Electronic Journal of Qualitative Theory of Differential Equations
2011, No. **62**, 1-7; **http://www.math.u-szeged.hu/ejqtde/**

A necessary and sufficient condition for existence and uniqueness of periodic solutions for a p-Laplacian Liénard equation

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Abstract. In this work, we investigate the following p-Laplacian Liénard equation:

$$(\varphi_p(x'(t)))' + f(x(t))x'(t) + g(x(t)) = e(t).$$

Under some assumption, a necessary and sufficient condition for the existence and uniqueness of periodic solutions of this equation is given by using Manásevich–Mawhin continuation theorem. Our results improve and extend some known results.

Keywords: Periodic solution; p-Laplacian; Liénard equation; Continuation theorem.
AMS 2000 Mathematics Subject Classification: 34C25.

1 Introduction

In this paper, we investigate the existence and uniqueness of periodic solutions of the following p-Laplacian Liénard equation

$$(\varphi_p(x'(t)))' + f(x(t))x'(t) + g(x(t)) = e(t), \tag{1.1}$$

where $p > 1$, $\varphi_p : \mathbb{R} \to \mathbb{R}$ is given by $\varphi_p(s) = |s|^{p-2}s$ for $s \neq 0$, $\varphi_p(0) = 0$, $f, g, e \in C(\mathbb{R}, \mathbb{R})$ and e is T-periodic with $T > 0$. If $p = 2$, (1.1) becomes the following forced Liénard equation:

$$x''(t) + f(x(t))x'(t) + g(x(t)) = e(t). \tag{1.2}$$

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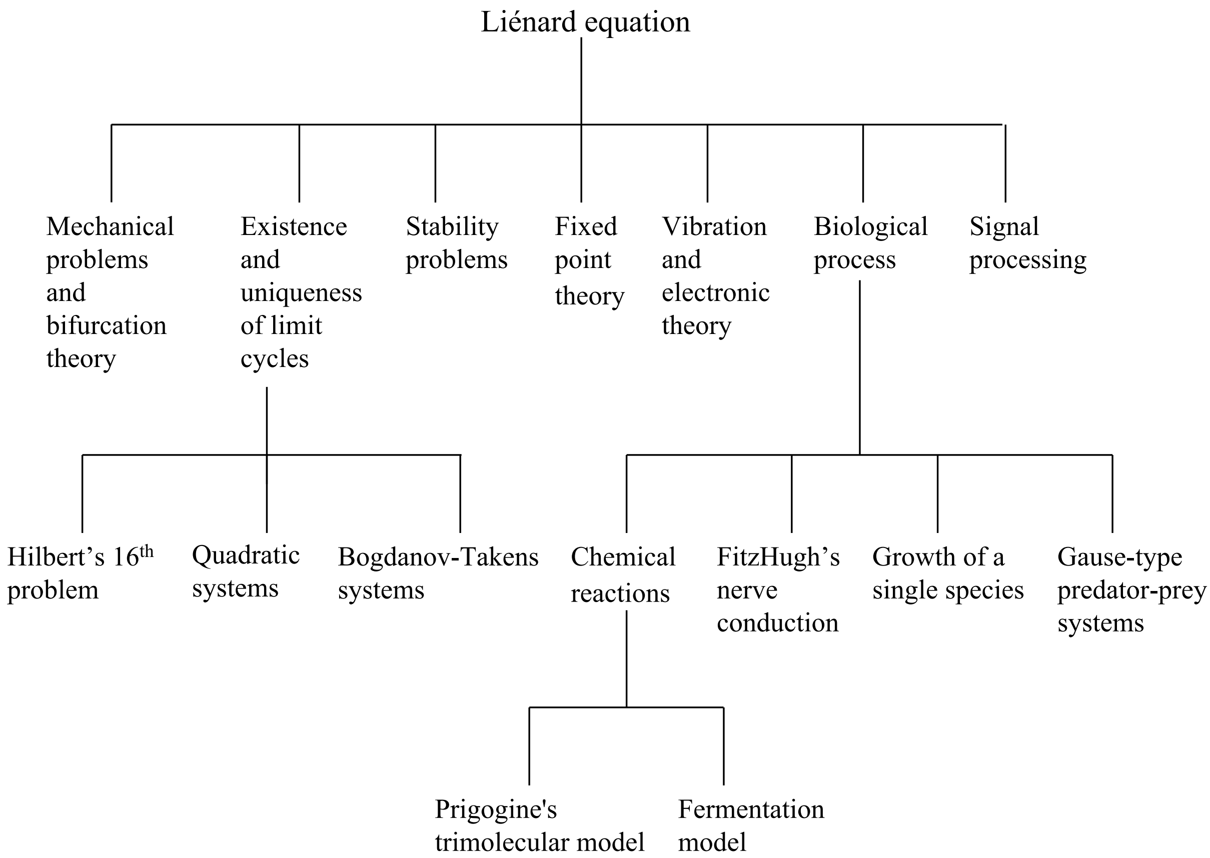

