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Existence and multiplicity of positive solutions for a singular system via sub-supersolution method and Mountain Pass Theorem

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Abstract: In this paper we show the existence and multiplicity of positive solutions using the sub-supersolution method and Mountain Pass Theorem in a general singular system which the operator is not homogeneous neither linear.

Keywords: $p\&q$-Laplacian operator, sub-supersolution method, singular system, Mountain Pass theorem.

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1 Introduction

In this paper we treat the question of the existence and multiplicity of positive solutions for the following class of singular systems of nonlinear elliptic equation

$$\begin{cases} -\mathrm{div}(a_1(|\nabla u|^{p_1})|\nabla u|^{p_1-2}\nabla u) = h_1(x)u^{-\gamma_1} + F_u(x,u,v) \text{ in } \Omega, \\ -\mathrm{div}(a_2(|\nabla v|^{p_2})|\nabla v|^{p_2-2}\nabla v) = h_2(x)v^{-\gamma_2} + F_v(x,u,v) \text{ in } \Omega, \\ u, v > 0 \text{ in } \Omega, \\ u = v = 0 \text{ on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary, $N \geq 3$ and $2 \leq p_1, p_2 < N$. For $i = 1, 2$, $\gamma_i > 0$ is a fixed constant, $a_i : \mathbb{R}^+ \to \mathbb{R}^+$ is a C^1-function and $h_i \geq 0$ is a nontrivial measurable function. More precisely, we suppose that the functions h_i and a_i satisfy the following assumptions:

(H) There exists $0 < \phi_0 \in C_0^1(\overline{\Omega})$ such that $h_i\phi_0^{-\gamma_i} \in L^\infty(\Omega)$.

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(A_1) There exist constants $k_1, k_2, k_3, k_4 > 0$ and $2 \leq p_i \leq q_i < N$ such that

$$k_1 t^{p_i} + k_2 t^{q_i} \leq a_i(t^{p_i})t^{p_i} \leq k_3 t^{p_i} + k_4 t^{q_i}, \quad \text{for all } t \geq 0.$$

(A_2) The functions

$$t \longmapsto a_i(t^{p_i})t^{p_i-2} \quad \text{are increasing.}$$

(A_3) The functions

$$t \longmapsto A_i(t^{p_i}) \quad \text{are strictly convex,}$$

where $A_i(t) = \int_0^t a_i(s)ds$.

(A_4) There exist constants μ_i, $\frac{1}{q_1^*} < \theta_s < \frac{1}{q_1}$ and $\frac{1}{q_2^*} < \theta_t < \frac{1}{q_2}$ such that

$$\frac{1}{\mu_i} a_i(t)t \leq A_i(t), \quad \text{for all } t \geq 0,$$

with $\frac{q_1}{p_1} \leq \mu_1 < \frac{1}{\theta_s p_1}$ and $\frac{q_2}{p_2} \leq \mu_2 < \frac{1}{\theta_t p_2}$.

Notice that the functions a_i satisfy suitable monotonicity conditions which allow to consider a larger class of $p\&q$ type problems. In order to illustrate the degree of generality of the kind of problems studied here, in the following we present some examples of functions a_i which are interesting from the mathematical point of view and have a wide range of applications in physics and related sciences.

Example 1.1. If $a_i \equiv 1$, for each $i = 1, 2$, our operator is the p-Laplacian and so problem (1.1) becomes

$$\begin{cases} -\Delta_{p_1} u = h_1(x)u^{-\gamma_1} + F_u(x,u,v) \text{ in } \Omega, \\ -\Delta_{p_2} v = h_2(x)v^{-\gamma_2} + F_v(x,u,v) \text{ in } \Omega, \\ u = v = 0 \text{ on } \partial\Omega, \end{cases}$$

with $q_i = p_i$, $k_1 + k_2 = 1$ and $k_3 + k_4 = 1$.

Example 1.2. If $a_i(t) = 1 + t^{\frac{q_i - p_i}{p_i}}$, for each $i = 1, 2$, we obtain

$$\begin{cases} -\Delta_{p_1} u - \Delta_{q_1} u = h_1(x)u^{-\gamma_1} + F_u(x,u,v) \text{ in } \Omega, \\ -\Delta_{p_2} v - \Delta_{q_2} v = h_2(x)v^{-\gamma_2} + F_v(x,u,v) \text{ in } \Omega, \\ u = v = 0 \text{ on } \partial\Omega, \end{cases}$$

with $k_1 = k_2 = k_3 = k_4 = 1$.

Example 1.3. Taking $a_i(t) = 1 + \frac{1}{(1+t)^{\frac{p_i-2}{p_i}}}$, for each $i = 1, 2$, we get

$$\begin{cases} -\text{div}\left(|\nabla u|^{p_1-2}\nabla u + \dfrac{|\nabla u|^{p_1-2}\nabla u}{(1+|\nabla u|^{p_1})^{\frac{p_1-2}{p_1}}} \right) = h_1(x)u^{-\gamma_1} + F_u(x,u,v) \text{ in } \Omega, \\ -\text{div}\left(|\nabla v|^{p_2-2}\nabla v + \dfrac{|\nabla v|^{p_2-2}\nabla v}{(1+|\nabla v|^{p_2})^{\frac{p_2-2}{p_2}}} \right) = h_2(x)v^{-\gamma_2} + F_v(x,u,v) \text{ in } \Omega, \\ u = v = 0 \text{ on } \partial\Omega, \end{cases}$$

with $q_i = p_i$, $k_1 + k_2 = 1$ and $k_3 + k_4 = 2$.

Example 1.4. If we consider $a_i(t) = 1 + t^{\frac{q_i - p_i}{p_i}} + \frac{1}{(1+t)^{\frac{p_i-2}{p_i}}}$, for each $i = 1, 2$, we obtain

$$\begin{cases} -\Delta_{p_1} u - \Delta_{q_1} u - \operatorname{div}\left(\dfrac{|\nabla u|^{p_1-2}\nabla u}{(1+|\nabla u|^{p_1})^{\frac{p_1-2}{p_1}}}\right) = h_1(x)u^{-\gamma_1} + F_u(x,u,v) \text{ in } \Omega, \\ -\Delta_{p_2} v - \Delta_{q_2} v - \operatorname{div}\left(\dfrac{|\nabla v|^{p_2-2}\nabla v}{(1+|\nabla v|^{p})^{\frac{p_2-2}{p_2}}}\right) = h_2(x)v^{-\gamma_2} + F_v(x,u,v) \text{ in } \Omega, \\ u = v = 0 \text{ on } \partial\Omega, \end{cases}$$

where $k_1 = k_2 = k_4 = 1$ and $k_3 = 2$.

Remark 1.5. Note that by hypothesis (H) we have $h_i \in L^\infty(\Omega)$ because

$$|h_i| = |h_i \phi_0^{-\gamma_i} \phi_0^{\gamma_i}| \le \|h_i \phi_0^{-\gamma_i}\|_\infty \phi_0^{\gamma_i}.$$

Here F is a function on $\overline{\Omega} \times \mathbb{R}^2$ of class C^1 satisfying

(F_1) There exists $0 < \delta < \frac{1}{2}$ such that

$$-h_1(x) \le F_s(x,s,t) \le 0 \quad \text{a.e. in } \Omega, \text{ for all } 0 \le s \le \delta$$

and

$$-h_2(x) \le F_t(x,s,t) \le 0 \quad \text{a.e. in } \Omega, \text{ for all } 0 \le t \le \delta.$$

It is worthwhile to point out that, since $p_i < q_i$ and by the boundedness of Ω, $W_0^{1,p_i}(\Omega) \cap W_0^{1,q_i}(\Omega) = W_0^{1,q_i}(\Omega)$. Thus, in order to show the existence and multiplicity of solutions to system (1.1), we define the Sobolev space $X = W_0^{1,q_1}(\Omega) \times W_0^{1,q_2}(\Omega)$ endowed with the norm

$$\|(u,v)\| = \|u\|_{1,q_1} + \|v\|_{1,q_2},$$

where

$$\|u\|_{1,q_i} = \left(\int_\Omega |\nabla u|^{q_i} dx\right)^{\frac{1}{q_i}}.$$

Moreover, we say that a pair $(u,v) \in X$ is a positive weak solution of system (1.1) if $u, v > 0$ in Ω and it verifies

$$\int_\Omega a_1(|\nabla u|^{p_1})|\nabla u|^{p_1-2}\nabla u \nabla \phi \, dx = \int_\Omega h_1(x)u^{-\gamma_1}\phi \, dx + \int_\Omega F_u(x,u,v)\phi \, dx$$

and

$$\int_\Omega a_2(|\nabla v|^{p_2})|\nabla v|^{p_2-2}\nabla v \nabla \varphi \, dx = \int_\Omega h_2(x)v^{-\gamma_2}\varphi \, dx + \int_\Omega F_v(x,u,v)\varphi \, dx,$$

for all $(\phi, \varphi) \in X$.

In our first theorem we apply the sub-supersolution method to establish the existence of a weak solution for system (1.1).

Theorem 1.6. *Suppose that (H), (F_1) and (A_1)–(A_3) are satisfied. Then system (1.1) has a positive weak solution if $\|h_i\|_\infty$ is sufficiently small, for $i = 1, 2$.*

Furthermore, we assume the conditions below to prove the existence of two solutions for problem (1.1).

