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Ergodic limits for inhomogeneous evolution equations

Dedicated to Professor Jeffrey R. L. Webb on the occasion of his 75th birthday

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Abstract. Let u satisfy an inhomogeneous wave equation such as

$$u''(t) + A^2u(t) = h(t), \qquad u(0) = f, \quad u'(0) = g.$$

We show that in many cases, the limit as $t \to \infty$ of $\frac{1}{t}\int_0^t u(s)ds$ exists, and can be calculated explicitly.

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1 Introduction

The mean ergodic theorem (MET) deals with the asymptotic behavior of semigroups governing

$$\frac{du}{dt} = Au, \qquad u(0) = f \tag{1.1}$$

and cosine functions governing

$$\frac{d^2u}{dt^2} = A^2u, \qquad u(0) = f, \quad u'(0) = 0. \tag{1.2}$$

The conclusion is that the unique mild solution u of (1.1) and of (1.2) both satisfy

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t u(s)ds \tag{1.3}$$

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exists and equals Pf, where P is a suitable projection onto the null space of A. Of course, some hypotheses are necessary, including the uniform boundedness of the solution semigroup or cosine function.

Our goal here is to obtain analogous results for solutions of the corresponding inhomogeneous problems

$$\frac{du}{dt} = Au + h(t), \qquad u(0) = f, \tag{1.4}$$

$$\frac{d^2u}{dt^2} = A^2u + h(t), \qquad u(0) = f, \quad \frac{du}{dt}(0) = g. \tag{1.5}$$

For (1.5) the ergodic limits do not always exist.

2 First order equations

Let A generate a uniformly bounded strongly continuous (or (C_0)) group $\{e^{tA} : t \in \mathbb{R}\} \subset L(X)$ on a Banach space X. For $f \in X$ and $h \in L^1(\mathbb{R}, X)$, the unique mild solution of (1.4) is given by the strongly continuous function

$$u(t) = e^{tA}f + \int_0^t e^{(t-s)A}h(s)ds, \qquad t \in \mathbb{R}. \tag{2.1}$$

For background on semigroups and cosine functions, see e.g. Goldstein [4]. The mild solution u is a strong solution in $C^1(\mathbb{R}, X)$ provided $f \in D(A)$ and either $h \in C^1(\mathbb{R}, X)$ or both h and Ah belong to $C(\mathbb{R}, X)$. We will assume $h \in L^1(\mathbb{R}, X)$ (or maybe $h \in L^1(\mathbb{R}^+, X)$, $\mathbb{R}^+ = [0, \infty)$ since we study (1.3)).

Let $X_0 := N(A) + \overline{R(A)}$, with N and R denoting null space and range, respectively. For $f \in N(A)$, $e^{tA}f = f$ for all $t \in \mathbb{R}$, while for $f = Ag \in R(A)$,

$$\frac{1}{t}\int_0^t e^{sA}f ds = \frac{1}{t}\int_0^t \frac{d}{ds}(e^{sA}g)ds = \frac{e^{tA}g - g}{t} \to 0$$

as $t \to \infty$, whence $N(A) \cap \overline{R(A)} = \{0\}$. Then the MET says that $\frac{1}{t}\int_0^t e^{sA}f ds \to Pf$ (strong convergence) as $t \to \infty$, for all $f = f_1 + f_2 \in N(A) + \overline{R(A)} =: X_0$ and $Pf = f_1$ where P is the projection of X_0 onto $N(A) = \overline{N(A)}$ along $\overline{R(A)}$.

