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Oscillation Criteria for Nonlinear Delay Differential Equations of Second Order*

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Abstract

We prove oscillation theorems for the nonlinear delay differential equation $\left(|y'(t)|^{\alpha-2}\,y'(t)\right)' + q(t)\,|y(\tau(t))|^{\beta-2}\,y(\tau(t)) = 0, \qquad t \geq t_* > 0,$ where $\beta > 1$, $\alpha > 1$, $q(t) \geq 0$ and locally integrable on $[t_*, \infty)$, $\tau(t)$ is a continuous function satisfiying $0 < \tau(t) \leq t$ and $\lim_{t\to\infty}\tau(t) = \infty$. The results obtained essentially improve the known results in the literature and can be applied to linear and half-linear delay type differential equations.

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1 Introduction

In the last decades, there has been an increasing interest in obtaining sufficient conditions for the oscillation and/or nonoscillation of solutions for different classes of second order differential equations with or without deviating arguments. For interested readers we refer to the papers [7, 8, 12, 13, 15] and the references quoted therein.

*Dedicated to Professor A. Okay Çelebi on the occasion of his 70th birthday
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Before we continue with the description of the content of this paper, we present a short survey of the most basic results in the literature.

Let us consider the following linear differential equation

$$y'' + q(t)y = 0, \qquad t \geq t_0 \geq t_* > 0, \tag{1}$$

where $q(t) \geq 0$ is locally integrable on $[t_0, \infty)$.

In 1948, Hille [6] established the following results:

Theorem A. *If $q \in L^1[t_0, \infty)$ and*

$$\limsup_{t\to\infty} \ t \int_t^\infty q(s)ds \leq \frac{1}{4}, \tag{2}$$

then equation (1) is nonoscillatory.

Theorem B. *If $q \in L^1[t_0, \infty)$ and*

$$\liminf_{t\to\infty} \ t \int_t^\infty q(s)ds > \frac{1}{4}, \tag{3}$$

then equation (1) is oscillatory.

In 1997, Huang [7] obtained the following interval criteria:

Theorem C. *If there exists $t_0 \geq t_*$ such that for each $n \in \mathbb{N}_0 = \{0, 1, 2, ...\}$, $q(t)$ satisfies*

$$\int_{2^n t_0}^{2^{n+1} t_0} q(s)ds \leq \frac{\theta_0}{2^{n+1} t_0}, \tag{4}$$

where $\theta_0 = 3 - 2\sqrt{2}$, then equation (1) is nonoscillatory.

Theorem D. *If there exists $t_0 \geq t_*$ such that for each $n \in \mathbb{N}_0 = \{0, 1, 2, ...\}$, $q(t)$ satisfies*

$$\int_{2^n t_0}^{2^{n+1} t_0} q(s)ds \geq \frac{\theta}{2^n t_0}, \tag{5}$$

where $\theta > \theta_0$, then equation (1) is oscillatory.

In 2004, by replacing the sequence $\{2^n\}$ in Theorems C and D by $\{\lambda^n\}$ with $\lambda > 1$, Wong [15] generalized Theorems C and D as follows:

Theorem E. *Let $\lambda > 1$. If there exists some t_0 such that for each $n \in \mathbb{N}_0 = \{0, 1, 2, ...\}$, $q(t)$ satisfies*

$$\int_{\lambda^n t_0}^{\lambda^{n+1} t_0} q(s)ds \leq \frac{\theta}{(\lambda - 1)\lambda^{n+1} t_0}, \tag{6}$$

where $\theta \leq k_0(\lambda) = (\sqrt{\lambda} - 1)^2$, then equation (1) is nonoscillatory.

Theorem F. *Let $\lambda > 1$. If there exists some t_0 such that for each $n \in \mathbb{N}_0 = \{0, 1, 2, ...\}$, $q(t)$ satisfies*

$$\int_{\lambda^n t_0}^{\lambda^{n+1} t_0} q(s)ds \geq \frac{\theta}{(\lambda - 1)\lambda^n t_0}, \tag{7}$$

where $\theta > k_0(\lambda)$, then equation (1) is oscillatory.

Furthermore, Wong [15] extended the oscillation criteria (3) and (7) for equation (1) to the following linear delay differential equation

$$y''(t) + q(t)y(\tau(t)) = 0, \qquad t \geq t_0, \tag{8}$$

where $q(t) \geq 0$ and locally integrable on $[t_0, \infty)$, and $\tau(t)$ is a continuous function satisfying $0 < \tau(t) \leq t$ and $\lim_{t\to\infty}\tau(t) = \infty$.

In 1987, Yan [16] proved the following result for equation (8), but Wong gave an alternative and simpler proof in [15].