(F_2) For $i = 1, 2$, there exists $1 < r < q_i^* = \frac{Nq_i}{(N-q_i)}$ ($q_i^* = \infty$ if $q_i \geq N$) such that

$$F_s(x, s, t) \leq h_1(x)(1 + s^{r-1} + t^{r-1}) \quad \text{a.e. in } \Omega, \text{ for all } s \geq 0$$

and

$$F_t(x, s, t) \leq h_2(x)(1 + s^{r-1} + t^{r-1}) \quad \text{a.e. in } \Omega, \text{ for all } t \geq 0.$$

(F_3) There exist $s_0, t_0 > 0$ such that

$$0 < F(x, s, t) \leq \theta_s s F_s(x, s, t) + \theta_t t F_t(x, s, t) \quad \text{a.e. in } \Omega, \text{ for all } s \geq s_0 \text{ and } t \geq t_0,$$

where θ_s and θ_t appeared in (A_4).

Theorem 1.7. *Suppose that (H), (F_1)–(F_3) and (A_1)–(A_4) are satisfied. Then system (1.1) has two positive weak solutions if $\|h_i\|_\infty$ is sufficiently small, for $i = 1, 2$.*

Singular problems has been much studied in last years. We are going to cite some authors in last ten years. System (1.1) with Laplacian operator in both equations was studied in [9], where it was investigated the questions of existence, non-existence and uniqueness for solutions. The results in [9] were complemented in [16]. The general operator as we consider in this paper was studied in [5] using continuous unbounded of solutions. The cases with Laplacian operator involving weights were studied in [7] and [11].

In this paper we complement the results that can be found in [5], [7], [9], [11] and [16] because we consider a general problem with singularity without restrictions in the exponents. Moreover, we are considering the sub-supersolution method for a system that involves a nonlinear and nonhomegeneous operator. The reader can see the generality of the operator in [5].

We would like to highlight that our theorems can be applied for the model nonlinearity

$$F(x, s, t) = h_1(x)\left(\frac{s^r}{r} - s\delta^{r-1}\right) + h_2(x)\left(\frac{t^r}{r} - t\delta^{r-1}\right).$$

This paper is organized in the following way. Section 2 is devoted to some preliminary results in order to prove the main results. The first theorem is proved in the Section 3 and the second theorem in the Section 4.

2 Preliminary results

The next lemma provides the uniqueness of solution to the linear problem. The proof can be found in [5, Lemma 1]. However, for the convenience of the reader, we also prove it here.

Lemma 2.1. *Assume that the conditions (A_1) and (A_2) hold. Then, there exists an unique solution $u_i \in W_0^{1,q_i}(\Omega)$ of the linear problem*

$$\begin{cases} -\operatorname{div}(a_i(|\nabla u_i|^{p_i})|\nabla u_i|^{p_i-2}\nabla u_i) = h_i(x) \text{ in } \Omega, \\ u_i = 0 \text{ on } \partial\Omega, \end{cases}$$

where $h_i \in (W_0^{1,q_i}(\Omega))'$, for all $i = 1, 2$ and $2 \leq p_i \leq q_i < N$.

Proof. Consider the operator $T_i : W_0^{1,q_i}(\Omega) \longrightarrow (W_0^{1,q_i}(\Omega))'$ given by

$$\langle T_i u_i, \phi_i \rangle = \int_\Omega a_i(|\nabla u_i|^{p_i})\, |\nabla u_i|^{p_i-2} \nabla u_i \nabla \phi_i \, dx.$$

In virtue of hypothesis (A_1), we can show that the operator T_i is well defined and it is continuous. Furthermore, by considering the hypothesis (A_2), we argument as [8, Lemma 2.4] to obtain the following inequality

$$C_i |u_i - v_i|^{p_i} \leq \langle a_i(|u_i|^{p_i})|u_i|^{p_i-2}u_i - a_i(|v_i|^{p_i})|v_i|^{p_i-2}v_i, u_i - v_i \rangle,$$

for some $C_i > 0$ and for all $i = 1, 2$. Therefore,

$$\langle T_i u_i - T_i v_i, u_i - v_i \rangle > 0, \text{ for all } u_i, v_i \in W_0^{1,q_i}(\Omega) \text{ with } u_i \neq v_i,$$

which implies that T_i is monotone. Moreover, using (A_1) again we get

$$\frac{\langle T_i u_i, u_i \rangle}{\|u_i\|_{1,q_i}} \geq k_2 \|u_i\|_{1,q_i}^{q_i - 1}$$

and hence

$$\lim_{\|u_i\|_{1,q_i} \to \infty} \frac{\langle T_i u_i, u_i \rangle}{\|u_i\|_{1,q_i}} = +\infty,$$

which shows that T_i is coercive. Thus, applying the Minty–Browder Theorem [2, Theorem 5.15] there exists an unique $u_i \in W_0^{1,q_i}(\Omega)$ such that $T_i u_i = h_i(x)$. □

Our approach in the study of system (1.1) rests heavily on the following Weak Comparison Principle for the $p\&q$-Laplacian operator. The proof of the result below for the scalar case can be found in [6, Lemma 2.1].

Lemma 2.2. *Let Ω a bounded domain and consider $u_i, v_i \in W_0^{1,q_i}(\Omega)$ satisfying*

$$\begin{cases} -\mathrm{div}(a_i(|\nabla u_i|^{p_i})|\nabla u_i|^{p_i-2}\nabla u_i) \leq -\mathrm{div}(a_i(|\nabla v_i|^{p_i})|\nabla v_i|^{p_i-2}\nabla v_i) \text{ in } \Omega, \\ u_i \leq v_i \text{ on } \partial\Omega, \end{cases}$$

then $u_i \leq v_i$ a.e. in Ω, for all $i = 1, 2$ and $2 \leq p_i \leq q_i < N$.

Proof. Using the test function $\phi_i = (u_i - v_i)^+ := \max\{u_i - v_i, 0\} \in W_0^{1,q_i}(\Omega)$, we get

$$\int_{\Omega \cap \{u_i > v_i\}} \langle a_i(|\nabla u_i|^{p_i})|\nabla u_i|^{p_i-2}\nabla u_i - a_i(|\nabla v_i|^{p_i})|\nabla v_i|^{p_i-2}\nabla v_i, \nabla u_i - \nabla v_i \rangle dx \leq 0.$$

From Lemma 2.1, $\|(u_i - v_i)^+\| \leq 0$, which implies that $u_i \leq v_i$ a.e. in Ω. □

Now, using Lemma 2.2, it is possible to repeat the same arguments of [13, Hopf's Lemma] to obtain the next result

Lemma 2.3. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary and $i = 1, 2$. If $u_i \in C^1(\overline{\Omega}) \cap W_0^{1,q_i}(\Omega)$, with $2 \leq p_i \leq q_i < N$, and*

$$\begin{cases} -\mathrm{div}(a_i(|\nabla u_i|^{p_i})|\nabla u_i|^{p_i-2}\nabla u_i) \geq 0 \text{ in } \Omega, \\ u_i > 0 \text{ in } \Omega, \\ u_i = 0 \text{ on } \partial\Omega. \end{cases}$$

Then, $\frac{\partial u_i}{\partial \eta} < 0$ on $\partial\Omega$, where η is the outwards normal to $\partial\Omega$.

We enunciate an iteration lemma due to Stampacchia that we will use to prove the L^∞-regularity of the solutions for this class of $p\&q$ type problems.

Lemma 2.4 (See [14]). *Assume that $\phi : [0,\infty) \to [0,\infty)$ is a nonincreasing function such that if $h > k > k_0$, for some $\alpha > 0$, $\beta > 1$, $\phi(h) \le \frac{C(\phi(k))^\beta}{(h-k)^\alpha}$. Then, $\phi(k_0 + d) = 0$, where $d^\alpha = C2^{\frac{\alpha\beta}{\beta-1}}\phi(k_0)^{\beta-1}$ and C is positive constant.*

Lemma 2.5. *Let $u_i \in W_0^{1,q_i}(\Omega)$ be solution to problem*

$$\begin{cases} -\operatorname{div}(a_i(|\nabla u_i|^{p_i})|\nabla u_i|^{p_i-2}\nabla u_i) = f_i \text{ in } \Omega, \\ u_i = 0 \text{ on } \partial\Omega, \end{cases} \tag{2.1}$$

such that $f_i \in L^{r_i}(\Omega)$ with $r_i > \frac{q_i^*}{q_i^* - q_i}$. Then, $u_i \in L^\infty$. In particular, if $\|f_i\|_{r_i}$ is small, then also $\|u_i\|_\infty$ is small, for all $i = 1,2$ and $2 \le p_i \le q_i < N$.

Proof. Since u_i is the weak solution to (2.1) we can write

$$\int_\Omega a_i(|\nabla u_i|^{p_i})|\nabla u_i|^{p_i-2}\nabla u_i \nabla\phi_i \, dx = \int_\Omega f_i\phi_i \, dx, \ \forall \phi_i \in W_0^{1,q_i}(\Omega).$$

For $k > 0$, we define the test function

$$v_i = sign(u_i)(|u_i| - k) = \begin{cases} u - k, \text{ if } u > k, \\ 0, \text{ if } u = k, \\ u + k, \text{ if } u < k. \end{cases}$$

Then, $u_i = v_i + k\,\text{sign}(u_i)$ and $\frac{\partial u_i}{\partial x_j} = \frac{\partial v_i}{\partial x_j}$ in the set $A(k) = \{x \in \Omega; |u(x)| > k\}$, $v_i = 0$ in $\Omega - A(k)$ and $v_i \in W_0^{1,q_i}(\Omega)$. By considering the test function v_i and using the Hölder inequality, we get

$$\int_{A(k)} a_i(|\nabla v_i|^{p_i})\, |\nabla v_i|^{p_i} dx = \int_\Omega f_i v_i \, dx \le \left(\int_{A(k)} |v_i|^{q_i^*} dx\right)^{\frac{1}{q_i^*}} \left(\int_{A(k)} |f_i|^{r_i} dx\right)^{\frac{1}{r_i}} |A(k)|^{1-\left(\frac{1}{q_i^*}+\frac{1}{r_i}\right)},$$

where $|A(k)|$ denotes the Lebesgue measure of $A(k)$. Moreover, applying (A_1) and Sobolev inequality we obtain

$$k_2 S \left(\int_{A(k)} |v_i|^{q_i^*} dx\right)^{\frac{q_i-1}{q_i^*}} \le \left(\int_{A(k)} |f_i|^{r_i} dx\right)^{\frac{1}{r_i}} |A(k)|^{1-\left(\frac{1}{q_i^*}+\frac{1}{r_i}\right)}, \tag{2.2}$$

where S is the best constant in the Sobolev inclusion.