Figure 1: Diagrammatic representation for applications of the Liénard equation (1.1).

Generalized Liénard equations appear in a number of physical models, and the problem concerning the periodic solutions for these equations has been studied extensively by lots of authors; see for example [1-12] and the references therein.

Here we are keen to dispel any perception that the mathematical proofs of existence and uniqueness that we present are merely verifying facts which might already be obvious in other disciplines, based on purely physical considerations. In particular, in many nonlinear problems arising in practical dynamical systems, physical reasoning alone is not sufficient or fully convincing. In these cases questions of existence and uniqueness are of importance in understanding the full range of solution behaviour possible, and represent a genuine mathematical challenge. The answers to these mathematical questions then provide the basis for obtaining the best numerical solutions to these problems, and determining other important practical aspects of the solution behaviour. Figure 1 shows the various applications of the Liénard equation (1.1).

The main purpose in this work is to give a necessary and sufficient condition for the existence and uniqueness of T-periodic solutions of (1.1) by using Manásevich–Mawhin continuation theorem. Our results improve and extend some results in [6] (see Remark 2 and Examples 1 and 2).

2 Lemmas

Let us start with some notations. Set $C_T^1 = \{x \in C^1(\mathbb{R},\mathbb{R}) : x \text{ is } T\text{-periodic}\}$, which is a Banach space endowed with the norm $\|\cdot\|$ defined by $\|x\| = max\{|x|_\infty, |x'|_\infty\}$, and

$$|x|_\infty = \max_{t\in[0,T]} |x(t)|, |x'|_\infty = \max_{t\in[0,T]} |x'(t)|, |x|_k = \left(\int_0^T |x(t)|^k dt\right)^{1/k}.$$

For the T-periodic function e, denote

$$\bar{e} = \frac{1}{T}\int_0^T e(t)dt.$$

The following conditions will be used later:

(H_1) $g \in C^1(\mathbb{R},\mathbb{R})$ and $g'(x) < 0$ for all $x \in \mathbb{R}$.

(H_2) $\bar{e} \in g(\mathbb{R})$.

Remark 1. Generally, x refers to the displacement, $f(x)x'$ refers to the damping term, $g(x)$ refers to the stiffness term and e refers to the forced term in a vibration system. It implies that the stiffness of the vibration system is monotone decreasing with regard to the displacement if $g'(x) < 0$.

Lemma 1 ([6]). *Suppose* (H_1) *holds. Then* (1.1) *has at most one T-periodic solution.*

Consider the homotopic equation of (1.1):

$$(\varphi_p(x'(t)))' + \lambda f(x(t))x'(t) + \lambda g(x(t)) = \lambda e(t), \lambda \in (0,1). \tag{2.1}$$

We have the following lemma.

Lemma 2. *Suppose* (H_1) *and* (H_2) *hold. Then the set of T-periodic solutions of* (2.1) *are bounded in C_T^1.*

