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On the double layer potential ansatz for the *n*-dimensional Helmholtz equation with Neumann condition

Dedicated to Professor Jeffrey R. L. Webb on the occasion of his 75th birthday

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Abstract. In the present paper we consider the Neumann problem for the n-dimensional Helmholtz equation. In particular we deal with the problem of representability of the solutions by means of double layer potentials.

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1 Introduction

Some time ago one of the authors proposed a method for treating the boundary integral equation of the first kind arising when you impose the Dirichlet condition for Laplace equation to a simple layer potential [2]. This method hinges on the theory of reducible operators and on the theory of differential forms, it does not use the theory of pseudodifferential operators and could be considered as an extension to higher dimensions of Muskhelishvili's method (see [3]). Later, this approach was extended to different BVPs for several partial differential equations and systems in simply and multiple connected domains (see [5] and the references therein).

Recently we have showed how to use this approach to solve the Dirichlet problem for the n-dimensional Helmholtz equation by means of a simple layer potential [6]. The aim of the present paper is to continue that investigation, showing how our method could be used to solve the Neumann problem for the same equation by means of a double layer potential. To this end, we make use of some fundamental results given by Colton and Kress in their celebrated monograph [7], in particular on the description of the traces on the boundary of eigensolutions of Dirichlet or Neumann problems. Colton and Kress proved their results in

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spaces of continuous functions on a C^2 boundary. As already remarked in [6], the same results can be established under more general assumptions by nowadays standard arguments in potential theory (see, e.g., [10]). In particular, they hold in L^p spaces on a Lyapunov boundary. When we consider their results, we shall always refer to them under these more general hypotheses.

Differently from [7], here we consider the Neumann problem with data in $L^p(\Sigma)$ and we obtain that the solution can be represented as a double layer potential with density in the Sobolev space $W^{1,p}(\Sigma)$.

We shall consider domains in $\mathbb{R}^n$, with $n \geq 3$. In principle our method could be applied also for $n = 2$ with some appropriate modifications, as to change fundamental solution and radiation condition (see [7, pp. 106–107]).

The paper is organized as follows. After summarizing notations and definitions in Section 2, we collect some preliminary results in Section 3. We mention that we prove a regularity result for the eigensolutions of a certain integral equation (see Proposition 3.2) without using the usual regularity properties of the double layer potential (see [8] for recent results in this direction and for an extensive bibliography). Our approach seems to be simpler and it is a consequence of some of our previous results on Laplace equation.

In the short Section 4 we recall the main result we have obtained in [6] for the Dirichlet problem. Section 5 is devoted to the main result of the present paper: we prove that the Neumann problem with data in $L^p(\Sigma)$ $(1 < p < \infty)$ can be represented by a double layer potential with density in $W^{1,p}(\Sigma)$ if and only if the data satisfies some necessary orthogonality conditions.

2 Notations and definitions

From now on Ω will be a bounded domain (open connected set) of $\mathbb{R}^n$ $(n \geq 3)$ whose boundary Σ is a Lyapunov hypersurface (i.e. Σ has a uniformly Hölder continuous normal field of some exponent $\lambda \in (0,1]$), and such that $\mathbb{R}^n \setminus \overline{\Omega}$ is connected; $\nu(x) = (\nu_1(x), \dots, \nu_n(x))$ denotes the outwards unit normal vector at the point $x = (x_1, \dots, x_n) \in \Sigma$. The Euclidean norm for elements of $\mathbb{R}^n$ is denoted by $|\cdot|$.

Now fix $1 < p < \infty$. By $L^p(\Sigma)$ we denote the space of p-integrable complex-valued functions defined on Σ. By $L^p_h(\Sigma)$ we mean the space of the differential forms of degree $h \geq 1$ whose components belong to $L^p(\Sigma)$.

The Sobolev space $W^{1,p}(\Sigma)$ can be defined as the space of functions in $L^p(\Sigma)$ such that their weak differential belongs to $L^p_1(\Sigma)$.