Note that if $0 < k < h$, then $A(h) \subset A(k)$ and

$$|A(k)|^{\frac{1}{q_i^*}}(h-k) = \left(\int_{A(h)} (h-k)^{q_i^*} dx\right)^{\frac{1}{q_i^*}} \le \left(\int_{A(h)} |v_i|^{q_i^*} dx\right)^{\frac{1}{q_i^*}} \le \left(\int_{A(k)} |v_i|^{q_i^*} dx\right)^{\frac{1}{q_i^*}}. \tag{2.3}$$

It follows from (2.2) and (2.3) that

$$|A(k)| \leq \frac{1}{(h-k)^{q_i^*}} \frac{1}{(k_2 S)^{\frac{q_i^*}{q_i-1}}} \|f_i\|_{r_i}^{\frac{q_i^*}{q_i-1}} |A(k)|^{\frac{q_i^*}{q_i-1}\left[1-\left(\frac{1}{q_i^*}+\frac{1}{r_i}\right)\right]}.$$

Since $r_i > \frac{q_i^*}{q_i^*-q_i}$ we have $\beta := \frac{q_i^*}{q_i-1}\left[1-\left(\frac{1}{q_i^*}+\frac{1}{r_i}\right)\right] > 1$. Therefore, if we define

$$\phi(h) = |A(h)|, \qquad \alpha = q_i^*, \qquad \beta := \frac{q_i^*}{q_i-1}\left[1-\left(\frac{1}{q_i^*}+\frac{1}{r_i}\right)\right], \qquad k_0 = 0,$$

we obtain that ϕ is a nonincreasing function and

$$\phi(h) \leq \frac{C(\phi(k))^\beta}{(h-k)^\alpha}, \quad \text{for all } h > k > 0.$$

By Lemma 2.4, we conclude that $\phi(d) = 0$ for $d = C\frac{\|f_i\|_{r_i}^{\frac{1}{q_i-1}}}{(k_2 S)^{\frac{1}{q_i-1}}}|\Omega|^{\frac{\beta-1}{\alpha}}$ and hence,

$$\|u_i\|_\infty \leq C\frac{\|f_i\|_{r_i}^{\frac{1}{q_i-1}}}{(k_2 S)^{\frac{1}{q_i-1}}}|\Omega|^{\frac{\beta-1}{\alpha}},$$

where β, α, S and C are constants that do not depend on f_i and u_i. □

Regarding the regularity of the solution of (2.1) the next result hold and the proof can be done repeating the same arguments of [10, Theorem 1].

Lemma 2.6. *Fix $h_i \in L^\infty(\Omega)$, for all $i = 1, 2$, and consider $u_i \in W_0^{1,q_i}(\Omega) \cap L^\infty(\Omega)$, with $2 \leq p_i \leq q_i < N$, satisfying the problem*

$$\begin{cases} -\operatorname{div}(a_i(|\nabla u_i|^{p_i})|\nabla u_i|^{p_i-2}\nabla u_i) = h_i \text{ in } \Omega, \\ u_i = 0 \text{ on } \partial\Omega, \end{cases}$$

Then, $u_i \in C^{1,\alpha}(\overline{\Omega})$, for some $\alpha \in (0,1)$.

The following result can be found in [12, Lemma 2.6]. The proof is presented for the completeness of the paper.

Lemma 2.7. *Let $\phi, \omega > 0$ be any functions on $C_0^1(\overline{\Omega})$. If $\frac{\partial \phi}{\partial \nu} > 0$ in $\partial\Omega$, where ν is the inwards normal to $\partial\Omega$, then there exists $C > 0$ such that*

$$\frac{\phi(x)}{\omega(x)} \geq C > 0, \quad \textit{for all } x \in \Omega.$$

Proof. For $\delta > 0$ sufficiently small, we consider the following set

$$\Omega_\delta = \{x \in \Omega;\ \operatorname{dist}(x, \partial\Omega) < \delta\}.$$

Since $\phi, \omega > 0$ in Ω and $\Omega \setminus \Omega_\delta$ is compact, there exists $m > 0$ such that

$$\frac{\phi(x)}{\omega(x)} \geq m, \quad \text{for all } x \in \Omega \setminus \Omega_\delta. \tag{2.4}$$

It follows from $\frac{\partial \phi}{\partial \nu} > 0$ in $\partial\Omega$ that $\frac{\partial \phi}{\partial \eta} < 0$, where η is the outwards normal to $\partial\Omega$. Furthermore, since $\Omega \subset \mathbb{R}^n$ is bounded domain, then $\partial\Omega$ is a compact set and consequently, there exists $C_1 < 0$ satisfying

$$\frac{\partial \phi(x)}{\partial \eta} \leq C_1, \quad \text{for all } x \in \overline{\Omega}_\delta.$$

Since $\omega \in C_0^1(\overline{\Omega})$, there exists $C_2 > 0$ such that

$$\left| \frac{\partial \omega(x)}{\partial \eta} \right| \leq C_2, \quad \text{for all } x \in \overline{\Omega}_\delta.$$

Consider $K_0 = \inf_{\overline{\Omega}_\delta} \frac{\partial \omega}{\partial \eta} < 0$ and define the function $H(x) = \alpha\omega(x) - \phi(x)$, for all $x \in \overline{\Omega}_\delta$ and $\alpha \in \mathbb{R}$ to be chosen later. Since $0 < \alpha < \frac{C_1}{K_0}$ we obtain

$$\frac{\partial H(x)}{\partial \eta} = \alpha \frac{\partial \omega(x)}{\partial \eta} - \frac{\partial \phi(x)}{\partial \eta} \geq \alpha K_0 - C_1 > 0, \quad \text{for all } x \in \overline{\Omega}_\delta.$$

Now, fix $x \in \overline{\Omega}_\delta$ and consider the function

$$f(x) = H(x + s\eta), \quad \text{for all } s \in \mathbb{R}.$$

For every $x \in \overline{\Omega}_\delta$, we choose an unique $\widetilde{x} \in \overline{\Omega}_\delta$ so that there exists $\widehat{s} > 0$ such that $x + \widehat{s}\eta = \widetilde{x} \in \partial\Omega$. Hence, since $H(\partial\Omega) \equiv 0$ we have

$$f(\widehat{s}) = H(x + \widehat{s}\eta) = H(\widetilde{x}) = 0.$$

Applying the Mean Value Theorem, there exists $\xi \in (0, \widehat{s})$ such that

$$f(\widehat{s}) - f(0) = f'(\xi)(\widehat{s} - 0),$$

which implies that

$$-H(x) = \frac{\partial H}{\partial \eta}(x + \xi\eta)\widehat{s} > 0 \quad \text{in } \overline{\Omega}_\delta.$$

Therefore, $H(x) \leq 0$ for all $x \in \overline{\Omega}_\delta$ and hence,

$$\alpha\omega(x) - \phi(x) \leq 0, \quad \text{for all } x \in \overline{\Omega}_\delta,$$

which result in

$$\alpha\omega(x) \leq \phi(x), \quad \text{for all } x \in \overline{\Omega}_\delta.$$

Thus,

$$\frac{\phi(x)}{\omega(x)} \geq \alpha > 0, \quad \text{for all } x \in \overline{\Omega}_\delta. \tag{2.5}$$

By virtue of (2.4) and (2.5), we conclude that there exists $C > 0$ so that

$$\frac{\phi(x)}{\omega(x)} \geq C, \quad \text{for all } x \in \Omega. \qquad \square$$

3 Proof of Theorem 1.6

In the proof of Theorem 1.6, we combine the sub-supersolution method with minimization arguments. Before this, we need of the following definition.

We say that $(\underline{u},\underline{v}),(\overline{u},\overline{v}) \in X$ form a pair of sub and supersolution for system (1.1) if $\underline{u},\overline{u},\underline{v},\overline{v} \in L^\infty(\Omega)$ with

(a) $\underline{u} \le \overline{u}$, $\underline{v} \le \overline{v}$ in Ω and $\underline{u} = 0 \le \overline{u}$, $\underline{v} = 0 \le \overline{v}$ on $\partial\Omega$,

(b) Given $(\phi,\varphi) \in X$, with $\phi,\varphi \ge 0$, we have

$$\begin{cases} \displaystyle\int_\Omega a_1(|\nabla \underline{u}|^{p_1})|\nabla \underline{u}|^{p_1-2}\nabla \underline{u}\nabla\phi\, dx \le \int_\Omega h_1(x)\underline{u}^{-\gamma_1}\phi\, dx + \int_\Omega F_u(x,\underline{u},w)\phi\, dx, \quad \text{for all } w \in [\underline{v},\overline{v}], \\ \displaystyle\int_\Omega a_2(|\nabla \underline{v}|^{p_2})|\nabla \underline{v}|^{p_2-2}\nabla \underline{v}\nabla\varphi\, dx \le \int_\Omega h_2(x)\underline{v}^{-\gamma_2}\varphi\, dx + \int_\Omega F_v(x,w,\underline{v})\varphi\, dx, \quad \text{for all } w \in [\underline{u},\overline{u}] \end{cases}$$

and

$$\begin{cases} \displaystyle\int_\Omega a_1(|\nabla \overline{u}|^{p_1})|\nabla \overline{u}|^{p_1-2}\nabla \overline{u}\ \nabla\phi\, dx \ge \int_\Omega h_1(x)\overline{u}^{-\gamma_1}\phi\, dx + \int_\Omega F_u(x,\overline{u},w)\phi\, dx, \quad \text{for all } w \in [\underline{v},\overline{v}] \\ \displaystyle\int_\Omega a_2(|\nabla \overline{v}|^{p_2})|\nabla \overline{v}|^{p_2-2}\nabla \overline{v}\ \nabla\varphi\, dx \ge \int_\Omega h_2(x)\overline{v}^{-\gamma_2}\varphi,\ dx + \int_\Omega F_v(x,w,\overline{v})\varphi\, dx, \quad \text{for all } w \in [\underline{u},\overline{u}]. \end{cases}$$

The next result is essential to provide the existence of a subsolution and a supersolution for system (1.1) whenever we fix the value of $\|h_i\|_\infty$ with $i = 1,2$.

