)$ is nonpositive, Dang and Luan [3] proved that problem (2.1) has a unique solution by reducing the problem to a nonlinear equation. Compared with [3], our proof in 2.1 is more concise.

3 Nonlinear eigenvalue problem

In this section, we study (1.1) with $f(u) \equiv u$, that is the nonlinear eigenvalue problem

$$\begin{cases} u''''(x) - (a + b\int_0^1 (u'(x))^2 dx) u''(x) = \lambda u(x), & x \in (0,1), \\ u(0) = u(1) = u''(0) = u''(1) = 0. \end{cases} \tag{3.1}$$

The solutions of (3.1) are closely related to the following linear eigenvalue problem:

$$\begin{cases} u''''(x) - Au''(x) = \lambda u(x), & x \in (0,1), \\ u(0) = u(1) = u''(0) = u''(1) = 0. \end{cases} \tag{3.2}$$

In [6], Del Pino and Manásevich proposed that: a pair of constants (λ, A) such that (3.2) possesses a nontrivial solution will be called an eigenvalue pair, and the corresponding nontrivial solution will be called an eigenfunction. Furthermore, they proved that the eigenvalue pair (λ, A) of (3.2) must satisfy

$$\frac{\lambda}{(k\pi)^4} - \frac{A}{(k\pi)^2} = 1, \quad \text{for some } k \in \mathbb{N},$$

and the corresponding eigenfunction is $\varphi_k = c \sin k\pi x (c \neq 0$ is an arbitrary constant).

Now, given a positive constant A, we use $\lambda_{1,A}$ to denote the principal eigenvalue of problem (3.2), then we have the following results:

Lemma 3.1. *(i) If A_1, A_2 are positive constants such that $A_1 < A_2$, then $\lambda_{1,A_1} < \lambda_{1,A_2}$.*
(ii) Let B, C be two fixed positive constants. Consider the map

$$\lambda_1(\mu) := \lambda_{1,B+\mu C}, \qquad \mu \geq 0,$$

then $\lambda_1(\cdot)$ is a continuous and strictly increasing function and

$$\lambda_1(0) = \lambda_{1,B}, \qquad \lim_{\mu \to +\infty} \lambda_1(\mu) = +\infty.$$

Proof. By [6], we know that the principal eigenvalue $\lambda_{1,A}$ of (3.2) satisfy

$$\frac{\lambda_{1,A}}{\pi^4} - \frac{A}{\pi^2} = 1, \tag{3.3}$$

and the corresponding first eigenfunction is $\varphi_1(x) = c \sin \pi x$, where $c \neq 0$ is an arbitrary constant. According to (3.3), $\lambda_{1,A} = (1 + \frac{A}{\pi^2})\pi^4$, then (i) and (ii) are immediate consequences. □

By using Lemma 3.1, we prove the following results on the nonlinear eigenvalue problem (3.1).

Theorem 3.2. *Problem (3.1) has a positive solution u_λ if and only if $\lambda \in (\lambda_{1,a}, +\infty)$, moreover, the solution u_λ is unique and satisfying*

$$\lim_{\lambda \to \lambda_{1,a}} \|u_\lambda\|_\infty = 0, \qquad \lim_{\lambda \to +\infty} \|u_\lambda\|_\infty = +\infty. \tag{3.4}$$

Proof. Assume that u is a positive solution of (3.1), then $\int_0^1 (u'(x))^2 dx > 0$, consequently by Lemma 3.1 (i) we have

$$\lambda = \lambda_{1,a+b\int_0^1 (u'(x))^2 dx} > \lambda_{1,a}.$$

To any $\lambda \in (\lambda_{1,a}, +\infty)$, by Lemma 3.1 (ii), there exists a unique $t_0(\lambda) > 0$ such that

$$\lambda_{1,a+bt_0} = \lambda,$$

moreover,

$$\lim_{\lambda \to \lambda_{1,a}} t_0(\lambda) = 0, \qquad \lim_{\lambda \to +\infty} t_0(\lambda) = +\infty. \tag{3.5}$$

For the fixed t_0, take appropriate principal eigenfunction $\varphi_1(x) = c \sin \pi x (c > 0)$ of (3.2) associated to $\lambda_{1,a+bt_0}$ such that

$$\int_0^1 (\varphi_1'(x))^2 dx = t_0. \tag{3.6}$$

Then it is easy to see that $u_\lambda = \varphi_1 > 0$ is a positive solution of (3.1).