If u is an h-form in Ω, the symbol du denotes the differential of u, while $*u$ denotes the dual Hodge form. Finally, we write $\underset{\Sigma}{*} w = w_0$ if w is an $(n-1)$-form on Σ and $w = w_0 d\sigma$.

Besides the theory of differential forms, the method we use hinges on the theory of reducible operators. Here we recall that, given two Banach spaces E and F, a continuous linear operator $S : E \to F$ can be reduced on the left if there exists a continuous linear operator $S' : F \to E$ such that $S'S = I + T$, I being the identity operator on E and T a compact operator on E. An operator S reducible on the right can be defined analogously. If S can be reduced (on the left or right), then its range is closed and, as a consequence the equation $S\alpha = \beta$ admits a solution if and only if $\langle \gamma, \beta \rangle = 0$, for any $\gamma \in F^*$ such that $S^*\gamma = 0$, where S^* is the adjoint of S. For more details we refer the readers, e.g., to [9] or [11].

We consider the n-dimensional Helmholtz equation

$$\Delta u + k^2 u = 0 \tag{2.1}$$

where $k \in \mathbb{C} \setminus \{0\}$, $\mathrm{Im}(k) \geq 0$, $u : \Omega \to \mathbb{C}$, and Δ is the Laplace operator. The fundamental solution of (2.1) is given by

$$\Phi(x) = \frac{i}{4} \left(\frac{k}{2\pi|x|} \right)^{(n-2)/2} H^{(1)}_{(n-2)/2}(k|x|)$$

where $H^{(1)}_{\mu}$ is the Hankel function of the first kind of order μ (see, e.g., [1, p. 42]). In what follows it will be useful to consider the auxiliary function

$$h(x) = \Phi(x) - s(x) \quad (x \in \mathbb{R}^n \setminus \{0\})\,,$$

where s is the fundamental solution of $-\Delta$, i.e. for $n \geq 3$ and $x \in \mathbb{R}^n \setminus \{0\}$,

$$s(x) = \frac{1}{(n-2)\omega_n} |x|^{2-n} \qquad \left(\omega_n = \frac{2\pi^{n/2}}{\Gamma(n/2)} \right).$$

We observe that (see [12, Lemma A.5, p. 571])

$$|\nabla h(x)| \leq c|x|^{3-n}, \quad \forall\, x \in \mathbb{R}^n \setminus \{0\}. \tag{2.2}$$

Hence, from (2.2), and recalling that $|\nabla s(x)| \leq c_1 |x|^{1-n}$, immediately we get

$$|\nabla \Phi(x)| \leq c_2 |x|^{1-n}\,. \tag{2.3}$$

Moreover,

$$\left| \frac{\partial^2 h(x)}{\partial x_j \partial x_l} \right| \leq c|x|^{2-n}, \quad \forall\, x \in \mathbb{R}^n \setminus \{0\},\ j,l = 1,\ldots,n. \tag{2.4}$$

As we shall see, we are interested to solve the Neumann problem related to the Helmholtz equation (2.1) in the class of potentials defined as follows.

Definition 2.1. We say that a function w belongs to the space $\mathcal{D}^p$ if and only if there exists $\psi \in W^{1,p}(\Sigma)$ such that w can be represented by means of a double layer potential with density ψ, i.e.

$$w(x) = \int_\Sigma \psi(y) \frac{\partial \Phi}{\partial \nu_y}(x-y)\, d\sigma_y, \quad x \in \Omega\,.$$

We also recall the following class of functions used in [6].

Definition 2.2. We say that a function u belongs to the space $\mathcal{S}^p$ if and only if there exists $\varphi \in L^p(\Sigma)$ such that u can be represented by means of a simple layer potential with density φ, i.e.

$$u(x) = \int_\Sigma \varphi(y)\, \Phi(x-y)\, d\sigma_y, \quad x \in \Omega\,. \tag{2.5}$$

We shall distinguish by indices $+$ and $-$ the nontangential limit obtained by approaching the boundary Σ from $\mathbb{R}^n \setminus \overline{\Omega}$ and Ω, respectively (see, e.g. [10, p. 293]).