Note that P is bounded because

$$\|P\| \leq \sup_{t \in \mathbb{R}} \|e^{tA}\| = M < \infty.$$

Also, $X_0 = X$ if X is reflexive. Moreover P is an orthogonal projection if $X = H$ is a Hilbert space and $M = 1$, i.e., $\{e^{tA} : t \in \mathbb{R}\}$ is a (C_0) unitary group. For the final term in (2.1),

$$\int_0^t e^{(t-s)A}h(s)ds = e^{tA}\int_0^t e^{-sA}h(s)ds,$$

$$k(t) := \int_0^t e^{-sA}h(s)ds \to \int_0^\infty e^{-sA}h(s)ds =: k_0 \tag{2.2}$$

as $t \to \infty$, and

$$\left\| e^{tA}\left[\int_0^t e^{-sA}h(s)ds - k_0\right]\right\| = \left\| e^{tA}\int_t^\infty e^{-sA}h(s)ds\right\| \to 0 \tag{2.3}$$

as $t \to \infty$ by the uniform boundedness of $\{e^{tA}\}$ and (2.2). Thus

$$\frac{1}{\tau}\int_0^\tau \left(\int_0^t e^{(t-s)A}h(s)ds\right)dt = \frac{1}{\tau}\int_0^\tau e^{tA}k_0 dt + o(1)$$

converges as $\tau \to \infty$ to Pk_0 by the MET and (2.3).

This proves

Theorem 2.1. *Let $\{e^{tA} : t \in \mathbb{R}\}$ be a uniformly bounded (C_0) group on X, let $X_0 = N(A) + \overline{R(A)}$, and P_0 be the (bounded) projection of X_0 onto $N(A)$ along $\overline{R(A)}$. Let $h \in L^1(\mathbb{R}, X)$. Let u, given by* (2.1)*, be the unique mild solution of* (1.4)*. Then*

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t u(s)ds = P(f + k_0)$$

where P is the projection of X_0 onto $N(A)$ along $\overline{R(A)}$ and

$$k_0 = \int_0^\infty e^{-sA}h(s)ds.$$

3 Second order case

In 1963, W. Littman [6] showed that the initial value problem for the wave equation $\frac{\partial^2 u}{\partial t^2} = \Delta u$ for $t \in \mathbb{R}$ and $x \in \mathbb{R}^n$ is wellposed (in the sense of existence, uniqueness and continuous dependence on the initial conditions) on a space based on $L^p(\mathbb{R}^n)$ iff $p = 2$ when $n \geq 2$. Earlier, K. Friedrichs had pointed out that wave propagation was intimately related to energy considerations, so again, Hilbert space was the optimal context for the study of waves. Still, some special equations can be studied in an L^p context, so we start this section in Hilbert space and later consider Banach spaces as well.

Let B generate a uniformly bounded (C_0) group on a Hilbert space $H_1 = (H, \langle\cdot,\cdot\rangle)$. Then there is as equivalent inner product $\langle\langle\cdot,\cdot\rangle\rangle$ such that on $H_2 = (H, \langle\langle\cdot,\cdot\rangle\rangle)$, B is a skewadjoint operator. This 1947 result is due to B. Sz.-Nagy [7]; cf. also [4]. Thus there is a bijective bounded linear operator $V : H_1 \to H_2$ with bounded inverse such that

$$e^{tB}|_{H_1} = V^{-1}(e^{tB}|_{H_2})V$$

and $\{e^{tB}|_{H_2} : t \in \mathbb{R}\}$ is a (C_0) unitary group on H_2. Then the P in Theorem 2.1 is an orthogonal projection in the H_2 context.

The selfadjoint operator $L = iB$ on H_2 determines the cosine function C given by

$$C(t) = \cos(tL) = \frac{1}{2}(e^{itL} + e^{-itL}), \qquad t \in \mathbb{R} \tag{3.1}$$

(see p.118 of [4]). The corresponding sine function can be defined by

$$\sin(tL) = \frac{1}{2i}(e^{itL} - e^{-itL}), \qquad t \in \mathbb{R}.$$

By a (now commonly accepted) abuse of notation, we define the modified sine function $S(t)$ (and omit the adjective "modified") by

$$S(t) = \frac{1}{2i}(e^{itL} - e^{-itL})L^{-1} \tag{3.2}$$

provided L is injective. But since $\frac{\sin(\lambda)}{\lambda} \to 1$ as $\lambda \to 0$, we can use the spectral theorem and the functional calculus to define $S(t)$ by (3.2) on $R(L)$ and $S(t) = tP$ on $N(A)$, because $v(t) = S(t)g$ is the unique solution of

