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Subharmonic bouncing solutions of generalized Lazer–Solimini equation

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Abstract. The paper deals with the singular differential equation $x'' + g(x) = p(t)$, with g having a weak singularity at $x = 0$ and 2π-periodic function p. For any positive integers m and n, the coexistence of $2m\pi$-periodic bouncing solutions having n impacts with the singularity and a classical positive periodic solution is proven.

Keywords: subharmonic solution, elastic impact, nonnegative solution, impulsive differential equation, generalized Lazer–Solimini equation, coexistence of solutions, weak singularity.

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1 Introduction

Investigation of Lazer–Solimini equation dates back to the year 1987 when the authors Lazer and Solimini published their existence results for the equation

$$x'' \pm \frac{1}{x^\alpha} = p(t), \quad \alpha > 0,\ p \text{ is a } 2\pi\text{-periodic function},$$

where they found necessary and sufficient conditions for the existence of periodic solution, see [3].

Later, many authors (e.g. see [1, 2, 8] or see an overview of the results in [10]) obtained existence results for equation with a generalized singular term

$$x'' + g(x) = p(t), \tag{1.1}$$

where $g : (0, \infty) \to \mathbb{R}$ has various types of singularity at $x = 0$. Two types of this singularity are distinguished:

- attractive, i.e. $\lim_{x\to 0+} g(x) = +\infty$, vs. repulsive, i.e. $\lim_{x\to 0+} g(x) = -\infty$

and

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- weak, i.e. $\int_0^1 g(x)\,\mathrm{d}x \in \mathbb{R}$ vs. strong, i.e. $\int_0^1 g(x)\,\mathrm{d}x = \pm\infty$.

It is a well-known result, that under the assumptions that g is positive, nonincreasing and continuous on $(0,\infty)$, $\lim_{x\to\infty} g(x) = 0$ and p is a 2π-periodic and continuous on $\mathbb{R}$, then the necessary and sufficient condition for the existence of a classical positive 2π-periodic solution of Eq. (1.1) is the assumption

$$\bar{p} := \frac{1}{2\pi}\int_0^{2\pi} p(t)\,\mathrm{d}t > 0,$$

where 2π is the period of the function p – see e.g. [1,8] (the necessity can be immediately seen by integrating Eq. (1.1) over the interval $[0,2\pi]$).

Otherwise, i.e. if $\bar{p} \le 0$, the Eq. (1.1) can be understood as an impact oscillator having a singularity at the obstacle. Therefore one can investigate another type of solution – so called *bouncing solution* – e.g. see [5]. It is a generalized solution of Eq. (1.1) in the sense that

- such function is a solution of Eq. (1.1) only on certain open intervals where it is positive,
- it satisfies certain impulsive conditions at those instants where the solution reaches zero – see Definition 2.2.

The problems of the existence of such solutions were investigated using Poincaré–Birkhoff Twist Map Theorem for an area preserving homeomorphism of an annulus, e.g. see [4–7,9].

In particular, in 2004, Qian and Torres [6] investigated Eq. (1.1) with an attractive weak singularity for the case $\bar{p} < 0$, i.e. if no classical solution exists. They found sufficient conditions for the existence of periodic and subharmonic solutions with prescribed number of bounces in each period. They suggested a possible existence of this type of solution even in the case when the classical solution exists, i.e. a classical solution would coexist with a bouncing one. In [9], this question was partially answered. Sufficient conditions ensuring the existence of at least two 2π-periodic bouncing solutions with one bounce in each period were given.

The purpose of this paper is to extend the results of [9] and find sufficient conditions guaranteeing the existence of the subharmonic solutions with prescribed number of bounces in each period. The proofs in [9] are based on the investigation of the area-preserving homeomorphism T which has been constructed just for one bounce in the period. But T looses some needed properties (e.g. the monotonicity of its first component T_1) if the construction of T is extended for more bounces in the period, and so the approach of [9] cannot be directly used. Therefore the proofs in this paper are based on the combination of the results obtained in [6] and [9].

The paper is organized as follows. In Section 2 we give necessary definitions of a classical and bouncing solution, the main result (Theorem 2.3) together with a consequent result for Lazer–Solimini equation (Corollary 2.4) and an example. In the third section we prove a slight modification of the existence theorem from [6] in order to apply it to the properly constructed auxiliary equation (3.4). In Section 4, the estimations of bouncing solutions of the auxiliary problem are given and subsequently the proof of the main result is finished.