We remark that by $\langle f, g \rangle$ we denote the bilinear form

$$\int_\Sigma f\, g\, d\sigma\,.$$

3 Preliminary results

Let us introduce the integral operators:

$$K : L^p(\Sigma) \to L^p(\Sigma)\,, \quad K\varphi(x) = 2\int_\Sigma \varphi(y)\frac{\partial\Phi}{\partial\nu_y}(x-y)\,d\sigma_y$$

and its adjoint

$$K^* : L^q(\Sigma) \to L^q(\Sigma)\,, \quad K^*\psi(x) = 2\int_\Sigma \psi(y)\frac{\partial\Phi}{\partial\nu_x}(x-y)\,d\sigma_y\,.$$

where $1 < p < \infty$ and $\frac{1}{p}+\frac{1}{q} = 1$. K and K^* are adjoint operators with respect to the duality

$$\langle\psi, K\varphi\rangle = \langle K^*\psi, \varphi\rangle.$$

Moreover, K and K^* are compact operators because of (2.3).

Here, we are interested in the kernels of the operators $I \pm K$ and $I \pm K^*$, where I is the identity operator on the relevant Lebesgue space. To this end, let us denote by $\mathcal{U}_0$ the space of solutions of

$$\begin{cases} u \in C^{1,\lambda}(\overline{\Omega}) \cap C^2(\Omega), & \\ \Delta u + k^2 u = 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial\nu} = 0 & \text{on } \Sigma \end{cases}$$

and by $\mathcal{V}_0$ the space of solutions of

$$\begin{cases} u \in C^{1,\lambda}(\overline{\Omega}) \cap C^2(\Omega), & \\ \Delta u + k^2 u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \Sigma\,. \end{cases}$$

Note that $\mathcal{U}_0 = \{0\}$ (resp. $\mathcal{V}_0 = \{0\}$) whenever k^2 is not an interior Neumann eigenvalue (resp. an interior Dirichlet eigenvalue).

It is known that (see [7, Theorem 3.17])

$$\mathcal{N}(I+K) = \{u_{|\Sigma} : u \in \mathcal{U}_0\} \tag{3.1}$$

and that (see [7, Theorem 3.22])

$$\mathcal{N}(I-K^*) = \left\{ \frac{\partial v}{\partial\nu}\Big|_\Sigma : v \in \mathcal{V}_0 \right\}\,. \tag{3.2}$$

Let $\dim\mathcal{N}(I+K) = m_N$ and $\dim\mathcal{N}(I-K) = m_D$. Note that, $m_N = 0$ if k^2 is not an interior Neumann eigenvalue, while $m_D = 0$ whenever k^2 is not an interior Dirichlet eigenvalue.

Moreover

$$\dim\mathcal{N}(I+K) = \dim\mathcal{N}(I+K^*) \quad \text{and} \quad \dim\mathcal{N}(I-K) = \dim\mathcal{N}(I-K^*)\,.$$

We have also the following lemma.

Lemma 3.1. $\mathcal{N}(I \pm K) \perp \mathcal{N}(I \mp K^*)$.

Proof. If $\alpha \in \mathcal{N}(I \pm K)$ and $\beta \in \mathcal{N}(I \mp K^*)$, then

$$\langle \alpha, \beta \rangle = \langle \mp K\alpha, \beta \rangle = \mp \langle \alpha, K^*\beta \rangle = -\langle \alpha, \beta \rangle ,$$

and hence $\langle \alpha, \beta \rangle = 0$. □

The next proposition shows that the functions in $\mathcal{N}(I - K)$ belong to the Sobolev space $W^{1,p}(\Sigma)$. As said in the introduction, this result could be deduced by regularizing properties of the double layer potential, but here we use a different approach which seems to be simpler.