Proof. Let $S \subset C_T^1$ be the set of T-periodic solutions of (2.1). If $S = \emptyset$, the proof is ended. Suppose $S \neq \emptyset$, and let $x \in S$. Noticing that $x(0) = x(T)$, $x'(0) = x'(T)$ and $\varphi_p(0) = 0$, it follows from (2.1) that

$$\int_0^T (g(x(t)) - e(t))dt = 0,$$

which implies that there exists $\tau \in [0,T]$ such that

$$g(x(\tau)) = \bar{e}.$$

By (H_1), we know that $g(x)$ is strictly decreasing in $\mathbb{R}$. So we have

$$x(\tau) = g^{-1}(\bar{e}) =: C.$$

Then, for $t \in [\tau, \tau+T]$,

$$|x(t)| = \left|x(\tau) + \int_\tau^t x'(s)ds\right| \le |C| + \int_\tau^{\tau+T} |x'(s)|ds = |C| + \int_0^T |x'(s)|ds,$$

which leads to

$$|x|_\infty \le |C| + |x'|_1. \tag{2.2}$$

By (H_2) we have $g(x_0) = \bar{e}$ for some $x_0 \in \mathbb{R}$. Let $d = |x_0|$, since $g(x)$ is strictly decreasing in $\mathbb{R}$, we get

$$x(g(x) - \bar{e}) < 0 \text{ for } |x| > d. \tag{2.3}$$

Define $E_1 = \{t : t \in [0,T], |x(t)| > d\}, E_2 = \{t : t \in [0,T], |x(t)| \le d\}$. Multiplying $x(t)$ and (2.1) and then integrating from 0 to T, by (2.3) we have

$$\begin{aligned}
\int_0^T |x'(t)|^p dt &= -\int_0^T (\varphi_p(x'(t)))' x(t) dt \\
&= \lambda \int_0^T (g(x(t)) - \bar{e}) x(t) dt - \lambda \int_0^T (e(t) - \bar{e}) x(t) dt \\
&= \lambda \int_{E_1} (g(x(t)) - \bar{e}) x(t) dt + \lambda \int_{E_2} (g(x(t)) - \bar{e}) x(t) dt - \lambda \int_0^T (e(t) - \bar{e}) x(t) dt \\
&\le \lambda \int_{E_2} (g(x(t)) - \bar{e}) x(t) dt - \lambda \int_0^T (e(t) - \bar{e}) x(t) dt \\
&\le \left(\max_{|x| \le d} |g(x) - \bar{e}| + |e - \bar{e}|_\infty \right) T |x|_\infty.
\end{aligned}$$

Let $M_0 = \left(\max_{|x| \le d} |g(x) - \bar{e}| + |e - \bar{e}|_\infty \right) T$. Then we obtain

$$|x'|_p \le M_0^{1/p} |x|_\infty^{1/p}. \tag{2.4}$$

Let $q > 1$ such that $1/p + 1/q = 1$. Then by the Hölder inequality we have

$$|x'|_1 \le |x'|_p |1|_q = T^{1/q} |x'|_p. \tag{2.5}$$

By (2.2), (2.4) and (2.5), we can get

$$|x'|_1 \le T^{1/q} M_0^{1/p} (|C| + |x'|_1)^{1/p},$$

which yields that there exists $M_1 > 0$ such that $|x'|_1 \le M_1$ since $p > 1$, and this together with (2.2) implies that $|x|_\infty \le |C| + M_1$.

Meanwhile, there exists $t_0 \in [0,T]$ such that $x'(t_0) = 0$ since $x(0) = x(T)$. Then by (2.1) we have, for $t \in [t_0, t_0 + T]$,

$$\begin{aligned}
|\varphi_p(x'(t))| &= \left| \int_{t_0}^t (\varphi_p(x'(s)))' ds \right| \\
&= \lambda \left| \int_{t_0}^t (f(x(s)) x'(s) + g(x(s)) + e(s)) ds \right| \\
&\le \int_0^T (|f(x(s))||x'(s)| + |g(x(s))| + |e(s)|) ds \\
&\le F M_1 + (G + |e|_\infty) T,
\end{aligned}$$

where $F = \max\{|f(x)| : |x| \le |C| + M_1\}$, $G = \max\{|g(x)| : |x| \le |C| + M_1\}$. So we obtain

$$|x'|_\infty = \max_{t \in [0,T]} \{|\varphi_p(x'(t))|^{1/(p-1)}\} \le (F M_1 + (G + |e|_\infty) T)^{1/(p-1)}.$$

Let $M = \max\{|C| + M_1, (F M_1 + (G + |e|_\infty) T)^{1/(p-1)}\}$. Then $\|x\| \le M$. This completes the proof. $\square$

For the periodic boundary value problem

$$(\varphi_p(x'(t)))' = h(t, x, x'), x(0) = x(T), x'(0) = x'(T), \tag{2.6}$$

where $h \in C(\mathbb{R}^3, \mathbb{R})$ is T-periodic in the first variable. The following continuation theorem can be induced directly from the theory in [9], and is cited as Lemma 1 in [12].