Lemma 3.1. *Suppose that (H), (F_1) and (A_1)–(A_2) are satisfied. If $\|h_i\|_\infty$ is small, for $i = 1,2$, then there exist $\underline{u},\overline{u},\underline{v},\overline{v} \in C^{1,\alpha}(\overline{\Omega})$, for some $\alpha \in (0,1)$, such that*

i) $h_1\underline{u}^{-\gamma_1}, h_2\underline{v}^{-\gamma_2} \in L^\infty(\Omega)$, $\|\underline{u}\|_\infty \le \delta$ and $\|\underline{v}\|_\infty \le \delta$, where δ is the constant that appeared in the hypothesis (F_1);

ii) $\|\overline{u}\|_\infty \le \delta$ and $\|\overline{v}\|_\infty \le \delta$, where δ is the constant that appeared in the hypothesis (F_1);

iii) $0 < \underline{u}(x) \le \overline{u}(x)$ a.e. in Ω and $0 < \underline{v}(x) \le \overline{v}(x)$ a.e. in Ω;

iv) $(\underline{u},\underline{v})$ is a subsolution and $(\overline{u},\overline{v})$ is a supersolution for system (1.1).

Proof. From Lemma 2.1 and maximum principle, there exists an unique positive solution $0 < \underline{u} \in W_0^{1,q_1}(\Omega)$ satisfying the problem below

$$\begin{cases} -\mathrm{div}(a_1(|\nabla \underline{u}|^{p_1})|\nabla \underline{u}|^{p_1-2}\nabla \underline{u}) = h_1(x) \text{ in } \Omega, \\ \underline{u} = 0 \text{ on } \partial\Omega. \end{cases} \tag{3.1}$$

Similary, there exists an unique positive solution $0 < \underline{v} \in W_0^{1,q_2}(\Omega)$ satisfying

$$\begin{cases} -\mathrm{div}(a_2(|\nabla \underline{v}|^{p_2})|\nabla \underline{v}|^{p_2-2}\nabla \underline{v}) = h_2(x) \text{ in } \Omega, \\ \underline{v} = 0 \text{ on } \partial\Omega. \end{cases} \tag{3.2}$$

Since $h_1, h_2 \in L^\infty(\Omega)$, it follows from Lemma 2.5 that $\underline{u}, \underline{v} \in L^\infty(\Omega)$ and there exist $C_1, C_2 > 0$ such that

$$\|\underline{u}\|_\infty \le C_1 \|h_1\|_\infty^{\frac{1}{p_1-1}} \quad \text{and} \quad \|\underline{v}\|_\infty \le C_2 \|h_2\|_\infty^{\frac{1}{p_2-1}},$$

where C_1 and C_2 are constants that does not depend on h_i, $\underline{u}$ and $\underline{v}$. Therefore, we may choose $\|h_i\|_\infty$ sufficiently small, with $i = 1,2$, so that

$$\|\underline{u}\|_\infty \le \delta < \frac{1}{2} \quad \text{and} \quad \|\underline{v}\|_\infty \le \delta < \frac{1}{2}.$$

Moreover, from Lemma 2.6 we have $\underline{u}, \underline{v} \in C^{1,\alpha}(\overline{\Omega})$, for some $\alpha \in (0,1)$. Thus, by virtue of Lemmas 2.3 and 2.7, there exist $C_3, C_4 > 0$ such that

$$\frac{\underline{u}(x)^{-\gamma_1}}{\phi_0(x)^{-\gamma_1}} \le C_3^{-\gamma_1} \quad \text{and} \quad \frac{\underline{v}(x)^{-\gamma_2}}{\phi_0(x)^{-\gamma_2}} \le C_4^{-\gamma_2}, \quad \text{for all } x \in \Omega.$$

Therefore, by (H) we get

$$|h_1 \underline{u}^{-\gamma_1}| \le C_3^{-\gamma_1} \|h_1 \phi_0^{-\gamma_1}\|_\infty \quad \text{and} \quad |h_2 \underline{v}^{-\gamma_2}| \le C_4^{-\gamma_2} \|h_2 \phi_0^{-\gamma_2}\|_\infty, \tag{3.3}$$

implying that $h_1 \underline{u}^{-\gamma_1}, h_2 \underline{v}^{-\gamma_2} \in L^\infty(\Omega)$, which ends the proof of condition (i).

In order to prove (ii), we invoke Lemma 2.1 and maximum principle once again to claim that there exists an unique positive solution $0 < \overline{u} \in W_0^{1,q_1}(\Omega)$ satisfying

$$\begin{cases} -\mathrm{div}(a_1(|\nabla \overline{u}|^{p_1})|\nabla \overline{u}|^{p_1-2}\nabla \overline{u}) = h_1(x)\underline{u}^{-\gamma_1} \text{ in } \Omega, \\ \overline{u} = 0 \text{ on } \partial\Omega, \end{cases} \tag{3.4}$$

and there exists an unique positive solution $0 < \overline{v} \in W_0^{1,q_2}(\Omega)$ satisfying

$$\begin{cases} -\mathrm{div}(a_2(|\nabla \overline{v}|^{p_2})|\nabla \overline{v}|^{p_2-2}\nabla \overline{v}) = h_2(x)\underline{v}^{-\gamma_2} \text{ in } \Omega, \\ \overline{v} = 0 \text{ on } \partial\Omega. \end{cases} \tag{3.5}$$

Since $h_1 \underline{u}^{-\gamma_1}, h_2 \underline{v}^{-\gamma_2} \in L^\infty(\Omega)$, we use Lemma 2.5 to obtain $\overline{u}, \overline{v} \in L^\infty(\Omega)$ and hence, from Lemma 2.6 we obtain $\overline{u}, \overline{v} \in C^{1,\alpha}(\overline{\Omega})$, for some $\alpha \in (0,1)$. Furthermore, note that using (3.3) we have

$$\|\overline{u}\|_\infty \le C_1^* \|h_1 \underline{u}^{-\gamma_1}\|_\infty^{\frac{1}{p_1-1}} \le C_1^* \|h_1\|_\infty^{\frac{1}{p_1-1}} C_3^{-\gamma_1\left(\frac{1}{p_1-1}\right)} \|\phi_0\|_\infty^{-\gamma_1\left(\frac{1}{p_1-1}\right)}$$

and

$$\|\overline{v}\|_\infty \le C_2^* \|h_2 \underline{v}^{-\gamma_2}\|_\infty^{\frac{1}{p_2-1}} \le C_2^* \|h_2\|_\infty^{\frac{1}{p_2-1}} C_4^{-\gamma_2\left(\frac{1}{p_2-1}\right)} \|\phi_0\|_\infty^{-\gamma_1\left(\frac{1}{p_1-1}\right)}.$$

So, choosing $\|h_i\|_\infty$ sufficiently small, with $i = 1,2$, we obtain

$$\|\overline{u}\|_\infty \le \delta < \frac{1}{2} \quad \text{and} \quad \|\overline{v}\|_\infty \le \delta < \frac{1}{2}.$$

Now, since $\|\underline{u}\|_\infty$ and $\|\underline{v}\|_\infty$ are small it follows from (3.1), (3.2), (3.4) and (3.5) that

$$\begin{aligned} -\mathrm{div}(a_1(|\nabla \overline{u}|^{p_1})|\nabla \overline{u}|^{p_1-2}\nabla \overline{u}) &= h_1(x)\underline{u}^{-\gamma_1} \ge h_1(x)\|\underline{u}\|_\infty^{-\gamma_1} \ge h_1(x) \\ &= -\mathrm{div}(a_1(|\nabla \underline{u}|^{p_1})|\nabla \underline{u}|^{p_1-2}\nabla \underline{u}) \end{aligned}$$

and

$$-\operatorname{div}(a_2(|\nabla\overline{v}|^{p_2})|\nabla\overline{v}|^{p_2-2}\nabla\overline{v}) = h_2(x)\underline{v}^{-\gamma_2} \geq h_2(x)\|\underline{v}\|_\infty^{-\gamma_2} \geq h_2(x)$$
$$= -\operatorname{div}(a_2(|\nabla\underline{v}|^{p_2})|\nabla\underline{v}|^{p_2-2}\nabla\underline{v}).$$

Therefore, applying the Weak Comparison Principle for the $p\&q$-Laplacian operator we conclude that

$$0 < \underline{u}(x) \leq \overline{u}(x) \quad \text{a.e. in } \Omega \quad \text{and} \quad 0 < \underline{v}(x) \leq \overline{v}(x) \quad \text{a.e. in } \Omega,$$

which proves (iii).

