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Singular Kneser solutions of higher-order quasilinear ordinary differential equations

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Abstract. In this paper we give a new sufficient condition in order that all nontrivial Kneser solutions of the quasilinear ordinary differential equation

$$D(\alpha_n, \alpha_{n-1}, \ldots, \alpha_1)x = (-1)^n p(t)|x|^\beta \operatorname{sgn} x, \quad t \geq a, \tag{1.1}$$

are singular. Here, $D(\alpha_n, \alpha_{n-1}, \ldots, \alpha_1)$ is the nth-order iterated differential operator such that

$$D(\alpha_n, \alpha_{n-1}, \ldots, \alpha_1)x = D(\alpha_n)D(\alpha_{n-1}) \cdots D(\alpha_1)x$$

and, in general, $D(\alpha)$ is the first-order differential operator defined by $D(\alpha)x = (d/dt)\,(|x|^\alpha \operatorname{sgn} x)$ for $\alpha > 0$. In the equation (1.1), the condition $\alpha_1\alpha_2 \cdots \alpha_n > \beta$ is assumed. If $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 1$, then one of the results of this paper yields a well-known theorem of Kiguradze and Chanturia.

Keywords: Kneser solutions, singular solutions, quasilinear equations.

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1 Introduction

For a positive constant α, let $D(\alpha)$ be the first-order differential operator defined by

$$D(\alpha)x = \frac{d}{dt}\left(|x|^\alpha \operatorname{sgn} x\right),$$

and for n positive constants $\alpha_1, \alpha_2, \ldots, \alpha_n$ let $D(\alpha_i, \alpha_{i-1}, \ldots, \alpha_1)$ be the ith-order iterated differential operator defined by

$$D(\alpha_i, \alpha_{i-1}, \ldots, \alpha_1)x = D(\alpha_i)D(\alpha_{i-1}) \cdots D(\alpha_1)x, \quad i = 0, 1, 2, \ldots, n.$$

Here, if $i = 0$, then $D(\alpha_i, \ldots, \alpha_1)x$ is interpreted as x.

In this paper we consider nth-order quasilinear ordinary differential equations of the form

$$D(\alpha_n, \alpha_{n-1}, \ldots, \alpha_1)x = (-1)^n p(t)|x|^\beta \operatorname{sgn} x, \quad t \geq a, \tag{1.1}$$

where it is assumed that

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(a) $n \geq 2$ is an integer;

(b) $\alpha_1, \alpha_2, \ldots, \alpha_n$ and β are positive constants;

(c) $p(t)$ is a continuous function on an interval $[a, \infty)$, and $p(t) \geq 0$ on $[a, \infty)$, and $p(t) \not\equiv 0$ on $[a_1, \infty)$ for any $a_1 \geq a$.

By a solution $x(t)$ of (1.1) on $[a, \infty)$ we mean that

$$D(\alpha_1)x(t), \quad D(\alpha_2)D(\alpha_1)x(t) = D(\alpha_2, \alpha_1)x(t), \ldots,$$
$$D(\alpha_n)D(\alpha_{n-1}) \cdots D(\alpha_1)x(t) = D(\alpha_n, \alpha_{n-1}, \ldots, \alpha_1)x(t)$$

are well-defined and continuous on $[a, \infty)$ and $x(t)$ satisfies (1.1) at every point $t \in [a, \infty)$. A function $x(t)$ is said to be a *Kneser solution* of (1.1) on $[a, \infty)$ if $x(t)$ is a solution of (1.1) on $[a, \infty)$ and satisfies

$$(-1)^i D(\alpha_i, \ldots, \alpha_1)x(t) \geq 0, \quad t \geq a, \quad i = 0, 1, 2, \ldots, n-1. \tag{1.2}$$

To shorten notation, we set

$$D(\alpha_i, \ldots, \alpha_1)x(t) = D_i x(t) \quad \text{for } i = 0, 1, 2, \ldots, n.$$

Then, the equation (1.1) may be expressed as

$$D_n x = (-1)^n p(t)|x|^\beta \operatorname{sgn} x, \quad t \geq a, \tag{1.3}$$

and the condition (1.2) is rewritten in the form

$$(-1)^i D_i x(t) \geq 0, \quad t \geq a, \quad i = 0, 1, 2, \ldots, n-1.$$

Suppose that $x(t)$ is a function on $[a, \infty)$ such that $D(\alpha)x(t)$, $\alpha > 0$, is well-defined and continuous on $[a, \infty)$. It is easily seen that if $D(\alpha)x(t) \geq 0$ [*resp.* $> 0, \leq 0, < 0$] on $[a, \infty)$, then $x(t)$ is increasing [*resp.* strictly increasing, decreasing, strictly decreasing] on $[a, \infty)$.

If $x(t)$ is a nonnegative solution of (1.3) on $[a, \infty)$, then $(-1)^n D_n x(t) = p(t)x(t)^\beta \geq 0$ on $[a, \infty)$. Therefore, if $x(t)$ is a Kneser solution of (1.3) on $[a, \infty)$, then $(-1)^i D_i x(t)$ is (nonnegative and) decreasing on $[a, \infty)$ $(i = 0, 1, 2, \ldots, n-1)$.