Theorem G. *Suppose that for all sufficiently large t, $q(t)$ satisfies*

$$\int_t^\infty q(s)\frac{\tau(s)}{s}ds \geq \frac{\theta}{t} \tag{9}$$

for some fixed constant $\theta > \frac{1}{4}$, then all solutions of equation (8) are oscillatory.

Wong also proved the extension of Theorem F for equation (8).

Theorem H. *Let $\lambda > 1$. If there exists $t_0 \geq t_*$ and for each $n \in \mathbb{N}_0 = \{0, 1, 2, ...\}$, $q(t)$ satisfies*

$$\int_{\lambda^n t_0}^{\lambda^{n+1}t_0} q(s)\frac{\tau(s)}{s}ds \geq \frac{\theta}{(\lambda-1)\lambda^n t_0}, \tag{10}$$

where $\theta > k_0(\lambda)$. Then all solutions of equation (8) are oscillatory.

Now, let us consider the following half-linear differential equation

$$\left(|y'(t)|^{\alpha-2}\, y'(t)\right)' + q(t)\,|y(t)|^{\alpha-2}\, y(t) = 0, \qquad t \geq t_0, \tag{11}$$

where $\alpha > 1$, $q(t) \geq 0$ is locally integrable on $[t_0, \infty)$.

In 1995, Kusano and Yoshida [9] generalized Theorems A and B as follows:

Theorem I. *If $q \in L^1[t_0, \infty)$, and*

$$\limsup_{t\to\infty}\ t^{\alpha-1}\int_t^\infty q(s)ds \leq \frac{(\alpha-1)^{\alpha-1}}{\alpha^\alpha}, \tag{12}$$

then equation (11) is nonoscillatory.

Theorem J. *If $q \in L^1[t_0, \infty)$, and*

$$\liminf_{t\to\infty}\ t^{\alpha-1}\int_t^\infty q(s)ds > \frac{(\alpha-1)^{\alpha-1}}{\alpha^\alpha}, \tag{13}$$

then equation (11) is oscillatory.

In 2004, Yang [17] extended Theorems I and J as follows:

Theorem K. *If $q \in L^1[t_0, \infty)$, and for large $t > t_0$,*

$$t^{\alpha-1}\int_t^\infty q(s)ds \leq \frac{(\alpha-1)^{\alpha-1}}{\alpha^\alpha}, \tag{14}$$

then equation (11) is nonoscillatory.

Theorem L. *If $q \in L^1[t_0, \infty)$, and for large $t > t_0$,*

$$t^{\alpha-1} \int_t^{\infty} q(s)ds \geq \alpha_0, \tag{15}$$

where $\alpha_0 > \frac{(\alpha-1)^{\alpha-1}}{\alpha^{\alpha}}$, then equation (11) is oscillatory.

In 2007, Kong [8] extended the results of Wong [15], namely Theorems E and F, for the linear differential equation (1) to the half-linear differential equation (11) as follows:

Theorem M. *Let* $\lambda > 1$ and $\xi^* = \xi^*(\alpha)$. *Assume there exists $t_0 \in (0, \infty)$ such that for each $n \in \mathbb{N}_0 = \{0, 1, 2, ...\}$, $q(t)$ satisfies*

$$\left(\int_{\lambda^n t_0}^{\lambda^{n+1} t_0} q(s)ds \right)^{\frac{1}{\alpha-1}} \leq \frac{\xi^*}{(\lambda-1)\lambda^{n+1} t_0}, \tag{16}$$

then equation (11) is nonoscillatory.

Theorem N. *Let* $\lambda > 1$ and $\xi^* = \xi^*(\alpha)$. *Assume there exists $t_0 \in (0, \infty)$ and $\xi > \xi^*$ such that for each $n \in \mathbb{N}_0 = \{0, 1, 2, ...\}$, $q(t)$ satisfies*

$$\left(\int_{\lambda^n t_0}^{\lambda^{n+1} t_0} q(s)ds \right)^{\frac{1}{\alpha-1}} \geq \frac{\xi}{(\lambda-1)\lambda^{n} t_0}, \tag{17}$$

then equation (11) is oscillatory.

In this paper, by using the same method in Wong [15], we extend Theorems G, H and N to the following nonlinear delay differential equation

$$\left(|y'(t)|^{\alpha-2} y'(t) \right)' + q(t) |y(\tau(t))|^{\beta-2} y(\tau(t)) = 0, \qquad t \geq t_0, \tag{18}$$

where $\beta > 1$, $\alpha > 1$, $q(t) \geq 0$ and locally integrable on $[t_0, \infty)$, $\tau(t)$ is continuous function satisfying $0 < \tau(t) \leq t$ and $\lim_{t\to\infty} \tau(t) = \infty$.