Our final task is to check that the condition (iv) holds. Indeed, we invoke (F_1), (i), (3.1) and (3.2) to obtain

$$-\operatorname{div}(a_1(|\nabla\underline{u}|^{p_1})|\nabla\underline{u}|^{p_1-2}\nabla\underline{u}) - h_1(x)\underline{u}^{-\gamma_1} - F_{\underline{u}}(x,\underline{u},v)$$
$$\leq 2h_1(x) - h_1(x)\underline{u}^{-\gamma_1} \leq h_1(x)(2 - \|\underline{u}\|_\infty^{-\gamma_1}) \leq 0$$

and

$$-\operatorname{div}(a_2(|\nabla\underline{v}|^{p_2})|\nabla\underline{v}|^{p_2-2}\nabla\underline{v}) - h_2(x)\underline{v}^{-\gamma_2} - F_{\underline{v}}(x,u,\underline{v}),$$
$$\leq 2h_2(x) - h_2(x)\underline{v}^{-\gamma_2} \leq h_2(x)(2 - \|\underline{v}\|_\infty^{-\gamma_2}) \leq 0,$$

which implies that $(\underline{u},\underline{v})$ is a subsolution for system (1.1). Finally, we use (F_1), (ii), (iii), (3.4) and (3.5) to get

$$-\operatorname{div}(a_1(|\nabla\overline{u}|^{p_1})|\nabla\overline{u}|^{p_1-2}\nabla\overline{u}) - h_1(x)\overline{u}^{-\gamma_1} - F_{\overline{u}}(x,\overline{u},v) \geq h_1(x)(\underline{u}^{-\gamma_1} - \overline{u}^{-\gamma_1}) \geq 0$$

and

$$-\operatorname{div}(a_2(|\nabla\overline{v}|^{p_2})|\nabla\overline{v}|^{p_2-2}\nabla\overline{v} - h_2(x)\overline{v}^{-\gamma_2} - F_{\overline{v}}(x,u,\overline{v}) \geq h_2(x)(\underline{v}^{-\gamma_2} - \overline{v}^{-\gamma_2}) \geq 0,$$

which shows that $(\overline{u},\overline{v})$ is a supersolution for system (1.1). □

Following the same idea in [4] (see also [3]), we introduce the truncation operators $T : W_0^{1,q_1}(\Omega) \to L^\infty(\Omega)$ and $S : W_0^{1,q_2}(\Omega) \to L^\infty(\Omega)$ given by

$$Tu(x) = \begin{cases} \overline{u}(x), & \text{if } u(x) > \overline{u}(x) \\ u(x), & \text{if } \underline{u}(x) \leq u(x) \leq \overline{u}(x) \\ \underline{u}(x), & \text{if } u(x) < \underline{u}(x) \end{cases} \tag{3.6}$$

and

$$Sv(x) = \begin{cases} \overline{v}(x), & \text{if } v(x) > \overline{v}(x) \\ v(x), & \text{if } \underline{v}(x) \leq v(x) \leq \overline{v}(x) \\ \underline{v}(x), & \text{if } v(x) < \underline{v}(x). \end{cases} \tag{3.7}$$

It is well that the truncation operators T and S are continuous and bounded. Now, we consider the following functions

$$G_u(x,u,v) = h_1(x)(Tu)^{-\gamma_1} + F_u(x,Tu,Sv) \tag{3.8}$$

and

$$G_v(x,u,v) = h_2(x)(Sv)^{-\gamma_2} + F_v(x,Tu,Sv) \tag{3.9}$$

and the auxiliary problem

$$\begin{cases} -\mathrm{div}(a_1(|\nabla u|^{p_1})|\nabla u|^{p_1-2}\nabla u) = G_u(x,u,v) \text{ in } \Omega, \\ -\mathrm{div}(a_2(|\nabla v|^{p_2})|\nabla v|^{p_2-2}\nabla v) = G_v(x,u,v) \text{ in } \Omega, \\ u, v > 0 \text{ in } \Omega, \\ u = v = 0 \text{ on } \partial\Omega. \end{cases} \tag{3.10}$$

Define the energy functional $\Phi : X \to \mathbb{R}$ associated with problem (3.10) by

$$\Phi(u,v) = \frac{1}{p_1}\int_\Omega A_1(|\nabla u|^{p_1})dx + \frac{1}{p_2}\int_\Omega A_2(|\nabla v|^{p_2})dx - \int_\Omega G(x,u,v)dx, \qquad \forall (u,v) \in X,$$

where $G(x,s,t) = \int_0^s G_\xi(x,\xi,t)d\xi + \int_0^t G_\xi(x,s,\xi)d\xi$.

It follows from Lemma 3.1 (i)–(iii), (3.8), (3.9) and (F_1) that

$$|G_u(x,u,v)| \le K_1 \quad \text{a.e. in } \Omega, \text{ for some } K_1 > 0, \forall (u,v) \in X. \tag{3.11}$$

Similarly,

$$|G_v(x,u,v)| \le K_2 \quad \text{a.e. in } \Omega, \text{ for some } K_2 > 0, \forall (u,v) \in X. \tag{3.12}$$

Consequently, we use (A_1) to show that the functional Φ is well defined and it is of class C^1 on Sobolev space X with

$$\begin{aligned} \Phi'(u,v)(\phi,\varphi) = & \int_\Omega \left[a_1(|\nabla u|^{p_1})|\nabla u|^{p_1-2}\nabla u\nabla\phi + a_2(|\nabla v|^{p_2})|\nabla v|^{p_2-2}\nabla v\nabla\varphi\right] dx \\ & - \int_\Omega G_u(x,u,v)\phi\, dx - \int_\Omega G_v(x,u,v)\varphi\, dx, \forall (u,v), (\phi,\varphi) \in X. \end{aligned}$$

Next, consider

$$M = \{(u,v) \in X; \underline{u} \le u \le \overline{u} \text{ a.e. in } \Omega \text{ and } \underline{v} \le v \le \overline{v} \text{ a.e. in } \Omega\}.$$

We claim that Φ is bounded from below in M. Indeed, for all $(u,v) \in X$, we use (A_1), (3.11), (3.12) and continuous embedding $W_0^{1,q_i}(\Omega) \hookrightarrow L^{1,q_i}(\Omega)$, for $i = 1,2$, to obtain that Φ is coercive in M. Moreover, since (A_3) holds and $G_u, G_v \in L^\infty(\Omega)$ we have that Φ is weak lower semi-continuous on M. Thus, as M is closed and convex in X, we use [15, Theorem 1.2] to conclude that Φ is bounded from below in M and attains it is infimum at a point $(u,v) \in M$.

Using the same arguments as in the proof of [15, Theorem 2.4], we see that this minimum point (u,v) is a weak solution of problem (3.10). Indeed, for all $\phi, \varphi \in C_0^\infty(\Omega)$ and $\varepsilon > 0$, let the functions $u_\varepsilon, v_\varepsilon \in M$ be given by

$$u_\varepsilon(x) = \begin{cases} \overline{u}(x), & u(x) + \varepsilon\phi(x) > \overline{u}(x) \\ u(x) + \varepsilon\phi(x), & \underline{u}(x) \le u(x) + \varepsilon\phi(x) \le \overline{u}(x) \\ \underline{u}(x), & u(x) + \varepsilon\phi(x) < \underline{u}(x) \end{cases}$$

and

$$v_\varepsilon(x) = \begin{cases} \overline{v}(x), & v(x) + \varepsilon\varphi(x) > \overline{v}(x) \\ v(x) + \varepsilon\varphi(x), & \underline{v}(x) \le v(x) + \varepsilon\varphi(x) \le \overline{v}(x) \\ \underline{v}(x), & v(x) + \varepsilon\varphi(x) < \underline{v}(x). \end{cases}$$

The functions u_ε and v_ε can be written as

$$u_\varepsilon = (u+\varepsilon\phi) - (\overline{\phi}_\varepsilon - \underline{\phi}_\varepsilon) \in M \quad \text{and} \quad v_\varepsilon = (v+\varepsilon\varphi) - (\overline{\varphi}_\varepsilon - \underline{\varphi}_\varepsilon) \in M,$$

where $\overline{\phi}_\varepsilon = \max\{0, u+\varepsilon\phi - \overline{u}\} \geq 0$, $\underline{\phi}_\varepsilon = -\min\{0, u+\varepsilon\phi - \underline{u}\} \geq 0$, $\overline{\varphi}_\varepsilon = \max\{0, v+\varepsilon\varphi - \overline{v}\} \geq 0$ and $\underline{\varphi}_\varepsilon = -\min\{0, v+\varepsilon\varphi - \underline{v}\} \geq 0$.