Now, for the positive constants $\alpha_1, \alpha_2, \ldots, \alpha_n$ appearing in (1.1), we put

$$\begin{aligned} \mu_n = {} & \alpha_2 + (\alpha_2\alpha_3 + \alpha_3) + (\alpha_2\alpha_3\alpha_4 + \alpha_3\alpha_4 + \alpha_4) \\ & + \cdots + (\alpha_2\alpha_3 \cdots \alpha_n + \alpha_3\alpha_4 \cdots \alpha_n + \cdots + \alpha_{n-1}\alpha_n + \alpha_n), \end{aligned} \tag{1.4}$$

$$\nu_n = \alpha_2\alpha_3 \cdots \alpha_n + \alpha_3\alpha_4 \cdots \alpha_n + \cdots + \alpha_{n-1}\alpha_n + \alpha_n, \tag{1.5}$$

$$\xi_n = \alpha_1 + \alpha_1\alpha_2 + \alpha_1\alpha_2\alpha_3 + \cdots + \alpha_1\alpha_2 \cdots \alpha_{n-1} + \alpha_1\alpha_2 \cdots \alpha_n. \tag{1.6}$$

Very recently, Naito and Usami ([6, Theorem 4.1]) have proved that, for each $A > 0$, the equation (1.1) has at least one Kneser solution $x(t)$ on $[a, \infty)$ such that $x(a) = A$. For the case $\alpha_1\alpha_2 \cdots \alpha_n \leq \beta$, any nontrivial Kneser solution $x(t)$ of (1.1) on $[a, \infty)$ satisfies

$$(-1)^i D_i x(t) > 0 \qquad (t \geq a) \quad \text{for } i = 0, 1, 2, \ldots, n-1$$

([6, the paragraph after the proof of Theorem 5.1]). However, for the case $\alpha_1\alpha_2 \cdots \alpha_n > \beta$, a Kneser solution $x(t)$ of (1.1) on $[a, \infty)$ may be singular in the sense that

$$x(t) > 0 \quad (a \leq t < b) \quad \text{and} \quad x(t) = 0 \quad (t \geq b)$$

for some finite number $b > a$. Such a solution is often said to be a *first kind singular* solution of (1.1). It is known ([6, Theorem 6.1]) that if $\alpha_1\alpha_2\cdots\alpha_n > \beta$ and $p(b) > 0$ $(b > a)$, then (1.1) always has at least one singular Kneser solution $x(t)$ such that

$$\begin{cases} (-1)^i D_i x(t) > 0 \quad (a \le t < b) \quad \text{for } i = 0,1,2,\dots,n-1, \quad \text{and} \\ x(t) = 0 \quad (t \ge b). \end{cases} \tag{1.7}$$

In particular, if $p(t)$ is positive on $[a,\infty)$, then for any b $(> a)$ (1.1) has a singular Kneser solution $x(t)$ which satisfies (1.7). Note that, by putting $x_i = (D_{i-1}x)^{\alpha_i *}$ $(i = 1,2,\dots,n)$, the scalar equation (1.1) is equivalent to the n-dimensional system

$$\begin{cases} x_1' = x_2^{(1/\alpha_2)*}, \\ \quad \vdots \\ x_{n-1}' = x_n^{(1/\alpha_n)*}, \\ x_n' = (-1)^n p(t) x_1^{(\beta/\alpha_1)*}. \end{cases}$$

Then, applying Theorem 1 of Čanturia [2] to this n-dimensional system, we find that if $p(t)$ is positive on $[a,\infty)$, then for any b $(> a)$ there is a' $(a \le a' < b)$ such that (1.1) has a singular Kneser solution which is defined on $[a',\infty)$ and satisfies (1.7) with a replaced by a'. Theorem 6.1 of [6] shows that a' can be taken as $a' = a$.

If $p(t)$ is large enough in a neighborhood of ∞, then *all* nontrivial Kneser solutions of (1.1) on $[a,\infty)$ are singular. In fact, making use of Theorem 2 of Čanturia [2], we have the following theorem.

Theorem A. *Let $\alpha_1\alpha_2\cdots\alpha_n > \beta$. Let ν_n be the number defined by (1.5). If*

$$\liminf_{t\to\infty} t^{\nu_n+1} p(t) > 0, \tag{1.8}$$

then all nontrivial Kneser solutions of (1.1) on $[a,\infty)$ are singular.

A different proof of Theorem A has been given by Naito and Usami [6, Theorem 6.8].

The main purpose of this paper is to show that Theorem A can be generalized as follows.

Theorem 1.1. *Let $\alpha_1\alpha_2\cdots\alpha_n > \beta$. Let μ_n, ν_n and ξ_n be the numbers defined by (1.4), (1.5) and (1.6), respectively. Suppose that there exist $\sigma > 0$ and $\tau > 0$ such that*

$$(\nu_n + 1)\sigma - \mu_n\tau - 1 \ge 0, \tag{1.9}$$

$$\left(\frac{\beta}{\alpha_1\alpha_2\cdots\alpha_n}\nu_n + 1\right)\sigma - \left(\mu_n - \frac{\nu_n\xi_n}{\alpha_1\alpha_2\cdots\alpha_n}\right)\tau - 1 \le 0, \tag{1.10}$$

and either

$$\int_{a^+}^{\infty} s^{-\mu_n\tau + (\nu_n+1)\sigma - 1} p(s)^\sigma ds = \infty \quad (a^+ > \max\{a,0\}), \tag{1.11}$$

or

$$\limsup_{t\to\infty} t^{\mu_n\tau} \int_t^\infty s^{-\mu_n\tau + (\nu_n+1)\sigma - 1} p(s)^\sigma ds > 0. \tag{1.12}$$

Then, all nontrivial Kneser solutions of (1.1) on $[a,\infty)$ are singular.