Note that the equation (18) with $\tau(t) = t$ is referred to as a super-half-linear equation, a sub-half-linear equation and an Emden-Fowler type equation for $\beta > \alpha$, $\beta < \alpha$ and $\beta \neq \alpha$, respectively. We refer the readers to the introductory books by Agarwal et al. [2] and by Došlý and Řehák [4] for the equation (18) with $\tau(t) = t$.

To present our results, we need the following lemma which is given by Erbe [5].

Lemma P. *Assume that $\tau \in C([t_0, \infty), \mathbb{R}^+)$, $0 < \tau(t) < t$ for $t \geq t_0$ and $\lim_{t\to\infty} \tau(t) = \infty$. Let $y \in C^2([t_0, \infty), \mathbb{R}^+)$ be such that $y''(t) \leq 0$ for $t \geq T \geq t_0$. Then for each constant $k \in (0, 1)$, there is a $T_k \geq T$ such that*

$$\frac{y(\tau(t))}{y(t)} \geq k \frac{\tau(t)}{t} \quad \text{for} \quad t \geq T_k. \tag{19}$$

2 Main Results

First, we obtain two theorems which concern the oscillatory behaviour of equation (18) with $\beta = \alpha$. Next, motivated by the ideas of Agarwal and Grace [1] and Çakmak [3], we present two other results for $\beta \neq \alpha$.

Theorem 1 *Suppose that for all sufficiently large t, $q(t)$ satisfies*

$$\int_t^\infty q(s)\left(\frac{\tau(s)}{s}\right)^{\alpha-1} ds \geq \frac{\alpha_1}{t^{\alpha-1}} \tag{20}$$

for some fixed constant $\alpha_1 > \frac{(\alpha-1)^{\alpha-1}}{\alpha^\alpha}$, then all solutions of equation (18) with $\beta = \alpha$ are oscillatory.

Proof. Assume on the contrary that equation (18) with $\beta = \alpha$ has a nontrivial nonoscillatory solution $y(t)$, we can assume without loss of generality that $y(t) > 0$ for $t \geq t_0$. Since $\lim_{t\to\infty} \tau(t) = \infty$, there exists $t_1 \geq t_0$ such that $y(\tau(t)) > 0$ for $t \geq t_1$. By equation (18) with $\beta = \alpha$, since $q(t) \geq 0$, $|y'(t)|^{\alpha-2} y'(t)$ is nonincreasing on $[t_1, \infty)$, so is $y'(t)$. This implies that $y'(t) > 0$ and $y''(t) \leq 0$ for $t \geq t_1$. Define $w(t) = \frac{|y'(t)|^{\alpha-2} y'(t)}{|y(t)|^{\alpha-2} y(t)}$, then $w(t)$ satisfies the equation

$$w'(t) + (\alpha - 1)\,|w(t)|^{\frac{\alpha}{\alpha-1}} + q(t)\left(\frac{y(\tau(t))}{y(t)}\right)^{\alpha-1} = 0 \tag{21}$$

on $[t_1, \infty)$. Thus, by Lemma P, for each constant $k \in (0, 1)$, there exists t_2, depending on k, such that for $t \geq t_2 \geq t_1$,

$$\frac{y(\tau(t))}{y(t)} \geq k\frac{\tau(t)}{t}. \tag{22}$$

Substituting (22) into (21), we find

$$w'(t) + (\alpha - 1)\,|w(t)|^{\frac{\alpha}{\alpha-1}} + \left(k\frac{\tau(t)}{t}\right)^{\alpha-1} q(t) \leq 0, \tag{23}$$

since $q(t) \geq 0$. It follows from the result of Li and Yeh [10, Theorem 3.2] that (23) implies the half-linear differential equation

$$\left(|u'(t)|^{\alpha-2} u'(t)\right)' + \left(k\frac{\tau(t)}{t}\right)^{\alpha-1} q(t)\,|u(t)|^{\alpha-2} u(t) = 0 \tag{24}$$

is nonoscillatory for every k, $0 < k < 1$. Note that $\mu = k^{\alpha-1} \in (0, 1)$ for every $0 < k < 1$ and $\alpha > 1$. Choose μ sufficiently close to 1 so that $\mu\alpha_1 > \frac{(\alpha-1)^{\alpha-1}}{\alpha^\alpha}$ which is possible since $\alpha_1 > \frac{(\alpha-1)^{\alpha-1}}{\alpha^\alpha}$; for example, choose $\mu = \frac{1}{2} + \frac{(\alpha-1)^{\alpha-1}}{2\alpha^\alpha \alpha_1} < 1$. Condition (20) implies

$$\mu \int_t^\infty \left(\frac{\tau(s)}{s}\right)^{\alpha-1} q(s)ds \geq \frac{\mu\alpha_1}{t^{\alpha-1}} \quad \text{and} \quad \mu\alpha_1 > \frac{(\alpha-1)^{\alpha-1}}{\alpha^\alpha}, \tag{25}$$

which in turn implies the oscillation criteria (15) given by Yang [17] that equation (24) is oscillatory. This is a contradiction, hence equation (18) with $\beta = \alpha$ is oscillatory. ■