Note that $\overline{\phi}_\varepsilon, \underline{\phi}_\varepsilon \in W_0^{1,q_1}(\Omega) \cap L^\infty(\Omega)$, $\overline{\varphi}_\varepsilon, \underline{\varphi}_\varepsilon \in W_0^{1,q_2}(\Omega) \cap L^\infty(\Omega)$ and Φ is differentiable in direction $(u_\varepsilon - u, v_\varepsilon - v)$. Since $(u,v) \in M$ minimizes the functional Φ in M, then

$$0 \leq \Phi'(u,v)(u_\varepsilon - u, v_\varepsilon - v) = \varepsilon\Phi'(u,v)(\phi,\varphi) - \Phi'(u,v)(\overline{\phi}_\varepsilon, \overline{\varphi}_\varepsilon) + \Phi'(u,v)(\underline{\phi}_\varepsilon, \underline{\varphi}_\varepsilon).$$

Thus,

$$\Phi'(u,v)(\phi,\varphi) \geq \frac{1}{\varepsilon}\left[\Phi'(u,v)(\overline{\phi}_\varepsilon, \overline{\varphi}_\varepsilon) - \Phi'(u,v)(\underline{\phi}_\varepsilon, \underline{\varphi}_\varepsilon)\right]. \tag{3.13}$$

Now, since $(\overline{u}, \overline{v})$ is a supersolution to system (1.1), we obtain

$$\begin{aligned}
\Phi'(u,v)(\overline{\phi}_\varepsilon, \overline{\varphi}_\varepsilon) &= \Phi'(\overline{u},\overline{v})(\overline{\phi}_\varepsilon, \overline{\varphi}_\varepsilon) + \left[\Phi'(u,v) - \Phi'(\overline{u},\overline{v})\right](\overline{\phi}_\varepsilon, \overline{\varphi}_\varepsilon) \\
&\geq \left[\Phi'(u,v) - \Phi'(\overline{u},\overline{v})\right](\overline{\phi}_\varepsilon, \overline{\varphi}_\varepsilon) \\
&= \int_{\Omega_\varepsilon} \left[a_1(|\nabla u|^{p_1})|\nabla u|^{p_1-2}\nabla u - a_1(|\nabla \overline{u}|^{p_1})|\nabla \overline{u}|^{p_1-2}\nabla \overline{u}\right] \nabla(u+\varepsilon\phi - \overline{u})dx \\
&\quad + \int_{\Omega_\varepsilon} \left[a_2(|\nabla v|^{p_2})|\nabla v|^{p_2-2}\nabla v - a_2(|\nabla \overline{v}|^{p_2})|\nabla \overline{v}|^{p_2-2}\nabla \overline{v}\right] \nabla(v+\varepsilon\varphi - \overline{v})dx \\
&\quad - \int_{\Omega_\varepsilon} \left[G_u(x,u,v) - G_{\overline{u}}(x,\overline{u},\overline{v})\right](u+\varepsilon\phi - \overline{u})dx \\
&\quad - \int_{\Omega_\varepsilon} \left[G_v(x,u,v) - G_{\overline{v}}(x,\overline{u},\overline{v})\right](v+\varepsilon\varphi - \overline{v})dx \\
&\geq \varepsilon \int_{\Omega_\varepsilon} \left[a_1(|\nabla u|^{p_1})|\nabla u|^{p_1-2}\nabla u - a_1(|\nabla \overline{u}|^{p_1})|\nabla \overline{u}|^{p_1-2}\nabla \overline{u}\right] \nabla\phi dx \\
&\quad + \varepsilon \int_{\Omega_\varepsilon} \left[a_2(|\nabla v|^{p_2})|\nabla v|^{p_2-2}\nabla v - a_2(|\nabla \overline{v}|^{p_2})|\nabla \overline{v}|^{p_2-2}\nabla \overline{v}\right] \nabla\varphi dx \\
&\quad - \varepsilon \int_{\Omega_\varepsilon} |G_u(x,u,v) - F_{\overline{u}}(x,\overline{u},\overline{v})|\,|\phi| dx - \varepsilon \int_{\Omega_\varepsilon} |G_v(x,u,v) - G_{\overline{v}}(x,\overline{u},\overline{v})|\,|\varphi| dx,
\end{aligned}$$

where $\Omega_\varepsilon = \{x \in \Omega; u(x) + \varepsilon\phi(x) > \overline{u}(x) \geq u(x) \text{ and } v(x) + \varepsilon\varphi(x) > \overline{v}(x) \geq v(x)\}$. Note that $|\Omega_\varepsilon| \to 0$ as $\varepsilon \to 0$. Then, $\Phi'(u,v)(\overline{\phi}_\varepsilon, \overline{\varphi}_\varepsilon) \geq o(\varepsilon)$, where $\frac{o(\varepsilon)}{\varepsilon} \to 0$ as $\varepsilon \to 0$. Similarly, we obtain $\Phi'(u,v)(\underline{\phi}_\varepsilon, \underline{\varphi}_\varepsilon) \leq o(\varepsilon)$ and consequently, by (3.13) we conclude that $\Phi'(u,v)(\phi,\varphi) \geq 0$, for all $\phi, \varphi \in C_0^\infty(\Omega)$. Repeating the above arguments for $(-\phi, -\varphi)$ we have $\Phi'(u,v)(\phi,\varphi) \leq 0$, for all $\phi, \varphi \in C_0^\infty(\Omega)$ and hence, $\Phi'(u,v)(\phi,\varphi) = 0$. Therefore, since $C_0^\infty(\Omega)$ is dense in $W_0^{q_i}$, $\forall i = 1,2$, we prove that $\Phi'(u,v) = 0$, which implies that (u,v) weakly solves (3.10).

Since $(u,v) \in M$ it follows from $G_u(x,u,v) = h_1(x)u^{-\gamma_1} + F_u(x,u,v)$ and $G_v(x,u,v) = h_2(x)v^{-\gamma_2} + F_v(x,u,v)$, for $(u,v) \in [\underline{u}, \overline{u}] \times [\underline{v}, \overline{v}]$, that $(u,v) \in X$ is precisely a positive weak solution for system (1.1).

4 Proof of Theorem 1.7

Let $(\underline{u}, \underline{v}) \in L^\infty(\Omega) \times L^\infty(\Omega)$ be the subsolution of system (1.1). Consider $T : W_0^{1,q_1}(\Omega) \to L^\infty(\Omega)$ and $S : W_0^{1,q_2}(\Omega) \to L^\infty(\Omega)$ the truncation operators given by

$$\widehat{T}u(x) = \begin{cases} u(x), & \text{if } u(x) > \underline{u}(x) \\ \underline{u}(x), & \text{if } u(x) \leq \underline{u}(x), \end{cases} \tag{4.1}$$

$$\widehat{S}v(x) = \begin{cases} v(x), & \text{if } v(x) > \underline{v}(x) \\ \underline{v}(x), & \text{if } v(x) \leq \underline{v}(x). \end{cases} \tag{4.2}$$

and the following functions

$$\widehat{G}_u(x,u,v) = h_1(x)(\widehat{T}u)^{-\gamma_1} + F_u(x, \widehat{T}u, \widehat{S}v) \tag{4.3}$$

and

$$\widehat{G}_v(x,u,v) = h_2(x)(\widehat{S}v)^{-\gamma_2} + F_v(x, \widehat{T}u, \widehat{S}v) \tag{4.4}$$

Next, consider the auxiliary problem

$$\begin{cases} -\text{div}(a_1(|\nabla u|^{p_1})|\nabla u|^{p_1-2}\nabla u) = \widehat{G}_u(x,u,v) \text{ in } \Omega, \\ -\text{div}(a_2(|\nabla v|^{p_2})|\nabla v|^{p_2-2}\nabla v) = \widehat{G}_v(x,u,v) \text{ in } \Omega, \\ u, v > 0 \text{ in } \Omega, \\ u = v = 0 \text{ on } \partial\Omega. \end{cases} \tag{4.5}$$

and define the functional $\widehat{\Phi} : X \to \mathbb{R}$ associated with problem (4.5) by

$$\widehat{\Phi}(u,v) = \frac{1}{p_1}\int_\Omega A_1(|\nabla u|^{p_1})dx + \frac{1}{p_2}\int_\Omega A_2(|\nabla v|^{p_2})dx - \int_\Omega \widehat{G}(x,u,v)dx,$$

where $\widehat{G}(x,s,t) = \int_0^s \widehat{G}_\xi(x,\xi,t)d\xi + \int_0^t \widehat{G}_\xi(x,s,\xi)d\xi$.

Note that, applying (4.3), (4.4), (F_1) and (F_2) we obtain

$$\widehat{G}_u(x,u,v) \leq h_1(x)\underline{u}^{-\gamma_1} + h_1(x)(1 + |u|^{r-1} + |\widehat{S}v|^{r-1}) \quad \text{a.e. in } \Omega, \forall u, v \geq 0 \tag{4.6}$$

Similarly,

$$\widehat{G}_v(x,u,v) \leq h_2(x)\underline{v}^{-\gamma_2} + h_2(x)(1 + |\widehat{T}u|^{r-1} + |v|^{r-1}) \quad \text{a.e. in } \Omega, \forall u, v \geq 0. \tag{4.7}$$

Again, using (A_1) its possible to prove that the functional $\widehat{\Phi} \in C^1(X, \mathbb{R})$ with the following Fréchet derivative

$$\begin{aligned} \widehat{\Phi}'(u,v)(\phi,\varphi) = & \int_\Omega \left[a_1(|\nabla u|^{p_1})|\nabla u|^{p_1-2}\nabla u \nabla\phi + a_2(|\nabla v|^{p_2})|\nabla v|^{p_2-2}\nabla v\nabla\varphi\right] dx \\ & - \int_\Omega \widehat{G}_u(x,u,v)\phi \, dx - \int_\Omega \widehat{G}_v(x,u,v)\varphi \, dx, \end{aligned}$$

for all $(u,v), (\phi,\varphi) \in X$. Furthermore, any critical point of $\widehat{\Phi}$ is a weak solution for auxiliary system (4.5).

In our next result we prove that the functional $\widehat{\Phi}$ satisfies the two geometries of the Mountain Pass Theorem [1].