If $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 1$, then

$$D_i x(t) = x^{(i)}(t) \quad (i = 0, 1, 2, \ldots, n),$$

and so (1.1) is reduced to

$$x^{(n)} = (-1)^n p(t)|x|^\beta \operatorname{sgn} x, \quad t \geq a. \tag{1.13}$$

If $n = 2$ and $\alpha_1 = 1$, $\alpha_2 = \alpha > 0$, then (1.1) is the second-order quasilinear differential equation

$$(|x'|^\alpha \operatorname{sgn} x')' = p(t)|x|^\beta \operatorname{sgn} x, \quad t \geq a. \tag{1.14}$$

Results on the problem of existence and asymptotic behavior of Kneser solutions of (1.13) are summarized and proved in the book of Kiguradze and Chanturia [3]. This problem has also been studied by Mizukami, Naito and Usami [4] for (1.14), and by Naito and Usami [6] for the general equation (1.1).

The proof of Theorem 1.1 is given in the next Section 2. In Section 3, Theorem 1.1 are restated in several ways, and some important corollaries are mentioned.

A function $x(t)$ is said to be a *strongly increasing solution* of the equation

$$D_n x = p(t)|x|^\beta \operatorname{sgn} x, \quad t \geq a, \tag{1.15}$$

on $[a, b)$ $(a < b \leq \infty)$ if $x(t)$ is a nontrivial solution of (1.15) on $[a, b)$ and satisfies

$$D_i x(t) \geq 0 \quad (a \leq t < b) \quad \text{for all} \quad i = 0, 1, 2, \ldots, n-1.$$

Suppose that $x(t)$ is a strongly increasing solution of (1.15) on $[a, b)$, and let $[a, b)$ be the maximal interval of existence of $x(t)$. If b is finite, then $x(t)$ is called *singular*. A singular strongly increasing solution is often said to be a *second kind singular* solution of (1.15). There is a remarkable duality between Kneser solutions of (1.3) and strongly increasing solutions of (1.15) (see [5,6]). In the paper [7] we have established a new sufficient condition in order that all strongly increasing solutions of (1.15) are singular. The present paper corresponds to [7].

2 Proof of Theorem 1.1

Let us begin with the proof of Theorem 1.1.

Proof of Theorem 1.1. The proof is done by contradiction. Suppose that (1.1) has a Kneser solution $x(t)$ on $[a, \infty)$ such that $x(t) > 0$ for $t \geq a$. As mentioned in the preceding section, $(-1)^i D_i x(t)$ is decreasing on $[a, \infty)$ $(i = 0, 1, 2, \ldots, n-1)$. Furthermore, by (1.1), we easily see that

$$(-1)^i D_i x(t) > 0, \quad t \geq a \quad (i = 0, 1, 2, \ldots, n-1). \tag{2.1}$$

Define $\lambda_1, \lambda_2, \ldots, \lambda_{n-1}$ and λ_n by

$$\begin{aligned}
\lambda_1 &= \frac{1}{\nu_n}\alpha_2 \cdots \alpha_n(1 - \sigma + \mu_n\tau) - \big(\alpha_2 + \alpha_2\alpha_3 + \cdots + \alpha_2 \cdots \alpha_{n-1}\alpha_n\big)\tau, \\
\lambda_2 &= \frac{1}{\nu_n}\alpha_3 \cdots \alpha_n(1 - \sigma + \mu_n\tau) - \big(\alpha_3 + \alpha_3\alpha_4 + \cdots + \alpha_3 \cdots \alpha_{n-1}\alpha_n\big)\tau, \\
&\vdots \\
\lambda_{n-1} &= \frac{1}{\nu_n}\alpha_n(1 - \sigma + \mu_n\tau) - \alpha_n\tau, \quad \text{and} \\
\lambda_n &= \sigma,
\end{aligned}$$

where σ and τ are positive constants satisfying (1.9) and (1.10). It is easy to see that

$$\lambda_1 + \lambda_2 + \cdots + \lambda_n = 1, \quad \text{and} \tag{2.2}$$

$$\lambda_i - \alpha_{i+1}\lambda_{i+1} = -\alpha_{i+1}\tau \quad (i = 1, 2, \ldots, n-2). \tag{2.3}$$

We have

$$\lambda_i > 0 \quad (i = 1, 2, \ldots, n). \tag{2.4}$$

To see this, note that the condition (1.10) is rewritten as

$$\frac{\beta}{\alpha_1}\sigma + \tau - \lambda_1 \le 0. \tag{2.5}$$

(The left-hand side of (1.10) multiplied by $(\alpha_2 \cdots \alpha_n)/\nu_n$ is equal to the left-hand side of (2.5).) It follows from (2.5) that

$$\lambda_1 \ge \frac{\beta}{\alpha_1}\sigma + \tau > 0.$$

By induction, (2.3) gives

$$\lambda_{i+1} = \frac{\lambda_i}{\alpha_{i+1}} + \tau > 0 \quad \text{for } i = 1, 2, \ldots, n-2.$$

Obviously, $\lambda_n = \sigma > 0$. Thus we have (2.4).

Next, define the function $y(t)$ by

$$y(t) = x(t)^{\alpha_1}[-D_1x(t)]^{\alpha_2}[D_2x(t)]^{\alpha_3} \cdots [(-1)^{n-1}D_{n-1}x(t)]^{\alpha_n}$$

for $t \ge a$. By (2.1), we have $y(t) > 0$ $(t \ge a)$. It is easy to find that the derivative $y'(t)$ of $y(t)$ is calculated as

$$y'(t) = -\left[\frac{-D_1x(t)}{x(t)^{\alpha_1}} + \frac{D_2x(t)}{[-D_1x(t)]^{\alpha_2}} + \cdots + \frac{(-1)^nD_nx(t)}{[(-1)^{n-1}D_{n-1}x(t)]^{\alpha_n}}\right]y(t), \quad t \ge a. \tag{2.6}$$