Proposition 3.2. *Let $\zeta \in L^p(\Sigma)$ be a solution of the equation $\zeta - K\zeta = 0$. Then ζ belongs to $W^{1,p}(\Sigma)$.*

Proof. Since $\zeta \in \mathcal{N}(I - K)$, the potential

$$v(x) = \int_\Sigma \zeta(y) \frac{\partial \Phi}{\partial \nu_y}(x - y)\, d\sigma_y$$

satisfies the condition $v_- = 0$ on Σ.

We can write the equation $\zeta - K\zeta = 0$ as

$$-\frac{1}{2}\zeta(x) + \int_\Sigma \zeta(y) \frac{\partial s}{\partial \nu_y}(x - y)\, d\sigma_y = T(x),$$

where

$$T(x) = -\int_\Sigma \zeta(y) \frac{\partial h}{\partial \nu_y}(x - y)\, d\sigma_y .$$

Thanks to (2.2) and (2.4), the function T belongs to $W^{1,p}(\Sigma)$. Therefore the harmonic function

$$a(x) = \int_\Sigma \zeta(y) \frac{\partial s}{\partial \nu_y}(x - y)\, d\sigma_y$$

satisfies the boundary condition $a = T$ on Σ. As proved in [2], the function a can be represented as a simple layer potential with density $A \in L^p(\Sigma)$:

$$a(x) = \int_\Sigma A(y)\, s(x - y)\, d\sigma_y .$$

This implies that there exists the normal derivative $\partial a/\partial \nu$ almost everywhere on Σ and it belongs to $L^p(\Sigma)$ (see [2, pp. 182–183]). It follows that the function ζ satisfies the condition

$$\frac{\partial}{\partial \nu_x} \int_\Sigma \zeta(y) \frac{\partial s}{\partial \nu_y}(x - y)\, d\sigma_y = \frac{\partial a}{\partial \nu}(x)$$

on Σ.

Thanks to [4, p.29] we can say that there exists a solution $\zeta_0 \in W^{1,p}(\Sigma)$ of this equation, since the right-hand side has zero mean value on Σ. Therefore

$$\frac{\partial}{\partial \nu_x} \int_\Sigma (\zeta(y) - \zeta_0(y)) \frac{\partial s}{\partial \nu_y}(x - y)\, d\sigma_y = 0$$

on Σ and the potential

$$\int_\Sigma (\zeta(y) - \zeta_0(y)) \frac{\partial s}{\partial \nu_y}(x - y)\, d\sigma_y$$

has to be constant in Ω. It follows $\zeta = \zeta_0 + c$ and this completes the proof. □

In the following theorem we collect some useful results contained in [7, Theorems 3.18 and 3.23].

Theorem 3.3.

(i) *Let* $\{\lambda_1, \dots, \lambda_{m_N}\}$ *be a basis of* $\mathcal{N}(I + K^*)$ *and define*

$$u_j(x) = \int_\Sigma \lambda_j(y)\Phi(x - y)\, d\sigma_y \quad x \in \mathbb{R}^n \setminus \Sigma\,, j = 1, \dots, m_N\,.$$

Then

$$\lambda_j = -\frac{\partial u_j}{\partial \nu^+} \quad \text{on } \Sigma\,, j = 1, \dots, m_N\,,$$

and the functions

$$\rho_j = -\overline{u}_{j,+} \quad \text{on } \Sigma\,, j = 1, \dots, m_N$$

form a basis of $\mathcal{N}(I + K)$*.*

Moreover, the determinant of the matrix $(\langle \rho_j, \lambda_l \rangle)_{j,l=1,\dots,m_N}$ *is nonzero.*

(ii) *Let* $\{\zeta_1, \dots, \zeta_{m_D}\}$ *be a basis of* $\mathcal{N}(I - K)$ *and define*

$$v_j(x) = \int_\Sigma \zeta_j(y) \frac{\partial \Phi}{\partial \nu_y}(x - y)\, d\sigma_y \quad x \in \mathbb{R}^n \setminus \Sigma\,, j = 1, \dots, m_D\,.$$