$$v'' + L^2 v = 0, \qquad v(0) = 0, \quad v'(0) = g$$

for $g \in N(L)$. It is easy to see that

$$S(t)f = \int_0^t C(s)f ds, \tag{3.3}$$

and this can be used to define $S(t) \in L(H_2)$ for $t \in \mathbb{R}$. The unique mild solution of

$$u'' + L^2 u = h(t), \qquad u(0) = f, \quad u'(0) = g \tag{3.4}$$

is given by

$$u(t) = C(t)f + S(t)g + \int_0^t S(t-s)h(s)ds. \tag{3.5}$$

It is a strong $C^2(\mathbb{R}, H_2)$ solution provided $f \in D(L^2)$, $g \in D(L)$ and $h \in C^1(\mathbb{R}, H_2)$.

Now suppose $A = iL$ generates a uniformly bounded (C_0) group on a Banach space X. Then (3.1) and (3.3) define C and S, and (3.5) gives the unique mild solution of (3.4).

Now let A be as in Theorem 2.1, so that (3.4) becomes

$$u'' = A^2 u + h(t), \qquad u(0) = f, \quad u'(0) = g. \tag{3.6}$$

We next state the analogue of Theorem 2.1 for second order equations.

Theorem 3.1. *Let A, X_0, P be as in Theorem 2.1. Let u, defined by (3.5), be the unique mild solution of (3.6), where we assume $(1+t)h(t) \in L^1(\mathbb{R}^+, X)$, $f \in D(A)$ and $g \in X_0$. Let $k_1 = \int_0^\infty Ph(s)ds \in N(A)$. If $k_1 \neq -Pg$, then*

$$\lim_{t\to\infty} \left\| \frac{1}{t} \int_0^t u(s)ds \right\| = \infty,$$

so that the ergodic limit $\lim_{t\to\infty} \frac{1}{t} \int_0^t u(s)ds$ fails to exist. If $k_1 = -Pg$, $k_0 = \int_0^\infty sPh(s)ds$ and if

$$\lim_{t\to\infty} t \left(Pg + \int_0^t Ph(s)ds \right) = k_2 \in N(A) \tag{3.7}$$

exists, then

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t u(s)ds = Pf + k_2 - k_0.$$

Proof. The unique mild solution of (3.6) is

$$u(t) = \sum_{j=1}^{3} u_j(t) := C(t)f + S(t)g + \int_0^t S(t-s)h(s)ds. \tag{3.8}$$

By the MET for cosine functions,

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t u_1(s)ds = Pf.$$

Now assume $(I-P)g, (I-P)h(s) \in R(A)$ for each $s \geq 0$. Then

$$u_2(t) = S(t)Ag_1 = \frac{1}{2}(e^{tA} - e^{-tA})g_1$$

and

$$\frac{1}{t}\int_0^t u_2(t)dt \to 0$$

as $t \to \infty$ by the MET for semigroups. Furthermore, we can approximate $(I-P)u_3(t)$ in $L^1(\mathbb{R}^+, X)$ by a sequence of the form

$$\int_0^t S(t-s)A\tilde{h}_n(s)ds$$

where $\tilde{h}_n(s) \in D(A)$ and $\tilde{h}_n \in L^1(\mathbb{R}^+, X)$. We omit writing the subscript n. Then

$$\begin{aligned}\int_0^t S(t-s)A\tilde{h}(s)ds &= \int_0^t \frac{1}{2}\left(e^{(t-s)A} - e^{(s-t)A}\right)\tilde{h}(s)ds \\ &= \frac{1}{2}\left[e^{tA}\int_0^t e^{-sA}\tilde{h}(s)ds - e^{-tA}\int_0^t e^{sA}\tilde{h}(s)ds\right] \\ &= \frac{1}{2}\left(e^{tA}l_- - e^{-tA}l_+\right) + o(1)\end{aligned}$$