Using the same argument as in the proof of Theorem 1, we can also prove the following result.

Theorem 2 *Let $\lambda > 1$ and $\xi^* = \xi^*(\alpha)$. Assume there exists $t_0 \in (0, \infty)$ and $\xi > \xi^*$ such that for each $n \in \mathbb{N}_0 = \{0, 1, 2, ...\}$, $q(t)$ satisfies*

$$\left(\int_{\lambda^n t_0}^{\lambda^{n+1} t_0} q(s) \left(\frac{\tau(s)}{s} \right)^{\alpha - 1} ds \right)^{\frac{1}{\alpha - 1}} \geq \frac{\xi}{(\lambda - 1)\lambda^n t_0}. \tag{26}$$

Then all solutions of equation (18) with $\beta = \alpha$ are oscillatory.

Proof. We follow the proof of Theorem 1 and conclude that the existence of a nonoscillatory solution of (18) with $\beta = \alpha$ lead to the conclusion that the half-linear differential equation (24) is nonoscillatory for every k, $0 < k < 1$. For every $0 < k < 1$ and $\alpha > 1$, we can again choose $\mu = k^{\alpha - 1} \in (0, 1)$ sufficiently close to 1 so that $\mu\xi > \xi^*$. Now the coefficient function of equation (24) satisfies

$$\mu \left(\int_{\lambda^n t_0}^{\lambda^{n+1} t_0} q(s) \left(\frac{\tau(s)}{s} \right)^{\alpha - 1} ds \right)^{\frac{1}{\alpha - 1}} \geq \frac{\xi_1}{(\lambda - 1)\lambda^n t_0}, \tag{27}$$

where $\xi_1 = \mu\xi > \xi^*$, so we can apply Theorem N given by Kong [8] to equation (24) and conclude that it is oscillatory for such μ, $0 < \mu < 1$, but $\mu\xi > \xi^*$. This contradicts the fact that equation (24) is nonoscillatory for all k, $0 < k < 1$. The proof is complete. ■

Remark 3 *When $\alpha = 2$, Theorems 1 and 2 reduce to Theorems G and H, respectively.*

Remark 4 *If the delayed argument is absent, i.e. $\tau(t) = t$, then Theorems 1 and 2 reduce to Theorems L and N, respectively. Furthermore, Theorem 1 is an extension of Theorem J.*

Remark 5 *Let $\alpha = 2$ and $\tau(t) = t$. In this case, Theorem 1 is an extension of Theorem B. Moreover, Theorem 2 (or Theorem 2 with $\lambda = 2$) reduces to Theorem F (or Theorem D).*

Theorem 6 *Suppose that for all sufficiently large t, $q(t)$ satisfies*

$$c \int_t^\infty q(s) \left(\frac{\tau(s)}{s} \right)^{\beta - 1} ds \geq \frac{\alpha_1}{t^{\alpha - 1}} \tag{28}$$

for some fixed constant $\alpha_1 > \frac{(\alpha-1)^{\alpha-1}}{\alpha^\alpha}$ and any constant $c > 0$, then the following assertions are true:

(i) *all unbounded solutions of equation (18) with $\beta > \alpha$ are oscillatory.*

(ii) *all bounded solutions of equation (18) with $\beta < \alpha$ are oscillatory.*

Proof. Assume on the contrary that equation (18) with $\beta \neq \alpha$ has a nontrivial nonoscillatory solution $y(t)$, we can assume without loss of generality that $y(t) > 0$ for $t \geq t_0$. Since $\lim_{t\to\infty} \tau(t) = \infty$, there exists $t_1 \geq t_0$ such that $y(\tau(t)) > 0$ for $t \geq t_1$. By equation (18) with $\beta \neq \alpha$, since $q(t) \geq 0$, $|y'(t)|^{\alpha-2} y'(t)$ is nonincreasing on $[t_1, \infty)$, so is $y'(t)$. This implies that $y'(t) > 0$ and $y''(t) \leq 0$ for $t \geq t_1$. Define $w(t) = \frac{|y'(t)|^{\alpha-2} y'(t)}{|y(t)|^{\alpha-2} y(t)}$, then $w(t)$ satisfies the equation

$$w'(t) + (\alpha - 1) |w(t)|^{\frac{\alpha}{\alpha-1}} + q(t) \left(\frac{y(\tau(t))}{y(t)} \right)^{\beta-1} (y(t))^{\beta-\alpha} = 0 \tag{29}$$

on $[t_1, \infty)$. Next, we consider the following two cases:

(i) If $y(t)$ is an unbounded nonoscillatory solution of equation (18) with $\beta > \alpha$ for $t \geq t_0$, then there exist a constant $k_1 > 0$ and $t_2 \geq t_1 \geq t_0$ such that $y(t) \geq k_1$ for $t \geq t_2$. Therefore,

$$(y(t))^{\beta-\alpha} \geq {k_1}^{\beta-\alpha} = c_1 \quad \text{for } t \geq t_2, \tag{30}$$

where c_1 is a constant. Using (30) in the equation (29), and proceeding as in the proof of Theorem 1, we arrive at the desired contradiction.

(ii) If $y(t)$ is a bounded nonoscillatory solution of equation (18) with $\beta < \alpha$ for $t \geq t_0$, then there exist a constant $k_2 > 0$ and $t_2 \geq t_1 \geq t_0$ such that $y(t) \leq k_2$ for $t \geq t_2$. Therefore,

$$(y(t))^{\beta-\alpha} \geq {k_2}^{\beta-\alpha} = c_2 \quad \text{for } t \geq t_2, \tag{31}$$

where c_2 is a constant. The rest of the proof is similar to that of previous case and, hence omitted. ■

Combining some ingredients of the proofs of Theorems 2 and 6, we give the following result for equation (18) with $\beta \neq \alpha$, the proof of which is similar to that of Theorem 6, and hence omitted.

Theorem 7 *Let $\lambda > 1$ and $\xi^* = \xi^*(\alpha)$. Assume there exists $t_0 \in (0, \infty)$ and $\xi > \xi^*$ such that for each $n \in \mathbb{N}_0 = \{0, 1, 2, ...\}$ and any constant $c > 0$, $q(t)$ satisfies*

$$\left(c \int_{\lambda^n t_0}^{\lambda^{n+1} t_0} q(s) \left(\frac{\tau(s)}{s} \right)^{\beta-1} ds \right)^{\frac{1}{\alpha-1}} \geq \frac{\xi}{(\lambda - 1)\lambda^n t_0}. \tag{32}$$

Then the following assertions are true:

(i) *all unbounded solutions of equation (18) with $\beta > \alpha$ are oscillatory.*

(ii) *all bounded solutions of equation (18) with $\beta < \alpha$ are oscillatory.*

Finally, we generalize above results for a class of more general nonlinear delay differential equations as follows:

Let α, β, $q(t)$, and $\tau(t)$ be as above, and consider

$$\left(|y'(t)|^{\alpha-2} y'(t) \right)' + f(t, y(\tau(t))) = 0, \qquad t \geq t_0, \tag{33}$$

where the function f satisfies

$$sf(t,s) \geq q(t)\,|s|^{\beta} \quad \text{for} \quad t \geq t_0 \quad \text{and } s \in \mathbb{R}. \tag{34}$$

The proof of the following results are exactly as in that of above theorems and hence omitted.

Theorem 8 *In addition to the conditions of Theorem 1 (or Theorem 2), if (34) with $\beta = \alpha$ holds, then all solutions of equation (33) are oscillatory.*

Theorem 9 *In addition to the conditions of Theorem 6 (or Theorem 7), if (34) with $\beta > \alpha$ holds, then all unbounded solutions of equation (33) are oscillatory.*

Theorem 10 *In addition to the conditions of Theorem 6 (or Theorem 7), if (34) with $\beta < \alpha$ holds, then all bounded solutions of equation (33) are oscillatory.*

Remark 11 *For another oscillation criteria contain for equation (18) with $\beta \geq \alpha$ and (33) with $f(t, y(\tau(t))) = F(y(\tau(t)))$ under different sufficient conditions, the reader is referred to [13].*

Remark 12 *In case the delay is bounded, i.e., $0 \leq t - \tau(t) \leq M$, then $\frac{\tau(t)}{t}$ in conditions (20), (26), (28) and (32) can be replaced by 1. In other words, Hille's oscillation criterion (3) is also valid oscillation criteria for equations (18) and (33) with $\beta = \alpha = 2$ and bounded delay; see [11, 14, 15].*

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