To prove the uniqueness, we assume that there exist two positive solutions $u \neq v$, since

$$\lambda = \lambda_{1,a+b\int_0^1 (u'(x))^2 dx} = \lambda_{1,a+b\int_0^1 (v'(x))^2 dx'}$$

then Lemma 3.1 (ii) guarantees that $\int_0^1 (u'(x))^2 dx = \int_0^1 (v'(x))^2 dx$ and u is proportional to v, which implies that $u = v$.

Finally, we prove (3.4). Since the unique positive solution of (3.1) is $u_\lambda = \varphi_1(x) = c_\lambda \sin \pi x$, where c_λ is a positive constant depending on λ, then by (3.6) and (3.5), we have

$$\lim_{\lambda \to \lambda_{1,a}} \int_0^1 (u_\lambda'(x))^2 dx = \lim_{\lambda \to \lambda_{1,a}} \int_0^1 [(c_\lambda \sin \pi x)']^2 dx = \lim_{\lambda \to \lambda_{1,a}} \frac{1}{2} c_\lambda^2 \pi^2 \to 0, \tag{3.7}$$

and similarly

$$\lim_{\lambda \to +\infty} \int_0^1 (u_\lambda'(x))^2 dx = \lim_{\lambda \to +\infty} \frac{1}{2} c_\lambda^2 \pi^2 \to \infty, \tag{3.8}$$

that is

$$\lim_{\lambda \to \lambda_{1,a}} c_\lambda \to 0, \qquad \lim_{\lambda \to +\infty} c_\lambda \to +\infty, \tag{3.9}$$

then (3.4) is an immediate consequence. □

4 The sublinear case

In this section, we study (1.1) when the nonlinear term f is sublinear which means that f satisfying:

(H1) $f : \mathbb{R} \to \mathbb{R}$ is continuous, $f(s) > 0$ for all $s > 0$, $f(0) = 0$ and $f_0 := \lim_{s \to 0+} \frac{f(s)}{s} = +\infty$;

(H2) $f_\infty := \lim_{s \to +\infty} \frac{f(s)}{s} = 0$.

The main tool we will use in this section is global bifurcation theory.

We first state some notation. Let $X := \{u \in C^2[0,1] : u(0) = u(1) = u''(0) = u''(1) = 0\}$ with the norm $\|u\|_X = \max\{\|u\|_\infty, \|u'\|_\infty, \|u''\|_\infty\}$. $B_\rho := \{u \in X : \|u\|_X < \rho\}$. For any $u \in X$, denote $u^+ = \max\{u, 0\}$. Define the operator $F : \mathbb{R} \times X \mapsto X$ by

$$F(\lambda, u)(x) := T(\lambda f(u^+(x))), \tag{4.1}$$

where T is the operator defined in Proposition 2.1. Obviously, if u is a nonnegative solution of (1.1), then u satisfies

$$u = F(\lambda, u). \tag{4.2}$$

On the other hand, if u is a solution of (4.2), we show that u must be a nonnegative solution of (1.1). In fact, by (H1) we have $f(u^+) \geq 0$ for any $u \in C[0,1]$. Then the positiveness of the operator T yields that the solution of (4.2) must be nonnegative or nonpositive according to $\lambda \geq 0$ or $\lambda \leq 0$. If we assume that the latter happens, that is, $u(x) \leq 0, \forall x \in [0,1]$, then

$f(u^+) \equiv 0$ and consequently (4.2) implies that $u \equiv 0$. From the above discussion, u is a nonnegative solution of (1.1) if and only if (4.2) holds.