Lemma 4.1. *Suppose that (H), (F_1)–(F_3) and (A_1)–(A_4) are satisfied. Then, for $\|h_i\|_\infty$ small, $\forall i = 1,2$, $\widehat{\Phi}$ satisfies*

$(\widehat{\Phi}_1)$ *There exist R, α, β with $R > \|(\underline{u}, \underline{v})\|$ and $\alpha < \beta$ such that*

$$\widehat{\Phi}(\underline{u}, \underline{v}) \leq \alpha < \beta \leq \inf_{\partial B_R(0)} \widehat{\Phi}.$$

$(\widehat{\Phi}_2)$ *There exists $e \in X \setminus \overline{B_R(0)}$ such that $\widehat{\Phi}(e) < \beta$.*

Proof. Since $(\underline{u}, \underline{v})$ is a subsolution of (1.1) it follows from Lemma 3.1(i), (F_1), (4.3) and (4.4) that

$$\widehat{G}(x, \underline{u}, \underline{v}) \geq \left[h_1(x)\underline{u}^{-\gamma_1} - h_1(x)\right] \underline{u} + \left[h_2(x)\underline{v}^{-\gamma_2} - h_2(x)\right] \underline{v} \quad \text{a.e. in } \Omega$$

and hence, in view of Lemma 3.1(i) again we obtain $0 < \alpha \in \mathbb{R}$ such that

$$\widehat{\Phi}(\underline{u}, \underline{v}) \leq \frac{1}{p_1} \int_\Omega A_1(|\nabla \underline{u}|^{p_1})\, dx + \frac{1}{p_2} \int_\Omega A_2(|\nabla \underline{v}|^{p_2})\, dx \equiv \alpha. \tag{4.8}$$

Now, without loss of generality, we can consider $q_1 \leq q_2$. So, using (H), (A_1), (4.6), (4.7), Lemma 3.1 and Sobolev embedding there exist positive constants such that

$$\begin{aligned} \widehat{\Phi}(u, v) \geq \frac{K}{2^{q_1}} \|(u,v)\|^{q_1} &- c_1 \|h_1 \underline{u}^{-\gamma_1}\|_\infty \|(u,v)\| - c_2 \|h_1\|_\infty \, \|(u,v)\| \\ &- c_3 \|h_1\|_\infty \|(u,v)\|^r - \|h_1\|_\infty \int_\Omega |\widehat{S}v|^{r-1} |u| dx - c_4 \|h_2 \underline{v}^{-\gamma_2}\|_\infty \|(u,v)\| \\ &- c_5 \|h_2\|_\infty \|(u,v)\| - c_6 \|h_2\|_\infty \|(u,v)\|^r - \|h_2\|_\infty \int_\Omega |\widehat{T}u|^{r-1} |v| dx, \end{aligned} \tag{4.9}$$

where $K = \min\left\{\frac{\tilde{k_2}}{q_1}, \frac{\tilde{k_2}}{q_2}\right\}$. Note that, invoking Young's inequality and Sobolev embedding we get

$$\begin{aligned} \|h_1\|_\infty \int_\Omega |\widehat{S}v|^{r-1} |u| dx &= \|h_1\|_\infty \int_{v \leq \underline{v}} |\underline{v}|^{r-1} |u| dx + \|h_1\|_\infty \int_{v > \underline{v}} |v|^{r-1} |u| dx \\ &\leq c_7 \|h_1\|_\infty \|\underline{v}\|_\infty^{r-1} \|(u,v)\| + c_8 \|h_1\|_\infty \|(u,v)\|^r + c_9 \|h_1\|_\infty \|(u,v)\|^r \end{aligned}$$

and

$$\begin{aligned} \|h_2\|_\infty \int_\Omega |\widehat{T}u|^{r-1} |v| dx &= \|h_2\|_\infty \int_{u \leq \underline{u}} |\underline{u}|^{r-1} |v| dx + \|h_2\|_\infty \int_{u > \underline{u}} |u|^{r-1} |v| dx \\ &\leq c_{10} \|h_2\|_\infty \|\underline{u}\|_\infty^{r-1} \|(u,v)\| + c_{11} \|h_2\|_\infty \|(u,v)\|^r + c_{12} \|h_2\|_\infty \|(u,v)\|^r. \end{aligned}$$

Thus, taking $\|(u,v)\| = R$ with $R > \max\{1, \|(\underline{u}, \underline{v})\|\}$ and $\|h_i\|_\infty$ sufficiently small, for $i = 1,2$, there exists $0 < \beta \in \mathbb{R}$, with $\beta > \alpha$, such that $\widehat{\Phi}(u,v) \geq \beta$, for all $(u,v) \in \partial B_R(0)$. Hence, the choices of α, β, R and $\|h_i\|_\infty$ combined with inequalities (4.8) and (4.9) result in

$$\widehat{\Phi}(\underline{u}, \underline{v}) \leq \alpha < \beta \leq \inf_{(u,v) \in \partial B_R(0)} \widehat{\Phi},$$

which shows the condition $\widehat{\Phi}_1$.

Now, by definition (4.3) we have

$$\widehat{G}_{s\underline{u}}(x, s\underline{u}, 0) \geq F_{s\underline{u}}(x, s\underline{u}, 0), \quad \text{for all } s \geq 1, \text{ a.e. in } \Omega$$

and invoking (A_1) we obtain

$$\widehat{\Phi}(s\underline{u}, 0) \leq \frac{k_3}{p_1} s^{p_1} \|\underline{u}\|_{1,p_1}^{p_1} + \frac{k_4}{q_1} s^{q_1} \|\underline{u}\|_{1,q_1}^{q_1} - \int_\Omega F(x, s\underline{u}, 0) dx.$$

The hypothesis (F_3) provides $d_1, d_2 > 0$ such that $F(x, s, 0) \geq d_1 s^{\frac{1}{\theta_s}} - d_2$, for all $s \geq \max\{1, s_0\}$, where s_0 is the constant that appeared in (F_3). Then, by Sobolev embedding there exist positive constants $d_3, d_4 > 0$ such that

$$\widehat{\Phi}(s\underline{u}, 0) \leq \frac{k_3}{p_1} s^{p_1} \|\underline{u}\|_{1,p_1}^{p_1} + \frac{k_4}{q_1} s^{q_1} \|\underline{u}\|_{1,q_1}^{q_1} - d_3 s^{\frac{1}{\theta_s}} \|\underline{u}\|^{\frac{1}{\theta_s}} + d_4.$$

Since $2 \leq p_1 \leq q_1 < \frac{1}{\theta_s} < q_1^*$, we conclude that $\widehat{\Phi}(s\underline{u}, 0) \to -\infty$ as $s \to +\infty$. So, we may find $s^* > 0$ with $e = s^*(\underline{u}, 0) \in X$ such that $\|e\| > R$ and $\widehat{\Phi}(e) < \beta$, which satisfies the condition $\widehat{\Phi}_2$. □

Lemma 4.2. *The functional $\widehat{\Phi}$ satisfies the Palais–Smale condition for all $c \in \mathbb{R}$.*

Proof. Consider $(u_n, v_n) \subset X$ a Palais–Smale sequence, i.e.,

$$\widehat{\Phi}(u_n, v_n) \to c \quad \text{and} \quad \widehat{\Phi}'(u_n, v_n) \to 0. \tag{4.10}$$

Thus, for all $n \in \mathbb{N}$ sufficiently large, there exists $C > 0$ such that

$$\widehat{\Phi}(u_n, v_n) - \left[\theta_{u_n} \widehat{\Phi}'(u_n, v_n)(u_n, 0) + \theta_{v_n} \widehat{\Phi}'(u_n, v_n)(0, v_n)\right] \leq C(1 + \|(u_n, v_n)\|).$$

On the other hand, we use (A_1) and (A_4) to obtain

$$\begin{aligned}
&\widehat{\Phi}(u_n, v_n) - \left[\theta_{u_n} \widehat{\Phi}'(u_n, v_n)(u_n, 0) + \theta_{v_n} \widehat{\Phi}'(u_n, v_n)(0, v_n)\right] \\
&\quad \geq \left(\frac{1}{p_1 \mu_1} - \theta_{u_n}\right) \tilde{k}_2 \|u_n\|_{1,q_1}^{q_1} + \left(\frac{1}{p_2 \mu_2} - \theta_{v_n}\right) \tilde{k}_2 \|v_n\|_{1,q_2}^{q_2} \\
&\qquad + \int_\Omega \left[\theta_{u_n} \widehat{G}_{u_n}(x, u_n, v_n) u_n + \theta_{v_n} \widehat{G}_{v_n}(x, u_n, v_n) v_n - \widehat{G}(x, u_n, v_n)\right].
\end{aligned}$$

Therefore, since $\theta_{u_n} < \frac{1}{\mu_1 p_1}$, $\theta_{v_n} < \frac{1}{\mu_2 p_2}$ and $q_1 \leq q_2$, without loss of generality, we have

$$\begin{aligned}
&C + (1 + \|(u_n, v_n)\|) \\
&\quad \geq \frac{\overline{K}}{2^{q_1}} \|(u_n, v_n)\|^{q_1} + \int_\Omega \left[\theta_{u_n} \widehat{G}_{u_n}(x, u_n, v_n) u_n + \theta_{v_n} \widehat{G}_{v_n}(x, u_n, v_n) v_n - \widehat{G}(x, u_n, v_n)\right],
\end{aligned} \tag{4.11}$$

where $\overline{K} = \min\left\{\tilde{k}_2\left(\frac{1}{p_1 \mu_1} - \theta_{u_n}\right), \tilde{k}_2\left(\frac{1}{p_2 \mu_2} - \theta_{v_n}\right)\right\}$.