as $t \to \infty$ where

$$l_\pm = \int_0^\infty e^{\mp sA}\tilde{h}(s)ds \in \overline{R(A)}.$$

Then

$$\begin{aligned}\frac{1}{\tau}\int_0^\tau \int_0^t S(t-s)A\tilde{h}(s)ds &= \frac{1}{2\tau}\int_0^\tau \left(e^{tA}l_- - e^{-tA}l_+\right)dt + o(1) \\ &\to 0\end{aligned}$$

by the MET for semigroups. This completes the portion of the proof dealing with $(I-P)u(t)$. Now we consider $Pu(t)$, using (3.8). Then

$$\begin{aligned}Pu(t) &= C(t)Pf + S(t)Pg + \int_0^t S(t-s)Ph(s)ds \\ &= C(t)Pf + tPg + \int_0^t (t-s)Ph(s)ds\end{aligned}$$

since $S(t) = tP$ on $N(A)$. Next

$$\frac{1}{t}\int_0^t Pu_1(s)ds = \frac{1}{t}\int_0^t C(s)Pfds \to Pf$$

as $t \to \infty$, and

$$\begin{aligned}w(t) := Pu_2(t) + Pu_3(t) &= tPg + t\int_0^t Ph(s)ds - \int_0^t sPh(s)ds \\ &= t\left(Pg + \int_0^t Ph(s)ds\right) - \int_0^\infty sPh(s)ds + o(1) \qquad (3.9)\end{aligned}$$

as $t \to \infty$. Let

$$k_1 = \int_0^\infty Ph(s)ds, \qquad k_0 = \int_0^\infty sPh(s)ds. \qquad (3.10)$$

If $Pg + k_1 \neq 0$, then $\|w(t)\| \to \infty$ as $t \to \infty$, whence

$$\left\|\frac{1}{t}\int_0^t w(s)ds\right\| \to \infty, \quad \text{as } t \to \infty.$$

Thus
$$\left\| \frac{1}{t} \int_0^t u(s)ds \right\| \to \infty, \quad \text{as } t \to \infty.$$

Now suppose $Pg + \int_0^\infty Ph(s)ds = 0$ and
$$\lim_{t\to\infty} t\Big(Pg + \int_0^t Ph(s)ds\Big) = k_2 \in N(A)$$
exists in X. Then
$$\lim_{t\to\infty} \frac{1}{t} \int_0^t Pu(s)ds = P(f+l) = Pf + k_2 - k_0$$
by (3.9), (3.10). Theorem 3.1 now follows. □

4 Examples

We conclude with some examples. The first is the Wentzell wave equation on a bounded domain Ω in $\mathbb{R}^n$.

Consider the wave equation
$$\frac{\partial^2 u}{\partial t^2} = \Delta u, \qquad x \in \Omega,\ t \in \mathbb{R}, \tag{4.1}$$
with initial conditions
$$u(x,0) = f(x), \qquad \frac{\partial u}{\partial t}(x,0) = g(x) \tag{4.2}$$
and dynamic boundary conditions
$$\frac{\partial^2 u}{\partial t^2} - \beta\frac{\partial u}{\partial n} - \gamma u + q\beta\Delta_{LB} u = 0, \qquad x \in \Omega,\ t \in \mathbb{R}, \tag{4.3}$$
where Ω is a $C^{2+\varepsilon}$ bounded domain in $\mathbb{R}^n$ with boundary $\partial\Omega$, $\varepsilon > 0$, $0 < \beta \in C^1(\partial\Omega)$, $0 \le \gamma \in C(\partial\Omega)$, $q \in [0,\infty)$, and Δ_{LB} is the Laplace–Beltrami operator on $\partial\Omega$. Assuming (4.1) holds for $x \in \partial\Omega$, then one can replace $\frac{\partial^2 u}{\partial t^2}$ by $\mathrm{tr}(\Delta u)$ in (4.3) and (4.3) then becomes a Wentzell boundary condition
$$\mathrm{tr}(\Delta u) - \beta\frac{\partial u}{\partial n} - \gamma u + q\beta\Delta_{LB} u = 0$$
on $\partial\Omega$. Let
$$X_2 = L^2(\Omega, dx) \oplus L^2(\partial\Omega, \frac{dS}{\beta(x)}),$$
$$S_0 = \begin{bmatrix} \Delta & 0 \\ -\beta\frac{\partial}{\partial n} & -\gamma + q\beta\Delta_{LB} \end{bmatrix},$$
$D(S_0) = \Big\{U = \left[\begin{smallmatrix} u \\ \mathrm{tr}(u) \end{smallmatrix}\right] =: u \in C^2(\overline{\Omega})\Big\}$, $S_1 = \overline{S_0}$. Then $S_1 = S_1^* \ge \varepsilon I$ on X_2 for some $\varepsilon > 0$, and
$$\frac{\partial^2 U}{\partial t^2} + S_1 U = h(x,t)$$
is the inhomogeneous Wentzell wave equation corresponding to (4.1)–(4.3). See [1–3].