Lemma 3 (Manásevich–Mawhin [9]). *Let $B = \{x \in C_T^1 : \|x\| < r\}$ for some $r > 0$. Suppose the following two conditions hold:*

(i) *For each $\lambda \in (0, 1)$ the problem $(\varphi_p(x'(t)))' = \lambda h(t, x, x')$ has no solution on ∂B.*

(ii) *The continuous function F defined on $\mathbb{R}$ by $F(a) = \frac{1}{T}\int_0^T h(t, a, 0)dt$ is such that $F(-r)F(r) < 0$. Then the periodic boundary value problem (2.6) has at least one T-periodic solution on $\bar{B}$.*

3 Main results

We are now in the position to give our main result.

Theorem 1. *Suppose* (H_1) *holds. Then* (1.1) *has a unique T-periodic solution if and only if* (H_2) *holds.*

Proof. Let $x(t)$ be a T-periodic solution of (1.1). By integrating the two sides of (1.1) from 0 to T, and noticing that $x(0) = x(T)$ and $x'(0) = x'(T)$, we have

$$\int_0^T (g(x(t)) - e(t))dt = 0.$$

Then there exists $\tau \in [0, T]$ such that

$$Tg(x(\tau)) = \int_0^T e(t)dt,$$

which implies $\bar{e} \in g(\mathbb{R})$, and the necessity is proved.

On the other hand, by Lemma 2.2, there exists $M > 0$ such that, for any solution $x(t)$ of (2.1),

$$||x|| \leq M. \tag{3.1}$$

Meanwhile, there exists $x_0 \in \mathbb{R}$ such that $g(x_0) = \bar{e}$ since $\bar{e} \in g(\mathbb{R})$. Let $h(t, x(t), x'(t)) = e(t) - g(x(t)) - f(x(t))x'(t)$ and $B = \{x \in C_T^1 : \|x\| < r\}$ with $r = \max\{M + 1, |x_0| + 1\}$. Then (3.1) implies that (2.1) has no solution on ∂B for all $\lambda \in (0, 1)$, and condition (i) of Lemma 3 is satisfied. Furthermore, we have

$$g(r) < \bar{e} < g(-r), \tag{3.2}$$

since $g(x)$ is strictly decreasing in $\mathbb{R}$. By the definition of F in Lemma 3 we get

$$F(a) = \frac{1}{T}\int_0^T h(t, a, 0)dt = \frac{1}{T}\int_0^T (e(t) - g(a))dt = \bar{e} - g(a).$$

This together with (3.2) yields that $F(r)F(-r) < 0$, i.e. condition (ii) of Lemma 3 is satisfied. Therefore, it follows from Lemma 3 that there exists a T-periodic solution $x(t)$ of (1.1). The uniqueness of this $x(t)$ is guaranteed by Lemma 1. This completes the proof. $\square$

Remark 2. In [6], The sufficient conditions for the existence of T-periodic solutions for (1.1) are (H_1) and the following condition:

(A$_2$) there exists constant $d > 0$ such that $x(g(x) - e(t)) < 0$ for $|x| > d$.

Noticing (2.3), it is easy to verify that the condition (H$_2$) is weaker than the condition (A$_2$) since $\min_{t\in\mathbb{R}} e(t) < \bar{e} < \max_{t\in\mathbb{R}} e(t)$ when $e(t) \neq$ constant. So our results improve and extend the main results in [6].

Finally, we close this work by two examples.

Example 1. Consider the following differential equation:

$$(\varphi_p(x'(t)))' + f(x(t))x'(t) - \arctan(x(t) + 1) = \frac{\pi}{4}(1 + 3\cos t), \tag{3.3}$$

where $f \in C(\mathbb{R}, \mathbb{R})$ and $p > 1$.