As a general inequality we have

$$u_1^{\lambda_1}u_2^{\lambda_2} \cdots u_n^{\lambda_n} \le \lambda_1u_1 + \lambda_2u_2 + \cdots + \lambda_nu_n$$

for $u_i \ge 0$, $\lambda_i > 0$, $\sum_{i=1}^n \lambda_i = 1$ (see, for example, [1, pp. 13–14]). This inequality may be written equivalently as

$$\Lambda v_1^{\lambda_1}v_2^{\lambda_2} \cdots v_n^{\lambda_n} \le v_1 + v_2 + \cdots + v_n \quad \text{with} \quad \Lambda = \lambda_1^{-\lambda_1}\lambda_2^{-\lambda_2} \cdots \lambda_n^{-\lambda_n} \tag{2.7}$$

for $v_i \ge 0$, $\lambda_i > 0$, $\sum_{i=1}^n \lambda_i = 1$. Therefore, by (2.6) and by (2.7) of the case

$$v_i = \frac{(-1)^iD_ix(t)}{[(-1)^{i-1}D_{i-1}x(t)]^{\alpha_i}} \quad (i = 1, 2, \ldots, n),$$

we get

$$\begin{aligned} y'(t) &\le -\Lambda\left[\frac{-D_1x(t)}{x(t)^{\alpha_1}}\right]^{\lambda_1}\left[\frac{D_2x(t)}{[-D_1x(t)]^{\alpha_2}}\right]^{\lambda_2} \cdots \left[\frac{(-1)^nD_nx(t)}{[(-1)^{n-1}D_{n-1}x(t)]^{\alpha_n}}\right]^{\lambda_n} y(t) \\ &= -\Lambda x(t)^{-\alpha_1\lambda_1}[-D_1x(t)]^{\lambda_1-\alpha_2\lambda_2} \cdots [(-1)^{n-2}D_{n-2}x(t)]^{\lambda_{n-2}-\alpha_{n-1}\lambda_{n-1}} \\ &\quad \times [(-1)^{n-1}D_{n-1}x(t)]^{\lambda_{n-1}-\alpha_n\lambda_n}[(-1)^nD_nx(t)]^{\lambda_n}y(t) \end{aligned}$$

for $t \geq a$. Then, on account of (1.3) and (2.3), we see that

$$y'(t) \leq -\Lambda x(t)^{-\alpha_1\lambda_1+\alpha_1\tau+\beta\lambda_n} x(t)^{-\alpha_1\tau}[-D_1x(t)]^{-\alpha_2\tau} \cdots [(-1)^{n-2}D_{n-2}x(t)]^{-\alpha_{n-1}\tau}[(-1)^{n-1}D_{n-1}x(t)]^{-\alpha_n\tau} \times [(-1)^{n-1}D_{n-1}x(t)]^{\alpha_n\tau+\lambda_{n-1}-\alpha_n\lambda_n} p(t)^{\lambda_n} y(t),$$

and, in consequence,

$$y'(t) \leq -\Lambda x(t)^{-\alpha_1\lambda_1+\alpha_1\tau+\beta\sigma}[(-1)^{n-1}D_{n-1}x(t)]^{\alpha_n\tau+\lambda_{n-1}-\alpha_n\sigma} p(t)^{\sigma} y(t)^{1-\tau} \tag{2.8}$$

for $t \geq a$. Since $x(t)$ is decreasing on $[a,\infty)$ and $-\alpha_1\lambda_1 + \alpha_1\tau + \beta\sigma \leq 0$ (see (2.5)), we have

$$x(t)^{-\alpha_1\lambda_1+\alpha_1\tau+\beta\sigma} \geq x(a)^{-\alpha_1\lambda_1+\alpha_1\tau+\beta\sigma}, \quad t \geq a. \tag{2.9}$$

Next, we will claim that

$$\lim_{t\to\infty} t^{\nu_n/\alpha_n}[(-1)^{n-1}D_{n-1}x(t)] = 0, \tag{2.10}$$

or equivalently

$$\varepsilon_{n-1}(t) \equiv t^{\alpha_2\alpha_3\cdots\alpha_{n-1}+\alpha_3\cdots\alpha_{n-1}+\cdots+\alpha_{n-1}+1}[(-1)^{n-1}D_{n-1}x(t)] \to 0 \tag{2.11}$$

as $t \to \infty$. Let $i = 0,1,2,\ldots,n-1$. Since $(-1)^iD_ix(t)$ is positive and decreasing on $[a,\infty)$, the limit

$$\lim_{t\to\infty}(-1)^iD_ix(t) = \ell_i$$

exists and is nonnegative. Assume that $\ell_i > 0$ for some $i = 1,2,\ldots,n-1$. Then it is easy to see that

$$\lim_{t\to\infty} \frac{[(-1)^{i-1}D_{i-1}x(t)]^{\alpha_i}}{t} = -\ell_i < 0.$$

This is a contradiction to the fact that $[(-1)^{i-1}D_{i-1}x(t)]^{\alpha_i}$ is positive on $[a,\infty)$. Hence we have

$$\lim_{t\to\infty}(-1)^iD_ix(t) = 0 \quad \text{for any } i = 1,2,\ldots,n-1, \quad \text{and} \tag{2.12}$$

$$\lim_{t\to\infty} x(t) = \ell_0 \geq 0. \tag{2.13}$$

It follows from (2.13) that

$$x(t)^{\alpha_1} - \ell_0^{\alpha_1} = \int_t^\infty [-D_1x(s)]ds, \quad t \geq a,$$

and so

$$x(t)^{\alpha_1} - \ell_0^{\alpha_1} \geq \int_t^{2t} [-D_1x(s)]ds \geq t[-D_1x(2t)], \quad t \geq a^+. \tag{2.14}$$