Then

$$\zeta_j = v_{j,+} \quad \text{on } \Sigma\,, j = 1, \dots, m_D\,,$$

and the functions

$$\mu_j = \frac{\partial v_j}{\partial \nu^+} \quad \text{on } \Sigma\,, j = 1, \dots, m_D \tag{3.3}$$

form a basis of $\mathcal{N}(I - K^*)$*.*

Moreover, the determinant of the matrix $(\langle \mu_j, \zeta_l \rangle)_{j,l=1,\dots,m_D}$ *is nonzero.*

Remark 3.4. Thanks to the Lyapunov property of the double layer potential (see [7, Theorem 2.21]), (3.3) is equivalent to

$$\mu_j = \frac{\partial v_j}{\partial \nu^-} \quad \text{on } \Sigma\,, j = 1, \dots, m_D\,. \tag{3.4}$$

4 The Dirichlet problem

In this section we describe the main lines of the method applied in [6] to the Dirichlet problem

$$\begin{cases} u \in S^p, \\ \Delta u + k^2 u = 0 & \text{in } \Omega, \\ u = f & \text{on } \Sigma\,, f \in W^{1,p}(\Sigma)\,. \end{cases} \tag{4.1}$$

First, we imposed the boundary condition to (2.5), obtaining

$$\int_\Sigma \varphi(y)\Phi(x - y)\, d\sigma_y = f(x), \quad x \in \Sigma\,. \tag{4.2}$$

Then, taking the exterior differential d of both sides of the integral equation of the first kind (4.2), we get the singular integral equation

$$S\varphi(x) = df(x), \quad \text{a.e. } x \in \Sigma, \tag{4.3}$$

where

$$S\varphi(x) = \int_\Sigma \varphi(y) d_x[\Phi(x-y)] d\sigma_y .$$

The singular integral operator $S : L^p(\Sigma) \to L^p_1(\Sigma)$ can be reduced on the left by the singular integral operator $J' : L^p_1(\Sigma) \longrightarrow L^p(\Sigma)$ defined as

$$J'\psi(z) = \underset{\Sigma}{*} \int_\Sigma \psi(x) \wedge d_z[s_{n-2}(z,x)], \quad z \in \Sigma,$$

with

$$s_{n-2}(x,y) = \sum_{j_1<\ldots<j_{n-2}} s(x-y) dx^{j_1} \ldots dx^{j_{n-2}} dy^{j_1} \ldots dy^{j_{n-2}}$$

being the Hodge double $(n-2)$-form (see [6, Theorem 2]).

Therefore, the range of S is closed and equation (4.3) has a solution $\varphi \in L^p(\Sigma)$ if and only if

$$\int_\Sigma \gamma \wedge df = 0$$

for every $\gamma \in W^{1,q}_{n-2}(\Sigma)$ $(q = p/(p-1))$[(∗)] such that $d\gamma = \frac{\partial v}{\partial \nu} d\sigma$, for all $v \in \mathcal{V}_0$ (see [6, Theorem 4]).

Using the above results, we proved the representability theorem for the Dirichlet problem via simple layer potentials, rewritten here in a new form.

Theorem 4.1. *Let $f \in W^{1,p}(\Sigma)$. There exists a solution of (4.1) if and only if f satisfies the compatibility conditions*

$$\int_\Sigma f \mu_j \, d\sigma = 0 \quad \textit{for every } j = 1, \ldots, m_D . \tag{4.4}$$

Proof. From [6, Theorem 5] it follows that there exists a solution of (4.1) if and only if f satisfies the compatibility conditions

$$\int_\Sigma f \frac{\partial v}{\partial \nu} d\sigma = 0 \quad \text{for all } v \in \mathcal{V}_0 . \tag{4.5}$$

Conditions (4.5) and (4.4) are equivalent because of (3.2), Theorem 3.3-(ii), and (3.4). □

5 The Neumann problem

In this section we consider the Neumann problem

$$\begin{cases} w \in \mathcal{D}^p, \\ \Delta w + k^2 w = 0 & \text{in } \Omega, \\ \dfrac{\partial w}{\partial \nu} = g & \text{on } \Sigma, \end{cases} \tag{5.1}$$

(∗) By $W^{1,q}_{n-2}(\Sigma)$ we denote the space of differential forms of degree $n-2$ whose coefficients belong to $W^{1,q}(\Sigma)$.

where $g \in L^p(\Sigma)$ satisfies

$$\int_\Sigma gu\, d\sigma = 0, \quad \forall\, u \in \mathcal{U}_0 . \tag{5.2}$$

Observe that conditions (5.2) are necessary for the solvability of the problem (5.1) because of Green's formulas.