In this example, $g(x) = -\arctan(x + 1)$, $e(t) = \frac{\pi}{4}(1 + 3\cos t)$ and $T = 2\pi$. It is obvious that the condition (A$_1$) of theorem 1 in [6] holds. However, it is easy to verify that the condition (A$_2$) does not hold. Therefore, Theorem 1 in [6] fails, while, our criterion in Theorem 1 in this paper remains applicable, as we now show. According to the above arguments, the condition (H$_1$) holds; since $\bar{e} = \frac{\pi}{4}$, it is easy to see that the condition (H$_2$) also holds. Hence, Theorem 1 in this paper shows that (3.3) has a unique 2π-periodic solution.

Example 2. Consider the following differential equation:

$$(\varphi_p(x'(t)))' + f(x(t))x'(t) - \arctan(x(t) + 1) = \pi(1 + 3\cos t), \tag{3.4}$$

where $f \in C(\mathbb{R}, \mathbb{R})$ and $p > 1$.

In this example, $g(x) = -\arctan(x + 1)$, $e(t) = \pi(1 + 3\cos t)$ and $T = 2\pi$. It is obvious that the condition (A$_1$) of theorem 1 in [6] holds. However, it is easy to verify that the condition (A$_2$) does not hold. Therefore, Theorem 1 in [6] fails, while, our criterion in Theorem 1 in this paper remains applicable, as we now show. According to the above arguments, the condition (H$_1$) holds; since $\bar{e} = \pi$, it is easy to see that the condition (H$_2$) does not hold. Hence, Theorem 1 in this paper shows that (3.4) has no 2π-periodic solutions.

4 Acknowledgements

This work was supported by the Sichuan Youth Science and Technology Fund (No. 2011JQ0044), the National Program on Key Basic Research Project (973 Program, Grant No. 2011CB201005), the Open Fund (PLN1003) of State Key Laboratory of Oil and Gas Reservoir Geology and Exploitation (Southwest Petroleum University) and the Scientific Research Fund (No. 10ZB113) of Sichuan Provincial Educational Department. The authors are grateful to the editor and referee for their careful reading of the manuscript and helpful suggestions on this work.

References

[1] F. Albrecht, G. Villari, On the uniqueness of the periodic solutions of certain Liénard equations, Nonlinear Anal. 11 (1987), 1267–1277.

[2] T. A. Burton, C. G. Townsend, On the generalized Liénard equation with forcing function, J. Diff. Equations 4(1968), 620–633.

[3] W. S. Cheung, J. L. Ren, Periodic solutions for p-Laplacian Liénard equation with a deviating argument, Nonlinear Anal. 59 (2004), 107–120.

[4] C. Fabry, D. Fayyad, Periodic solutions of second order differential equations with a p-Laplacian and asymmetric nonlinearities, Rend. Istit. Univ. Trieste 24 (1992), 207–227.

[5] J. R. Graef, On the generalized Liénard equation with negative damping, J. Diff. Equations 12(1972), 34–62.

[6] B. Liu, Existence and uniqueness of periodic solutions for a kind of Liénard type p-Laplacian equation, Nonlinear Anal. 69 (2008), 724–729.

[7] B. Liu, Periodic solutions for Liénard type p-Laplacian equation with a deviating argument, J. Comput. Appl. Math. 214 (2008), 13–18.

[8] S. Lu, Existence of periodic solutions to a p-Laplacian Liénard differential equation with a deviating argument, Nonlinear Anal. 68 (2008), 1453–1461.

[9] R. Manásevich, J. Mawhin, Periodic solutions for nonlinear systems with p-Laplacian-like operators, J. Differential Equations. 145 (1998), 367–393.

[10] Y. Wang, X.-Z. Dai, New Results on Existence of Asymptotically Stable Periodic Solutions of a Forced Liénard Type Equation, Res. Math. 54 (2009), 359–375.

[11] Y. Wang, J. Tian, Periodic solutions for a Liénard equation with two deviating arguments, Elec. J. Diff. Eq. 140 (2009), 1–12.

[12] F. Zhang, Y. Li, Existence and uniqueness of periodic solutions for a kind of duffing type p-Laplacian equation, Nonlinear Anal.: Real World Appl. 9 (2008), 985–989.

(Received April 8, 2011)