Here, a^+ is a number such that $a^+ > \max\{a,0\}$. In the same manner, it follows from (2.12) that

$$[(-1)^iD_ix(t)]^{\alpha_{i+1}} \geq t[(-1)^{i+1}D_{i+1}x(2t)], \quad t \geq a^+, \tag{2.15}$$

for $i = 1,2,\ldots,n-2$. By (2.14) and (2.15), we can check with no difficulty that

$$\begin{aligned}
&[x(t)^{\alpha_1} - \ell_0^{\alpha_1}]^{\alpha_2\alpha_3\cdots\alpha_{n-1}} \\
&\quad \geq t^{\alpha_2\alpha_3\cdots\alpha_{n-1}}(2t)^{\alpha_3\cdots\alpha_{n-1}} \cdots (2^{n-3}t)^{\alpha_{n-1}}(2^{n-2}t)[(-1)^{n-1}D_{n-1}x(2^{n-1}t)] \\
&\quad = 2^{\alpha_3\cdots\alpha_{n-1}} \cdots (2^{n-3})^{\alpha_{n-1}}2^{n-2}t^{\alpha_2\alpha_3\cdots\alpha_{n-1}+\alpha_3\cdots\alpha_{n-1}+\cdots+\alpha_{n-1}+1}[(-1)^{n-1}D_{n-1}x(2^{n-1}t)]
\end{aligned}$$

for $t \geq a^{+}$. Then, by (2.13), it is seen that (2.11) or equivalently (2.10) holds.

According to (2.10), there is $a_1 > a^{+}$ such that

$$(-1)^{n-1} D_{n-1} x(t) \leq t^{-\nu_n/\alpha_n}, \quad t \geq a_1. \tag{2.16}$$

Observe that (1.9) implies

$$\alpha_n \tau + \lambda_{n-1} - \alpha_n \sigma = -\frac{\alpha_n}{\nu_n}[(\nu_n + 1)\sigma - \mu_n \tau - 1] \leq 0,$$

and so (2.16) gives

$$[(-1)^{n-1} D_{n-1} x(t)]^{\alpha_n \tau + \lambda_{n-1} - \alpha_n \sigma} \geq t^{(\nu_n+1)\sigma - \mu_n \tau - 1}, \quad t \geq a_1. \tag{2.17}$$

Then it follows from (2.8), (2.9) and (2.17) that

$$y'(t) \leq -L t^{(\nu_n+1)\sigma - \mu_n \tau - 1} p(t)^{\sigma} y(t)^{1-\tau}, \quad t \geq a_1,$$

where $L = \Lambda x(a)^{-\alpha_1 \lambda_1 + \alpha_1 \tau + \beta \sigma}$ is a positive constant. From this inequality it follows that

$$y(t')^{\tau} - y(t)^{\tau} \leq -\tau L \int_t^{t'} s^{(\nu_n+1)\sigma - \mu_n \tau - 1} p(s)^{\sigma} ds$$

for any t and t' such that $a_1 \leq t \leq t'$. Then, letting $t' \to \infty$, we find that

$$\int_{a_1}^{\infty} s^{(\nu_n+1)\sigma - \mu_n \tau - 1} p(s)^{\sigma} ds < \infty \tag{2.18}$$

and

$$y(t)^{\tau} \geq \tau L \int_t^{\infty} s^{(\nu_n+1)\sigma - \mu_n \tau - 1} p(s)^{\sigma} ds, \quad t \geq a_1. \tag{2.19}$$

Of course, (2.18) contradicts (1.11). It will be showed that (2.19) is a contradiction to (1.12). By the definition of $y(t)$, the inequality (2.19) gives

$$\begin{aligned} &\Big[x(t)^{\alpha_1}[-D_1 x(t)]^{\alpha_2}[D_2 x(t)]^{\alpha_3} \cdots [(-1)^{n-1} D_{n-1} x(t)]^{\alpha_n}\Big]^{\tau} \\ &\qquad \geq \tau L \int_t^{\infty} s^{(\nu_n+1)\sigma - \mu_n \tau - 1} p(s)^{\sigma} ds, \quad t \geq a_1. \end{aligned} \tag{2.20}$$

As in the proof of (2.11), we can find that

$$\begin{aligned} \varepsilon_{n-2}(t) &\equiv t^{\alpha_2 \alpha_3 \cdots \alpha_{n-2} + \alpha_3 \cdots \alpha_{n-2} + \cdots + \alpha_{n-2} + 1}[(-1)^{n-2} D_{n-2} x(t)] \to 0, \\ &\ \vdots \\ \varepsilon_2(t) &\equiv t^{\alpha_2 + 1}[(-1)^2 D_2 x(t)] \to 0, \\ \varepsilon_1(t) &\equiv t[-D_1 x(t)] \to 0, \end{aligned}$$

as $t \to \infty$. Set $\varepsilon_0(t) = x(t)$. From (2.20) and the definition of $\varepsilon_i(t)$ $(i = 0, 1, 2, \ldots, n-1)$ it follows that

$$\begin{aligned} &\Big[\varepsilon_0(t)^{\alpha_1}[t^{-1}\varepsilon_1(t)]^{\alpha_2}[t^{-\alpha_2 - 1}\varepsilon_2(t)]^{\alpha_3} \cdots [t^{-\alpha_2 \alpha_3 \cdots \alpha_{n-1} - \cdots - \alpha_{n-1} - 1}\varepsilon_{n-1}(t)]^{\alpha_n}\Big]^{\tau} \\ &\qquad \geq \tau L \int_t^{\infty} s^{(\nu_n+1)\sigma - \mu_n \tau - 1} p(s)^{\sigma} ds, \quad t \geq a_1, \end{aligned}$$

and, hence,

$$
\begin{aligned}
&[\varepsilon_0(t)^{\alpha_1}\varepsilon_1(t)^{\alpha_2}\varepsilon_2(t)^{\alpha_3}\cdots\varepsilon_{n-1}(t)^{\alpha_n}]^\tau \\
&\quad \geq \tau L t^{[\alpha_2+(\alpha_2+1)\alpha_3+\cdots+(\alpha_2\alpha_3\cdots\alpha_{n-1}+\cdots+\alpha_{n-1}+1)\alpha_n]\tau}\int_t^\infty s^{(\nu_n+1)\sigma-\mu_n\tau-1}p(s)^\sigma ds \\
&\quad = \tau L t^{\mu_n\tau}\int_t^\infty s^{(\nu_n+1)\sigma-\mu_n\tau-1}p(s)^\sigma ds, \quad t\geq a_1.
\end{aligned}
$$