Since the map from X into $U := P \cup (-P)$ defined by $u \mapsto \lambda f(u^+)$ is continuous, and $C^4[0,1] \cap X$ is compactly imbedded in X, then by Proposition 2.1, the operator $F : \mathbb{R} \times X \mapsto X$ as in (4.1) is completely continuous. In order to prove the main result of this section, we need the following lemmas.

Lemma 4.1. *For any fixed $\lambda < 0$, there exists a number $\rho > 0$ such that*

$$\deg(I - F(\lambda, \cdot), B_\rho(0), 0) = 1.$$

Proof. First, we claim that there exists $\delta > 0$ such that

$$u \neq tF(\lambda, u) = tT(\lambda f(u^+)) \quad \text{for all } u \in \overline{B_\delta},\ u \neq 0 \quad \text{and } t \in [0,1].$$

Suppose on the contrary that there exists a sequence $\{u_n\}$ in $X \setminus 0$ with $\|u_n\|_X \longrightarrow 0$ and $\{t_n\}$ in $[0,1]$ such that

$$u_n = t_n F(\lambda, u_n) = t_n T(\lambda f(u_n^+)),$$

that is

$$u_n''''(x) - \left(a + b\int_0^1 (u_n'(x))^2 dx\right) u_n''(x) = t_n \lambda f(u_n^+(x)) \leq 0, \qquad x \in (0,1). \tag{4.3}$$

Set $w_n = -u_n''$, then by (4.3) we can get an inequality for w_n similar to (2.18), which can deduce that $w_n \leq 0$. Consequently, $-u_n'' = w_n \leq 0$ and $u_n(0) = u_n(1) = 0$ guarantee that $u_n \leq 0$, which implies $f(u_n^+) \equiv 0$ according to (H1). Then by Proposition 2.1, (4.3) has only a unique solution $u_n \equiv 0$, a contradiction with $u_n \in X \setminus 0$.

Take $\rho \in (0, \delta]$, according to the homotopy invariance of topological degree and the normalization property, we have

$$\deg(I - F(\lambda, \cdot), B_\rho(0), 0) = \deg(I, B_\rho(0), 0) = 1.$$ □

Lemma 4.2. *For any fixed $\lambda > 0$, there exists a number $\rho > 0$ such that*

$$\deg(I - F(\lambda, \cdot), B_\rho(0), 0) = 0.$$

Proof. First, take a $\psi \in X, \psi > 0$, we claim that there exists $\delta > 0$ such that

$$u \neq T(\lambda f(u^+) + t\psi) \quad \text{for all } u \in \overline{B_\delta},\ u \neq 0 \quad \text{and } t \in [0,1].$$

Suppose on the contrary that there exists a sequence $\{u_n\}$ in $X \setminus 0$ with $\|u_n\|_X \longrightarrow 0$ and $\{t_n\}$ in $[0,1]$ such that

$$u_n = T(\lambda f(u_n^+) + t_n\psi),$$

that is

$$u_n''''(x) - \left(a + b\int_0^1 (u_n'(x))^2 dx\right) u_n''(x) = \lambda f(u_n^+(x)) + t_n\psi(x), \qquad x \in (0,1), \tag{4.4}$$

Since $t_n\psi(x) > 0, \forall x \in (0,1)$, from the similar argument in Lemma 4.1 we have that $u_n(x) > 0, \forall x \in (0,1)$.

On the other hand, $\|u_n\|_X \longrightarrow 0$ implies that

$$\int_0^1 (u_n'(x))^2 dx \leq C$$

for some positive constant C. Hence, according to Lemma 3.1 we have that

$$\lambda_{1,a+b\int_0^1 (u_n'(x))^2 dx} \leq \lambda_{1,a+bC} =: \Lambda.$$

Fix this value of Λ, since $\|u_n\|_\infty \longrightarrow 0$, then according to (H1), for n large we have that $\lambda f(u_n^+(x)) > \Lambda u_n(x), \forall x \in (0,1)$. Combining this with $u_n''(x) \leq 0, \forall x \in [0,1]$ we can conclude that for any $x \in (0,1)$ the following inequality holds

$$\begin{aligned} u_n''''(x) - (a+bC)u_n''(x) \geq u_n''''(x) - \left(a + b\int_0^1 (u_n'(x))^2 dx\right) u_n''(x) \\ = \lambda f(u_n^+(x)) + t_n\psi(x) > \Lambda u_n(x), \end{aligned}$$

which implies that $\lambda_{1,a+bC} > \Lambda$, a contradiction.