Moreover, conditions (5.2) can be rewritten as

$$\int_\Sigma g\rho_j\, d\sigma = 0, \quad j = 1, \dots, m_N . \tag{5.3}$$

We begin by stating some preliminary results.

Proposition 5.1. *Consider $u \in \mathcal{S}^p$ with density $\varphi \in L^p(\Sigma)$ and let $W_0 \in \mathcal{D}^p$ with density u:*

$$W_0(x) = \int_\Sigma u(y)\, \frac{\partial \Phi}{\partial \nu_y}(x-y)\, d\sigma_y, \quad x \in \Omega .$$

Then

$$\frac{\partial W_0}{\partial \nu}(x) = -\frac{1}{4}\varphi(x) + \frac{1}{4}K^{*2}\varphi(x) . \tag{5.4}$$

for almost every $x \in \Sigma$.

Proof. First observe that u solves equation (2.1), and hence (see [7, Theorem 3.1])

$$u(x) = \int_\Sigma \left\{ \Phi(x-y)\frac{\partial u}{\partial \nu}(y) - u(y)\frac{\partial \Phi}{\partial \nu_y}(x-y) \right\} d\sigma_y, \quad x \in \Omega .$$

Moreover, for u the following jump relation holds (see [7, Theorem 2.19])

$$\frac{\partial u}{\partial \nu^-}(x) = \lim_{\substack{y \to x \\ y \in \nu_x^-}} \frac{\partial u}{\partial \nu}(y) = \frac{1}{2}\,\varphi(x) + \int_\Sigma \varphi(y)\frac{\partial \Phi}{\partial \nu_x}(x-y)\, d\sigma_y,$$

almost everywhere on Σ. We have also

$$\begin{aligned}
\frac{\partial W_0}{\partial \nu}(x) &= \frac{\partial}{\partial \nu}\left\{ -u(x) + \int_\Sigma \Phi(x-y)\frac{\partial u}{\partial \nu}(y) d\sigma_y \right\} \\
&= -\frac{\partial u}{\partial \nu}(x) + \frac{\partial}{\partial \nu_x}\int_\Sigma \Phi(x-y)\frac{\partial u}{\partial \nu}(y) d\sigma_y \\
&= \left(\frac{1}{2} - 1\right)\frac{\partial u}{\partial \nu}(x) + \int_\Sigma \frac{\partial u}{\partial \nu}(y)\frac{\partial \Phi}{\partial \nu_x}(x-y)\, d\sigma_y \\
&= -\frac{1}{2}\left\{ \frac{1}{2}\varphi(x) + \int_\Sigma \varphi(y)\frac{\partial \Phi}{\partial \nu_x}(x-y)\, d\sigma_y \right\} \\
&\quad + \int_\Sigma \left\{ \frac{1}{2}\varphi(y) + \int_\Sigma \varphi(z)\frac{\partial \Phi}{\partial \nu_y}(y-z)\, d\sigma_z \right\} \frac{\partial \Phi}{\partial \nu_x}(x-y)\, d\sigma_y \\
&= -\frac{1}{4}\varphi(x) + \int_\Sigma \varphi(z)\, d\sigma_z \int_\Sigma \frac{\partial \Phi}{\partial \nu_y}(y-z)\frac{\partial \Phi}{\partial \nu_x}(x-y)\, d\sigma_y.
\end{aligned}$$

Hence formula (5.4) is proved. □

Lemma 5.2. *The Fredholm equation*

$$-\varphi + K^{*2}\varphi = 4g , \tag{5.5}$$

where $g \in L^p(\Sigma)$, admits a solution $\varphi \in L^p(\Sigma)$ if and only if conditions

$$\int_\Sigma g\rho_j \, d\sigma = 0, \quad j = 1, \dots, m_N \tag{5.6}$$

and

$$\int_\Sigma g\zeta_i \, d\sigma = 0, \quad i = 1, \dots, m_D \tag{5.7}$$

are satisfied.