Since $\varepsilon_0(t) = x(t)$ is bounded on $[a,\infty)$ and $\varepsilon_i(t) \to 0$ as $t \to \infty$ $(i = 1, 2, \ldots, n-1)$, we conclude that

$$
\lim_{t\to\infty} t^{\mu_n\tau}\int_t^\infty s^{(\nu_n+1)\sigma-\mu_n\tau-1}p(s)^\sigma ds = 0,
$$

which is a contradiction to (1.12). This finishes the proof of Theorem 1.1. □

For the case $n = 2$, $\alpha_1 = 1$ and $\alpha_2 = \alpha > 0$, the equation (1.1) becomes (1.14). In this case we have

$$
\mu_2 = \alpha, \quad \nu_2 = \alpha \quad \text{and} \quad \xi_2 = 1 + \alpha.
$$

Therefore Theorem 1.1 gives an extension of Theorem 3.4 of [4]. The lim inf in the condition (3.3) of Theorem 3.4 of [4] can be replaced to lim sup.

Theorem A can easily be derived from Theorem 1.1. To see this, we first remark that

$$
\nu_n\xi_n - \alpha_1\alpha_2\cdots\alpha_n\mu_n > 0, \tag{2.21}
$$

where μ_n, ν_n and ξ_n are defined by (1.4), (1.5) and (1.6), respectively. Therefore the term $\mu_n - [(\nu_n\xi_n)/(\alpha_1\alpha_2\cdots\alpha_n)]$ appearing in (1.10) is a negative number. Then we find that the set of all pairs $(\sigma, \tau) \in (0,\infty)\times(0,\infty)$ satisfying (1.9) and (1.10) is nonempty. More precisely, the set is a triangle in the $\sigma\tau$ plane. Now, to prove Theorem A, suppose that (1.8) holds. There is a constant $c > 0$ such that $p(t) \geq ct^{-\nu_n-1}$ for all large t. Take a pair $(\sigma,\tau) \in (0,\infty)\times(0,\infty)$ satisfying (1.9) and (1.10). Then we get

$$
t^{-\mu_n\tau+(\nu_n+1)\sigma-1}p(t)^\sigma \geq c^\sigma t^{-\mu_n\tau-1}
$$

for all large t. If (1.11) does not hold, then the above inequality implies

$$
\int_t^\infty s^{-\mu_n\tau+(\nu_n+1)\sigma-1}p(s)^\sigma ds \geq \frac{c^\sigma}{\mu_n\tau}t^{-\mu_n\tau}
$$

for all large t, and, in consequence, the condition (1.12) is satisfied. Therefore we conclude from Theorem 1.1 that all nontrivial Kneser solutions of (1.1) on $[a,\infty)$ are singular.

3 Other forms of Theorem 1.1

For simplicity, we put

$$
\zeta_n = \frac{\nu_n\xi_n}{\alpha_1\alpha_2\cdots\alpha_n\mu_n} - 1.
$$

By (2.21), ζ_n is a positive number.

Now, let $\alpha_1\alpha_2\cdots\alpha_n > \beta$. It is easy to check that $\sigma > 0$ and $\tau > 0$ satisfy (1.9) and (1.10) if and only if

$$
\frac{1}{\nu_n+1} < \sigma < \frac{1}{[\beta/(\alpha_1\alpha_2\cdots\alpha_n)]\,\nu_n+1} \tag{3.1}
$$

and

$$0 < \tau \le \frac{1}{\mu_n} \min\left\{(\nu_n+1)\sigma - 1,\ \frac{1}{\zeta_n}\left[1 - \left(\frac{\beta}{\alpha_1\alpha_2\cdots\alpha_n}\nu_n + 1\right)\sigma\right]\right\}. \tag{3.2}$$

Suppose that $\sigma > 0$ satisfies (3.1). Next, choose $\tau > 0$ so that the equality holds in the latter inequality of (3.2), and put $\tau = \tau(\sigma)$, that is to say, we define the number $\tau(\sigma)$ by

$$\tau(\sigma) = \frac{1}{\mu_n} \min\left\{(\nu_n+1)\sigma - 1,\ \frac{1}{\zeta_n}\left[1 - \left(\frac{\beta}{\alpha_1\alpha_2\cdots\alpha_n}\nu_n + 1\right)\sigma\right]\right\}. \tag{3.3}$$

For this choice, the conditions (1.11) and (1.12) become

$$\int_{a^+}^{\infty} s^{-\mu_n\tau(\sigma)+(\nu_n+1)\sigma-1} p(s)^\sigma ds = \infty \quad (a^+ > \max\{a, 0\}) \tag{3.4}$$

and

$$\limsup_{t\to\infty} t^{\mu_n\tau(\sigma)} \int_t^{\infty} s^{-\mu_n\tau(\sigma)+(\nu_n+1)\sigma-1} p(s)^\sigma ds > 0, \tag{3.5}$$

respectively. Therefore Theorem 1.1 produces the following result.