Considering s_0 and t_0 given in (F_3), it follows from (4.3), (4.4) and (F_3) that there exists $\widehat{C} > 0$ such that

$$
\begin{aligned}
&\int_\Omega \left[\theta_{u_n}\widehat{G}_{u_n}(x,u_n,v_n)u_n + \theta_{v_n}\widehat{G}_{v_n}(x,u_n,v_n)v_n - \widehat{G}(x,u_n,v_n)\right] \\
&\quad \geq \int_\Omega \left[\theta_{u_n}h_1(x)(\widehat{T}_{u_n})^{-\gamma_1}u_n + \theta_{v_n}h_2(x)(\widehat{S}_{v_n})^{-\gamma_2}v_n - \int_0^{u_n} h_1(x)(\widehat{T}_{u_n})^{-\gamma_1} - \int_0^{v_n} h_2(x)(\widehat{S}_{v_n})^{-\gamma_2}\right] - \widehat{C} \\
&\quad = \int_{\{u_n \leq \underline{u}\}\cup\{v_n \leq \underline{v}\}} \left[(\theta_{u_n}-1)\,h_1(x)\underline{u}^{1-\gamma_1} + (\theta_{v_n}-1)\,h_2(x)\underline{v}^{1-\gamma_2}\right] \\
&\qquad + \int_{\{u_n > \underline{u}\}\cup\{v_n > \underline{v}\}} \left[\left(\theta_{u_n} - \frac{1}{1-\gamma_1}\right)h_1(x)u_n^{1-\gamma_1} + \left(\theta_{v_n} - \frac{1}{1-\gamma_2}\right)h_2(x)u_n^{1-\gamma_2}\right] - \widehat{C}. \qquad (4.12)
\end{aligned}
$$

Now, using (A_4), Lemma 3.1(i), (4.11) and (4.12) we consider the following cases below:
Case 1: If $\gamma_1, \gamma_2 > 1$, then there exists $M > 0$ such that

$$
M + C\|(u_n, v_n)\| \geq \frac{\overline{K}}{2^{q_1}}\|(u_n, v_n)\|^{q_1}.
$$

Case 2: If $0 < \gamma_1, \gamma_2 < 1$, we apply Hölder's inequality in (4.12) to obtain

$$
\begin{aligned}
M{+}C\|(u_n,v_n)\| &+ \left(\frac{1}{1-\gamma_1} - \theta_{u_n}\right)\|h_1\|_{1,q_1}^{q_1+(\gamma_1-1)}\|u_n\|_{1,q_1}^{1-\gamma_1} \\
&+ \left(\frac{1}{1-\gamma_2} - \theta_{v_n}\right)\|h_2\|_{1,q_2}^{q_2+(\gamma_2-1)}\|v_n\|_{1,q_2}^{1-\gamma_2} \geq \frac{\overline{K}}{2^{q_1}}\|(u_n, v_n)\|^{q_1},
\end{aligned}
$$

Case 3: If $\gamma_1 > 1$ and $0 < \gamma_2 < 1$, we get

$$
M{+}C\|(u_n,v_n)\| + \left(\frac{1}{1-\gamma_2} - \theta_{v_n}\right)\|h_2\|_{1,q_2}^{q_2+(\gamma_2-1)}\|v_n\|_{1,q_2}^{1-\gamma_2} \geq \frac{\overline{K}}{2^{q_1}}\|(u_n, v_n)\|^{q_1}.
$$

Case 4: If $\gamma_2 > 1$ and $0 < \gamma_1 < 1$, then

$$
M{+}C\|(u_n,v_n)\| + \left(\frac{1}{1-\gamma_1} - \theta_{u_n}\right)\|h_1\|_{1,q_1}^{q_1+(\gamma_1-1)}\|u_n\|_{1,q_1}^{1-\gamma_1} \geq \frac{\overline{K}}{2^{q_1}}\|(u_n, v_n)\|^{q_1}.
$$

Case 5: Making $\gamma_1, \gamma_2 = 1$ in (4.3) and (4.4) we have

$$
M + C\|(u_n, v_n)\| + \|h_1\|_\infty\|u_n\| + \|h_2\|_\infty\|v_n\| \geq \frac{\overline{K}}{2^{q_1}}\|(u_n, v_n)\|^{q_1}.
$$

So, analyzing all cases above, we conclude that (u_n, v_n) is a bounded sequence in X. Thus, up to subsequence, there exists $(u, v) \in X$ such that

$$
\begin{cases}
u_n \rightharpoonup u_2 \text{ in } W_0^{1,q_1}(\Omega), \\
u_n \to u_2 \text{ in } L^s(\Omega),\ 1 \leq s < q_1^*, \\
u_n(x) \to u_2(x) \text{ a.e. in } \Omega
\end{cases} \qquad (4.13)
$$

and

$$
\begin{cases}
v_n \rightharpoonup v_2 \text{ in } W_0^{1,q_1}(\Omega), \\
v_n \to v_2 \text{ in } L^t(\Omega),\ 1 \leq t < q_2^*, \\
v_n(x) \to v_2(x) \text{ a.e. in } \Omega.
\end{cases} \qquad (4.14)
$$

Using (A_2), Lemma 4.1(i), (4.10), (4.13) and (4.14) we can argue as in [5, Lemma 1] to obtain

$$C_{q_1}\|u_n-u\|_{1,q_1}^{q_1}+C_{q_2}\|v_n-v\|_{1,q_2}^{q_2} \leq \int_\Omega \left[\widehat{G}_{u_n}(x,u_n,v_n)(u_n-u)+\widehat{G}_{v_n}(x,u_n,v_n)(v_n-v)\right]dx. \quad (4.15)$$

Moreover, we invoke (4.6), (4.7), (4.13), (4.14) and Lebesgue's Dominated Convergence Theorem to get

$$\int_\Omega \left[\widehat{G}_{u_n}(x,u_n,v_n)(u_n-u)+\widehat{G}_{v_n}(x,u_n,v_n)(v_n-v)\right]dx \to 0, \quad \text{as } n\to+\infty. \quad (4.16)$$

Note that, without loss of generality, we can consider $q_1 \geq q_2$. It follows from (4.15) and (4.16) that $(u_n,v_n)\to(u,v)$ in X. □

Next, let $(\underline{u},\underline{v})$ and $(\overline{u},\overline{v})$ be the subsolution and the supersolution, respectively, of system (1.1) given in Lemma 3.1 and (u_1,v_1) a weak solution of system (1.1) obtained in Theorem 1.6. Invoking Lemmas 4.1 and 4.4, it follows from Mountain Pass Theorem that there exists $(u_2,v_2)\in X$ such that

$$\beta<\widehat{\Phi}(u_2,v_2)=c,$$

where c is the minimax value of $\widehat{\Phi}$. Furthermore, since $G_u(x,u,v)=\widehat{G}_u(x,u,v)$ and $G_v(x,u,v)=\widehat{G}_v(x,u,v)$, for all $(u,v)\in[0,\overline{u}]\times[0,\overline{v}]$, then $\Phi(u,v)=\widehat{\Phi}(u,v)$, for all $(u,v)\in[0,\overline{u}]\times[0,\overline{v}]$. Thus, $\widehat{\Phi}(u_1,v_1)=\inf_M\Phi$, where $(u_1,v_1)\in[\underline{u},\overline{u}]\times[\underline{v},\overline{v}]$ and M is given in the proof of Theorem 1.6. Thus, auxiliary system (4.5) has two positive weak solutions $(u_1,v_1),(u_2,v_2)\in X$ such that

$$\widehat{\Phi}(u_1,v_1)\leq\widehat{\Phi}(\underline{u},\underline{v})\leq\alpha<\beta\leq\widehat{\Phi}(u_2,v_2)=c.$$

Finally, let's show that $\underline{u}\leq u_2$ and $\underline{v}\leq v_2$. Indeed, taking $((\underline{u}-u_2)^+,(\underline{v}-v_2)^+)$ as test function and defining $\{(u_2,v_2)<(\underline{u},\underline{v})\}:=\{x\in\Omega; u_2(x)<\underline{u}(x) \text{ and } v_2(x)<\underline{v}(x)\}$, we have

$$\begin{aligned}\int_\Omega & a_1(|\nabla u_2|^{p_1})|\nabla u_2|^{p_1-2}\nabla u_2\nabla(\underline{u}-u_2)^+dx+\int_\Omega a_2(|\nabla v_2|^{p_2})|\nabla v_2|^{p_2-2}\nabla v_2\nabla(\underline{v}-v_2)^+dx\\ &=\int_{\{u_2<\underline{u}\}}\left[h_1(x)\underline{u}^{-\gamma_1}+F_{u_2}(x,\underline{u},\widehat{S}v_2)\right](\underline{u}-u_2)^+dx\\ &\quad+\int_{\{v_2<\underline{v}\}}\left[h_2(x)\underline{v}^{-\gamma_2}+F_{v_2}(x,\widehat{T}u_2,\underline{v})\right](\underline{v}-v_2)^+dx.\end{aligned}$$

Since $(\underline{u},\underline{v})$ is subsolution for system (1.1), then

$$\int_\Omega a_1(|\nabla\underline{u}|^{p_1})|\nabla\underline{u}|^{p_1-2}\nabla\underline{u}\nabla(\underline{u}-u_2)^+dx-\int_\Omega a_1(|\nabla u_2|^{p_1})|\nabla u_2|^{p_1-2}\nabla u_2\nabla(\underline{u}-u_2)^+dx\leq 0$$

and

$$\int_\Omega a_2(|\nabla\underline{v}|^{p_2})|\nabla\underline{v}|^{p_2-2}\nabla\underline{v}\nabla(\underline{v}-v_2)^+dx-\int_\Omega a_2(|\nabla v_2|^{p_2})|\nabla v_2|^{p_2-2}\nabla v_2\nabla(\underline{v}-u_2)^+dx\leq 0,$$

which implies that $(\underline{u} - u_2)^+ = 0$ and $(\underline{v} - v_2)^+ = 0$. Therefore, we conclude that $0 < \underline{u} \leq u_2$ a.e. in Ω and $0 < \underline{v} \leq v_2$ a.e. in Ω, as claimed. It follows from (4.3) and (4.4) that

$$\widehat{G}_{u_2}(x, u_2, v_2) = h(x)u_2^{-\gamma_1} + F_{u_2}(x, u_2, v_2) \quad \text{in } \Omega$$

and

$$\widehat{G}_{v_2}(x, u_2, v_2) = h(x)v_2^{-\gamma_2} + F_{v_2}(x, u_2, v_2) \quad \text{in } \Omega.$$

Then, (u_1, v_1) and (u_2, v_2) are two positive weak solutions for system (1.1).

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