Take $\rho \in (0, \delta]$, since the equation $u - T(\lambda f(u) + \psi) = 0$ has no solution in $B_\rho(0)$, then according to the homotopy invariance of topological degree we have

$$\deg(I - F(\lambda, \cdot), B_\rho(0), 0) = \deg(I - T(\lambda f(\cdot) + \psi), B_\rho(0), 0) = 0. \qquad \square$$

Now, we are ready to consider the bifurcation of positive solutions of (1.1) from the line of trivial solutions $\{(\lambda, 0) \in \mathbb{R} \times X : \lambda \in \mathbb{R}\}$.

Theorem 4.3. *Assume that (H1) and (H2) hold. Then from* $(0,0)$ *there emanate an unbounded continuum* $\mathcal{C}_0$ *of positive solutions of* (4.2) *in* $\mathbb{R} \times X$.

Proof. First, we show that $(0,0)$ is a bifurcation point from the line of trivial solutions $\{(\lambda, 0) \in \mathbb{R} \times X : \lambda \in \mathbb{R}\}$ for the equation (1.1). In fact, this can be obtained following from a simple modification of the global bifurcation theorem of Rabinowitz [22, Theorem 1.3], and the similar arguments has been used in [2, Proposition 3.5] Suppose on the contrary that $(0,0)$ is not a bifurcation point for (4.2), then there is a neighborhood of $(0,0)$ containing no nontrivial solutions of (4.2). In particular there exists a small $\epsilon > 0$ such that there are no solutions of (4.2) on $[-\epsilon, \epsilon] \times \partial B_\epsilon(0)$. Then $\deg(I - F(\lambda, \cdot), B_\epsilon(0), 0)$ is well defined for $\lambda \in [-\epsilon, \epsilon]$ and, by the homotopy invariance property of degree we have

$$\deg(I - F(\lambda, \cdot), B_\epsilon(0), 0) \equiv \text{constant}, \qquad \forall \lambda \in [-\epsilon, \epsilon],$$

which is a contradict with Lemma 4.1 and 4.2.

Then according to Rabinowitz's global bifurcation theorem, an continuum $\mathcal{C}_0$ of positive solutions of (4.2) emanates from $(0,0)$, and either

(i) $\mathcal{C}_0$ is unbounded in $\mathbb{R} \times X$, or

(ii) $\mathcal{C}_0 \cap [(\mathbb{R} \setminus 0) \times \{0\}] \neq \emptyset$.

To prove the unboundedness of $\mathcal{C}_0$, we only need to show that the case (ii) cannot occur, that is: $\mathcal{C}_0$ can not meet $(\lambda, 0)$ for any $\lambda \neq 0$. It is easy to see that for $\lambda < 0$ problem (1.1) does not possess a positive solution. For the case $\lambda > 0$, we assume on the contrary that there exist some $\lambda_0 > 0$ and a sequence of parameters $\{\lambda_n\}$ and corresponding positive solutions

$\{u_n\}$ of (1.1) such that $\lambda_n \longrightarrow \lambda_0$ and $\|u_n\|_X \longrightarrow 0$. Since $\|u_n\|_\infty \longrightarrow 0$, then by (H1), for fixed $\varepsilon \in (0, \lambda_0)$ there exists $n_0 \in \mathbb{N}$ such that when $n > n_0$ we have

$$u_n''''(x) - \left(a + b\int_0^1 (u_n'(x))^2 dx\right) u_n''(x)$$
$$= \lambda_n f(u_n(x)) \geq (\lambda_0 - \varepsilon) f(u_n(x)) > \Lambda u_n(x), \qquad \forall x \in (0,1),$$

where Λ is defined as in Lemma 4.2. Now, we can get a contradiction in a similar way that in the proof of Lemma 4.2. $\square$

The main result of this section is following:

Theorem 4.4. *Assume that (H1) and (H2) hold, then* (1.1) *has a positive solution if and only if $\lambda > 0$. In addition, if f is monotone increasing and there exists $\alpha \in (0,1)$ such that*

$$f(\tau s) \geq \tau^\alpha f(s) \tag{4.5}$$

for any $\tau \in (0,1)$ and $s > 0$, then the positive solution of (1.1) *is unique.*

Proof. By Theorem 4.3, there exists an unbounded continuum $\mathcal{C}_0 \in \mathbb{R} \times X$ of positive solutions of (1.1). We will show that $\|u\|_X$ is bounded for any fixed $\lambda > 0$, that is, $\mathcal{C}_0$ can not blow up at finite $\lambda \in (0, +\infty)$. To do this, we first prove $\|u\|_\infty$ is bounded for any fixed $\lambda > 0$. Assume on the contrary that there exist $\lambda_0 > 0$ and a sequence of parameters $\{\lambda_n\}$ and corresponding positive solutions $\{u_n\}$ of (1.1) such that $\lambda_n \longrightarrow \lambda_0$, $\|u_n\|_\infty \longrightarrow \infty$. Since

$$u_n''''(x) - \left(a + b\int_0^1 (u_n'(x))^2 dx\right) u_n''(x) = \lambda_n f(u_n), \tag{4.6}$$

divide (4.6) by $\|u_n\|_\infty$ and set $v_n = \frac{u_n}{\|u_n\|_\infty}$, then we get

$$v_n''''(x) - \left(a + b\int_0^1 (u_n'(x))^2 dx\right) v_n''(x) = \lambda_n \frac{f(u_n(x))}{\|u_n\|_\infty}. \tag{4.7}$$

Multiplying (4.7) by v_n and integrating it over $[0,1]$, based on boundary conditions and integration by parts we obtain

$$\int_0^1 (v_n'(x))^2 dx = \frac{\int_0^1 \lambda_n \frac{f(u_n(x))}{\|u_n\|_\infty} v_n(x) dx - \int_0^1 (v_n''(x))^2 dx}{a + b\int_0^1 (u_n'(x))^2 dx}. \tag{4.8}$$

Since $\|v_n\|_\infty \equiv 1$, $\{\lambda_n\}$ is bounded and (H2) guarantees that $\frac{f(u_n(x))}{\|u_n\|_\infty} \longrightarrow 0$ as $n \longrightarrow \infty$, then (4.8) implies

$$0 \leq \int_0^1 (v_n'(x))^2 dx \leq \frac{\int_0^1 \lambda_n \frac{f(u_n(x))}{\|u_n\|_\infty} v_n(x) dx}{a} \longrightarrow 0 \quad \text{as } n \longrightarrow \infty,$$

that is $\|v_n'\|_\infty \longrightarrow 0$. By the boundary conditions $v_n(0) = v_n(1) = 0$, there exist $\xi_n \in (0,1)$ such that $v_n(x) = \int_{\xi_n}^x v_n'(t) dt$, $\forall x \in [0,1]$. Combining this with $\|v_n'\|_\infty \longrightarrow 0$ we can conclude that $\|v_n\|_\infty \longrightarrow 0$. This contradicts $\|v_n\|_\infty \equiv 1$, and then we get the boundedness of $\|u\|_\infty$. Next, we show that the boundedness of $\|u\|_\infty$ can deduce the boundedness of $\|u'\|_\infty$ and $\|u''\|_\infty$. Since

$$u''''(x) - \left(a + b\int_0^1 (u'(x))^2 dx\right) u''(x) = \lambda f(u(x)), \tag{4.9}$$

multiplying (4.9) by u and integrating it over $[0,1]$, similarly we can obtain

$$\int_0^1 (u'(x))^2 dx = \frac{\int_0^1 \lambda f(u(x))u(x)dx - \int_0^1 (u''(x))^2 dx}{a + b\int_0^1 (u'(x))^2 dx} \le \frac{\int_0^1 \lambda f(u(x))u(x)dx}{a}. \tag{4.10}$$

(4.10) implies that $\|u'\|_\infty$ is bounded, and consequently, $\|u''\|_\infty$ is bounded too. According to the definition of $\|u\|_X$, the above conclusion means that $\|u\|_X$ is bounded for any fixed $\lambda > 0$. Combining this with the unboundedness of $\mathcal{C}_0$, we conclude that $\sup\{\lambda | \ (\lambda, u) \in \mathcal{C}_0\} = \infty$, then for any $\lambda > 0$ there exists a positive solution for (1.1).