Theorem 3.1. *Let $\alpha_1\alpha_2\cdots\alpha_n > \beta$. Suppose that σ satisfies (3.1). Define $\tau(\sigma)$ by (3.3). If either (3.4) or (3.5) holds, then all nontrivial Kneser solutions of (1.1) on $[a,\infty)$ are singular.*

As an example, consider the fourth-order equation

$$(|x''|^\alpha \operatorname{sgn} x'')'' = \kappa t^{-2(\alpha+1)}(1+\sin t)|x|^\beta \operatorname{sgn} x, \quad t \ge 1, \tag{3.6}$$

where $\alpha > \beta > 0$, and κ is a positive constant. The equation (3.6) is a special case of (1.1) with $n = 4$, $\alpha_1 = 1$, $\alpha_2 = 1$, $\alpha_3 = \alpha$, $\alpha_4 = 1$, and $p(t) = \kappa t^{-2(\alpha+1)}(1+\sin t)$. Then we have

$$\mu_4 = 2(2\alpha+1), \quad \nu_4 = 2\alpha+1, \quad \xi_4 = 2(\alpha+1), \quad \zeta_4 = \frac{1}{\alpha}.$$

We can choose $\varepsilon_0 > 0$ sufficiently small so that

$$\frac{1}{2(\alpha+1)} < \frac{1+\varepsilon_0}{2(\alpha+1)} < \frac{1}{[\beta/\alpha](2\alpha+1)+1}$$

and

$$\varepsilon_0 < \alpha\left[1 - \left(\frac{\beta}{\alpha}(2\alpha+1)+1\right)\frac{1+\varepsilon_0}{2(\alpha+1)}\right].$$

For such $\varepsilon_0 > 0$, put

$$\sigma = \frac{1+\varepsilon_0}{2(\alpha+1)}.$$

Then, σ satisfies (3.1), and the number $\tau(\sigma)$ is given by

$$\begin{aligned}\tau(\sigma) &= \frac{1}{2(2\alpha+1)} \min\left\{2(\alpha+1)\sigma - 1,\ \alpha\left[1 - \left(\frac{\beta}{\alpha}(2\alpha+1)+1\right)\sigma\right]\right\} \\ &= \frac{\varepsilon_0}{2(2\alpha+1)}.\end{aligned}$$

Moreover, we have

$$\begin{aligned}&t^{\mu_4\tau(\sigma)} \int_t^{\infty} s^{-\mu_4\tau(\sigma)+(\nu_4+1)\sigma-1} p(s)^\sigma ds \\ &\qquad = \kappa^{(1+\varepsilon_0)/[2(\alpha+1)]} t^{\varepsilon_0} \int_t^{\infty} s^{-1-\varepsilon_0}(1+\sin s)^{(1+\varepsilon_0)/[2(\alpha+1)]} ds.\end{aligned}$$

If $m = 1, 2, \ldots$, then

$$\int_{2m\pi}^{\infty} s^{-1-\varepsilon_0}(1+\sin s)^{(1+\varepsilon_0)/[2(\alpha+1)]}ds \geq \sum_{i=0}^{\infty} \int_{2(m+i)\pi}^{(2(m+i)+1)\pi} s^{-1-\varepsilon_0}ds$$
$$\geq \sum_{i=0}^{\infty} \left[(2(m+i)+1)\pi\right]^{-1-\varepsilon_0}\pi \geq \pi^{-\varepsilon_0}\int_m^{\infty} \frac{1}{(2s+1)^{1+\varepsilon_0}}ds$$
$$= \frac{\pi^{-\varepsilon_0}}{2\varepsilon_0}(2m+1)^{-\varepsilon_0},$$

and so

$$\liminf_{m\to\infty} (2m\pi)^{\varepsilon_0} \int_{2m\pi}^{\infty} s^{-1-\varepsilon_0}(1+\sin s)^{(1+\varepsilon_0)/[2(\alpha+1)]}ds \geq \frac{1}{2\varepsilon_0} > 0.$$

Consequently, we find that

$$\limsup_{t\to\infty} t^{\mu_4\tau(\sigma)} \int_t^{\infty} s^{-\mu_4\tau(\sigma)+(\nu_4+1)\sigma-1}p(s)^{\sigma}ds > 0.$$

By Theorem 3.1, it is concluded that all nontrivial Kneser solutions of (3.6) on $[1,\infty)$ are singular. Note that Theorem A cannot be applied to (3.6) since the lower limit as $t \to \infty$ of $t^{\nu_4+1}p(t)$ is equal to 0.

Now, let $\alpha_1\alpha_2\cdots\alpha_n > \beta$, and set

$$\sigma_n = \frac{\zeta_n + 1}{[\beta/(\alpha_1\alpha_2\cdots\alpha_n)]\nu_n + 1 + \zeta_n(\nu_n+1)}. \tag{3.7}$$

We have

$$\frac{1}{\nu_n+1} < \sigma_n < \frac{1}{[\beta/(\alpha_1\alpha_2\cdots\alpha_n)]\,\nu_n + 1}.$$

It is easily seen that if σ satisfies

$$\sigma_n \leq \sigma < \frac{1}{[\beta/(\alpha_1\alpha_2\cdots\alpha_n)]\,\nu_n + 1}, \tag{3.8}$$

then the number $\tau(\sigma)$ which is defined by (3.3) is

$$\tau(\sigma) = \frac{1}{\mu_n\zeta_n}\left[1 - \left(\frac{\beta}{\alpha_1\alpha_2\cdots\alpha_n}\nu_n + 1\right)\sigma\right]. \tag{3.9}$$

Therefore Theorem 3.1 produces the following result.