Proof. Assume that (5.6) and (5.7) are satisfied and rewrite equation (5.5) as

$$(I + K^*)(-I + K^*)\varphi = 4g.$$

Observe that the equation $(I + K^*)\gamma = 4g$ admits a solution because of (5.6). Denote by γ_0 such a solution and consider

$$(-I + K^*)\varphi = \gamma_0. \tag{5.8}$$

The last equation is solvable if and only if $\langle \gamma_0, \zeta_i \rangle = 0$ for every $\zeta_i \in \mathcal{N}(I - K)$, $i = 1, \dots, m_D$. We have

$$\langle \gamma_0, \zeta_i \rangle = \langle \gamma_0, K\zeta_i \rangle = \langle K^*\gamma_0, \zeta_i \rangle = -\langle \gamma_0, \zeta_i \rangle + \langle 4g, \zeta_i \rangle,$$

and then, thanks to (5.7),

$$\langle \gamma_0, \zeta_i \rangle = \langle 2g, \zeta_i \rangle = 0, \quad i = 1, \dots, m_D.$$

This shows that there exists a solution φ of (5.8). Therefore φ satisfies (5.5).

Conversely, if φ is such that (5.5) holds, we have

$$(-I + K^*)(I + K^*)\varphi = 4g.$$

In particular, $4g \in \mathcal{R}(I - K^*) = \mathcal{N}(I - K)^\perp$, and then conditions (5.7) are satisfied. On the other hand, $(I + K^*)(-I + K^*)\varphi = 4g$, hence $4g \in \mathcal{R}(I + K^*) = \mathcal{N}(I + K)^\perp$, and then all conditions in (5.6) hold. □

Lemma 5.3. *Given $\psi \in W^{1,p}(\Sigma)$ there exist $\varphi \in L^p(\Sigma)$ and $c_1, \dots, c_{m_D} \in \mathbb{C}$ such that*

$$\psi(x) = \int_\Sigma \varphi(y)\Phi(x - y) \, d\sigma_y + \sum_{i=1}^{m_D} c_i\zeta_i(x), \quad x \in \Sigma. \tag{5.9}$$

The vector $(c_1, \dots, c_{m_D})$ is the unique solution of the system

$$\sum_{i=1}^{m_D} c_i \langle \zeta_i, \mu_j \rangle = \langle \psi, \mu_j \rangle, \quad j = 1, \dots, m_D. \tag{5.10}$$

Proof. Let $\psi \in W^{1,p}(\Sigma)$. In view of Proposition 3.2 the function $\psi - \sum_{i=1}^{m_D} c_i\zeta_i$ belongs to $W^{1,p}(\Sigma)$ for any $c_1, \dots, c_{m_D}$. Thanks to Theorem 4.1, there exists $\varphi \in L^p(\Sigma)$ satisfying (5.9) if and only if

$$\int_\Sigma \left(\psi - \sum_{i=1}^{m_D} c_i\zeta_i \right) \mu_j d\sigma = 0, \quad j = 1, \dots, m_D,$$

that is, $(c_1, \dots, c_{m_D})$ is solution of system (5.10). Note that the constants $c_1, \dots, c_{m_D}$ are uniquely determined since the determinant of the matrix $(\langle \mu_j, \zeta_l \rangle)_{j,l=1,\dots,m_D}$ is nonzero (see Theorem 3.3). □

Theorem 5.4. *There exists a solution of* (5.1) *if and only if g satisfies* (5.2).