Now, we prove that if f is monotone increasing and satisfies (4.5), then (1.1) has only a unique positive solution. Assume that there exist two positive solutions $u \neq v$ corresponding to some fixed $\lambda > 0$. If $\int_0^1 (u'(x))^2 dx = \int_0^1 (v'(x))^2 dx = R > 0$, consider the problem

$$\begin{cases} \omega''''(x) - \left(a + b\int_0^1 (u'(x))^2 dx\right)\omega''(x) = \lambda f(\omega(x)), & x \in (0,1), \\ \omega(0) = \omega(1) = \omega''(0) = \omega''(1) = 0, \end{cases} \tag{4.11}$$

and its corresponding integral operator $H : P \to P$ given by

$$H(\omega) = T(\lambda f(\omega)) = \lambda \int_0^1 \int_0^1 G_1(x,t)G_{2,R}(t,s)f(\omega(s))dsdt.$$

By the monotonicity of f and (4.5), H is an increasing α-concave operator according to [31, Definition 2.3], then by [31, Theorem 2.1, Remark 2.1], the operator equation $H(\omega) = \omega$ has a unique solution, which is also the unique positive solution of (4.11). That is, $u = v$.

If we assume that $\int_0^1 (u'(x))^2 dx > \int_0^1 (v'(x))^2 dx$, since $v'' \le 0$, we have

$$\begin{aligned} &v''''(x) - \left(a + b\int_0^1 (u'(x))^2 dx\right) v''(x) \\ &\qquad \ge v''''(x) - \left(a + b\int_0^1 (v'(x))^2 dx\right) v''(x) = \lambda f(v(x)), \end{aligned} \tag{4.12}$$

which means that v is actually an upper solution of (4.11). Constructing an iterative sequence $v_{n+1} = Hv_n, n = 0,1,2,\ldots$, where $v_0 = v$, then (4.12) and the monotonicity of f guarantee that $\{v_n\}$ is decreasing. Moreover, by [31, Theorem 2.1, Remark 2.1], $\{v_n\}$ must converge to the unique solution u of (4.11), and consequently we have

$$0 \le u(x) \le v(x), \qquad \forall x \in [0,1]. \tag{4.13}$$

On the other hand, based on boundary conditions and integration by parts, from the assumption $\int_0^1 (u'(x))^2 dx > \int_0^1 (v'(x))^2 dx$ we have that

$$\begin{aligned} \int_0^1 (u'(x))^2 dx - \int_0^1 (v'(x))^2 dx &= \int_0^1 [u'(x) + v'(x)][u'(x) - v'(x)]dx \\ &= -\int_0^1 (u(x) - v(x))(u''(x) + v''(x))dx > 0, \end{aligned} \tag{4.14}$$

since $-(u''(x) + v''(x)) \ge 0$ following from (2.15), then (4.14) contradicts with (4.13). This concludes the proof. $\square$

Remark 4.5. If c_1, c_2 are nonnegative constants satisfying $c_1^2 + c_2^2 \neq 0$, $0 < p, q < 1$, then it is easy to check that the function

$$f(u) = c_1 u^p + c_2 u^q$$

is increasing and satisfies (H1),(H2) and (4.5). Consequently, Theorem 4.4 guarantees that the problem

$$\begin{cases} u''''(x) - (a + b\int_0^1 (u'(x))^2 dx) u''(x) = \lambda(c_1 u^p(x) + c_2 u^q(x)), & x \in (0,1), \\ u(0) = u(1) = u''(0) = u''(1) = 0, \end{cases}$$

has a positive solution if and only if $\lambda > 0$, moreover, the positive solution is unique.

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