The operator S_1 has a compact resolvent and has an orthonormal basis $\{\varphi_k\}_{k=0}^\infty$ of eigenfunctions corresponding to eigenvalues $0 < \lambda_0 < \lambda_1 \le \lambda_2 \le \cdots \to \infty$, with λ_0 a simple eigenvalue and $\varphi_0 > 0$ in Ω, the "ground state eigenfunction". Now let $A = i(S_1 - \lambda_0)^{\frac{1}{2}}$, so that iA

is selfadjoint on X_2 and $N(A) = \operatorname{span}\{\varphi_0\}$, a one dimensional space. For $F = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \in X_2$, PF is the constant function with value $\langle F, \varphi_0 \rangle_{X_2} = \int_\Omega f_1(x)\varphi_0(x)dx + \int_{\partial\Omega} f_2(x)\varphi_0(x)\frac{dS}{\beta(x)}$. Theorem 3.1 applies. The initial condition $\frac{\partial u}{\partial t}(0) = g \in X_2$ is in $\overline{R(A)}$ iff $\langle g, \varphi_0 \rangle_{X_2} = 0$. The ergodic limits of Theorem 3.1 will all exist if the limit (3.7) exists, that is,

$$\lim_{t\to\infty} t \left(\langle g, \varphi_0 \rangle_{X_2} + \int_0^t \langle h(s), \varphi_0 \rangle_{X_2} ds \right) \tag{4.4}$$

exists. Since $\int_0^\infty \langle h(s), \varphi_0 \rangle_{X_2} ds$ exists, the existence of (4.4) means, when

$$\int_0^\infty \langle h(s), \varphi_0 \rangle_{X_2} ds = -\langle g, \varphi_0 \rangle_{X_2},$$

that the integral in (4.4) converges fast enough as $t \to \infty$.

For non Hilbert space examples, we look at the one dimensional wave equation,

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + h(x,t), \qquad u(x,0) = f(x), \quad u_t(x,0) = g(x) \tag{4.5}$$

for $x, t \in \mathbb{R}$. Let $w \in BUC(\mathbb{R})$ be a weight function which satisfies $0 < \varepsilon \le w(x) \le \frac{1}{\varepsilon} < \infty$ for all $x \in \mathbb{R}$. Let $X_p = L^p(\mathbb{R}, w(x)dx)$, $X_\infty = BUC_w(\mathbb{R})$ with norm $\|f\|_{w_\infty} = \sup_{x\in\mathbb{R}} |f(x)|w(x)$.

Let $A = \frac{d}{dx}$, $e^{tA}f(x) = f(x+t)$. The unique mild solution of (4.5) in X_p, $1 \le p \le \infty$, is

$$u(x,t) = \frac{1}{2}\Big(f(x+t) + f(x-t)\Big) + \frac{1}{2}\int_{x-t}^{x+t} g(s)ds + \frac{1}{2}\int_0^t \int_{x-t+s}^{x+t-s} h(r,x)drds.$$

Then A generates a uniformly (C_0) group on X_p which is not isometric if $w \ne$ constant.

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