Proof. Assume that g satisfies (5.2). Let $(c_1, \ldots, c_{m_D})$ be the solution of the system

$$\sum_{i=1}^{m_D} c_i \int_\Sigma \mu_i \zeta_j d\sigma = \int_\Sigma g \zeta_j d\sigma\,, \quad j = 1, \ldots, m_D \tag{5.11}$$

and consider the potential

$$w(x) = \int_\Sigma \left(\int_\Sigma \varphi(z) \Phi(y-z) d\sigma_z \right) \frac{\partial \Phi}{\partial \nu_y}(x-y) d\sigma_y + \sum_{i=1}^{m_D} c_i \int_\Sigma \zeta_i(y) \frac{\partial \Phi}{\partial \nu_y}(x-y)\, d\sigma_y\,, \quad x \in \Omega\,,$$

where $\varphi \in L^p(\Sigma)$ has to be determined. By imposing the boundary condition we obtain

$$\begin{aligned} &\frac{\partial}{\partial \nu_x} \int_\Sigma \left(\int_\Sigma \varphi(z) \Phi(y-z) d\sigma_z \right) \frac{\partial \Phi}{\partial \nu_y}(x-y) d\sigma_y + \sum_{i=1}^{m_D} c_i \frac{\partial}{\partial \nu_x} \int_\Sigma \zeta_i(y) \frac{\partial \Phi}{\partial \nu_y}(x-y)\, d\sigma_y \\ &\quad = -\frac{1}{4}\varphi(x) + \frac{1}{4} K^{*2} \varphi(x) + \sum_{i=1}^{m_D} c_i \mu_i(x) = g(x)\,, \quad x \in \Sigma\,, \end{aligned}$$

because of (5.4), (3.3), and (3.4). Then w satisfies the boundary conditions if and only if

$$-\varphi + K^{*2}\varphi = 4\left(g - \sum_{i=1}^{m_D} c_i \mu_i \right) \quad \text{on } \Sigma\,.$$

By virtue of Lemma 5.2, there exists a solution $\varphi \in L^p(\Sigma)$ of this equation if and only if

$$\int_\Sigma \left(g - \sum_{i=1}^{m_D} c_i \mu_i \right) \rho_j d\sigma = 0, \quad j = 1, \ldots, m_N \tag{5.12}$$

and

$$\int_\Sigma \left(g - \sum_{i=1}^{m_D} c_i \mu_i \right) \zeta_j d\sigma = 0, \quad j = 1, \ldots, m_D\,. \tag{5.13}$$

Conditions (5.12) are satisfied because

$$\int_\Sigma \left(g - \sum_{i=1}^{m_D} c_i \mu_i \right) \rho_j d\sigma = -\sum_{i=1}^{m_D} c_i \int_\Sigma \mu_i \rho_j d\sigma = 0$$

thanks to (5.3) and Lemma 3.1. On the other hand, conditions (5.13) hold in view of (5.11).

Conversely, let $w \in \mathcal{D}^p$ be a solution of (5.1) with density $\psi \in W^{1,p}(\Sigma)$. From Lemma 5.3, ψ can be written as in (5.9). Therefore,

$$-\varphi + K^{*2}\varphi + 4\sum_{i=1}^{m_D} c_i \mu_i = 4g \quad \text{on } \Sigma\,.$$

Now we consider $u \in \mathcal{U}_0$. From (3.1), $u|_\Sigma \in \mathcal{N}(I+K)$ and, from Lemma 3.1, $\int_\Sigma \mu_i u\, d\sigma = 0$.

On the other hand, $-\varphi + K^{*2}\varphi \in \mathcal{R}(I+K^*) = \mathcal{N}(I+K)^\perp$, and hence we have that $\int_\Sigma (-\varphi + K^{*2}\varphi) u\, d\sigma = 0$.

Accordingly,

$$\int_\Sigma 4gu\, d\sigma = \int_\Sigma (-\varphi + K^{*2}\varphi) u\, d\sigma + 4\sum_{i=1}^{m_D} c_i \int_\Sigma \mu_i u\, d\sigma = 0$$

and condition (5.2) is fulfilled. □

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