Theorem 3.2. *Let $\alpha_1\alpha_2\cdots\alpha_n > \beta$. Let σ be a number satisfying (3.8), where σ_n is given by (3.7), and define $\tau(\sigma)$ by (3.9). If either (3.4) or (3.5) holds, then all nontrivial Kneser solutions of (1.1) on $[a,\infty)$ are singular.*

We have derived Theorem 3.1 from Theorem 1.1, and Theorem 3.2 from Theorem 3.1. We remark here that Theorem 1.1 can be derived from Theorem 3.2. In this sense, these three theorems are essentially identical. The following is a brief proof of the fact that Theorem 1.1 is derived from Theorem 3.2. Let $\sigma > 0$ and $\tau > 0$ be numbers which satisfy (1.9) and (1.10). As stated before, this is equivalent to the statement that σ and τ satisfy (3.1) and (3.2). Choose $\sigma^* > 0$ such that $\sigma = \sigma^*$ satisfies (3.8) and $\tau(\sigma^*)/\sigma^* < \tau/\sigma$ and $\sigma < \sigma^*$. Here, $\tau(\sigma^*)$ is given by (3.9) with $\sigma = \sigma^*$. If σ^* is taken sufficiently close to $1/\{[\beta/(\alpha_1\alpha_2\cdots\alpha_n)]\,\nu_n + 1\}$, then it is possible to choose such a number σ^*. By the Höder inequality we find that

$$\int_{a^+}^{t} s^{-\mu_n\tau+(\nu_n+1)\sigma-1} p(s)^\sigma ds \le K_1 \left(\int_{a^+}^{t} s^{-\mu_n\tau(\sigma^*)+(\nu_n+1)\sigma^*-1} p(s)^{\sigma^*} ds\right)^{\sigma/\sigma^*}, \quad t \ge a^+,$$

and

$$t^{\mu_n\tau} \int_{t}^{\infty} s^{-\mu_n\tau+(\nu_n+1)\sigma-1} p(s)^\sigma ds \le K_2 \left(t^{\mu_n\tau(\sigma^*)} \int_{t}^{\infty} s^{-\mu_n\tau(\sigma^*)+(\nu_n+1)\sigma^*-1} p(s)^{\sigma^*} ds\right)^{\sigma/\sigma^*}, \quad t \ge a^+,$$

where K_1 and K_2 are certain positive constants. Therefore, (1.11) implies (3.4) with $\sigma = \sigma^*$, and (1.12) implies (3.5) with $\sigma = \sigma^*$. This means that Theorem 1.1 is derived from Theorem 3.2 of the case $\sigma = \sigma^*$.

It is also clear that if σ satisfies

$$\frac{1}{\nu_n + 1} < \sigma \le \sigma_n, \tag{3.10}$$

then the number $\tau(\sigma)$ defined by (3.3) is

$$\tau(\sigma) = \frac{1}{\mu_n}[(\nu_n + 1)\sigma - 1]. \tag{3.11}$$

Therefore, by Theorem 3.1, we have the following result.

Corollary 3.3. *Let $\alpha_1\alpha_2\cdots\alpha_n > \beta$. Let σ be a number satisfying (3.10), where σ_n is given by (3.7), and define $\tau(\sigma)$ by (3.11). If either (3.4) or (3.5) holds, then all nontrivial Kneser solutions of (1.1) on $[a,\infty)$ are singular.*

As mentioned before, if $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 1$, then $D_i x(t) = x^{(i)}(t)$ $(i = 0,1,2,\dots,n)$, and (1.1) is reduced to (1.13). Note that the singularity condition (1.7) is rewritten in the form

$$(-1)^i x^{(i)}(t) > 0 \quad \text{on } [a,b) \quad (i = 0,1,2,\dots,n-1) \quad \text{and} \quad x(t) = 0 \quad (t \ge b).$$

Moreover, in the case $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 1$, we have

$$\mu_n = \frac{n(n-1)}{2}, \quad \nu_n = n-1, \quad \xi_n = n, \quad \zeta_n = 1.$$

Therefore Theorem 3.2 yields the following result. For simplicity, we set $\rho_n(\sigma) = \mu_n\tau(\sigma)$.

Corollary 3.4. *Consider the equation (1.13). Let $0 < \beta < 1$. Let σ be a number satisfying $2/[n + (n-1)\beta + 1] \le \sigma < 1/[(n-1)\beta + 1]$, and set $\rho_n(\sigma) = 1 - [(n-1)\beta + 1]\sigma$. If either*

$$\int_{a^+}^{\infty} s^{-\rho_n(\sigma)+n\sigma-1} p(s)^\sigma ds = \infty \quad (a^+ > \max\{a,0\})$$

or

$$\limsup_{t\to\infty} t^{\rho_n(\sigma)} \int_{t}^{\infty} s^{-\rho_n(\sigma)+n\sigma-1} p(s)^\sigma ds > 0,$$

then all nontrivial Kneser solutions of (1.13) on $[a,\infty)$ are singular.

Corollary 3.4 has been formulated in the book of Kiguradze and Chanturia [3, Theorem 11.2 ($m = 0$, $k = 1$)].

By Corollary 3.3, we have the following result.

Corollary 3.5. *Consider the equation (1.13). Let $0 < \beta < 1$. Let σ be a number satisfying $1/n < \sigma \le 2/[n + (n-1)\beta + 1]$. If either*

$$\int_{a}^{\infty} p(s)^\sigma ds = \infty \quad \text{or} \quad \limsup_{t\to\infty} t^{n\sigma-1} \int_{t}^{\infty} p(s)^\sigma ds > 0,$$

then all nontrivial Kneser solutions of (1.13) on $[a,\infty)$ are singular.

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