2 Problem formulation and main results

We investigate the differential equation of the second order (1.1) under the following assumptions:

$$g \text{ is locally Lipschitz continuous function, positive and nonincreasing on } (0,\infty), \tag{2.1}$$

$$\lim_{x\to 0+} g(x) = \infty, \quad \int_0^1 g(x)\,\mathrm{d}x < \infty, \tag{2.2}$$

$$p \text{ is continuous function, } 2\pi\text{-periodic on } \mathbb{R}, \tag{2.3}$$

$$\frac{1}{2\pi}\int_0^{2\pi} p(s)\,\mathrm{d}s =: \bar{p} > g(\infty) := \lim_{x\to\infty} g(x). \tag{2.4}$$

Let us precisely define the types of solutions of Eq. (1.1) used in this article. To emphasize the concept of bouncing solution, we start with a classical solution.

Definition 2.1. We say that x is *a (classical) solution of Eq.* (1.1) *on an interval* $J \subset \mathbb{R}$ iff x is a positive, twice continuously differentiable function on J, and x satisfies the differential equation (1.1) on J.

Definition 2.2. We say that $x : \mathbb{R} \to \mathbb{R}$ is *a bouncing solution of Eq.* (1.1) iff there exists a doubly infinite sequence $\{t_i\}_{i\in\mathbb{Z}}$, $t_i < t_{i+1}$, $i \in \mathbb{Z}$ such that

(i) $x(t_i) = 0$,

(ii) $x'(t_i+) = -x'(t_i-)$,

(iii) x is a classical solution of Eq. (1.1) on (t_i, t_{i+1}).

We call t_i *the bounces of the solution* x.

We can see that a bouncing solution consists of several maximal classical solutions separated by bounces.

To state the main result of the paper, we introduce the notation

$$p_{\max} = \max_{s\in\mathbb{R}} p(s), \quad p_{\min} = \min_{s\in\mathbb{R}} p(s),$$

and denote by K a positive constant satisfying

$$g(K) > p_{\max}. \tag{2.5}$$

The existence of such K follows from assumptions (2.1)–(2.3).

Theorem 2.3 (Main result: Coexistence of bouncing and classical periodic solutions)**.** *Let* (2.1)–(2.4) *hold and let*

$$\left(\frac{K}{m}\right)^2 + 2\pi^2 p_{\min} K \geq 2\pi^2 \int_0^K g(x)\,\mathrm{d}x, \tag{2.6}$$

where K fulfills (2.5)*, $m \in \mathbb{N}$. Then*

(i) there exists a classical solution of Eq. (1.1) *greater than K,*

(ii) there exist at least two $2m\pi$-periodic bouncing solutions of Eq. (1.1) *with one bounce in each period such that their maximal values are lower than K,*

(iii) for any $n \in \mathbb{N}$, $n > 1$ there exist at least one $2m\pi$-periodic bouncing solution of Eq. (1.1) *with exactly n bounces in each period, which has the maximum value lower than K.*

We give even more effective sufficient conditions for the existence of solutions of Lazer–Solimini equation

$$x'' + x^{-\alpha} = p(t), \tag{2.7}$$

with $\alpha \in (0, 1)$.

Corollary 2.4. *Let $p : \mathbb{R} \to \mathbb{R}$ be a continuous, 2π-periodic function, $\bar{p} > 0$, $\alpha \in (0,1)$ and $m \in \mathbb{N}$ be such that*

$$m < \frac{1}{\pi}\sqrt{\frac{1-\alpha}{2p_{\max}^{\frac{1}{\alpha}}(p_{\max} - (1-\alpha)p_{\min})}}. \tag{2.8}$$

Then

(i) there exists a classical solution of Eq. (2.7) greater or equal to $p_{\max}^{-\frac{1}{\alpha}}$,

(ii) there exist at least two $2m\pi$-periodic bouncing solutions of Eq. (2.7) with one bounce in each period such that their maximal values are lower than $p_{\max}^{-\frac{1}{\alpha}}$,

(iii) for any $n \in \mathbb{N}$, $n > 1$ there exist at least one $2m\pi$-periodic bouncing solution of Eq. (2.7) with exactly n bounces in each period, which has the maximum value lower than $p_{\max}^{-\frac{1}{\alpha}}$.

Proof. We apply Theorem 2.3 on Eq. (2.7). The assumptions (2.1)–(2.4) are trivially satisfied for $g(x) = x^{-\alpha}$, $x > 0$. It remains to find a positive K satisfying (2.5) and (2.6). These conditions are satisfied iff

$$K^{-\alpha} > p_{\max} \tag{2.9}$$

and

$$\left(\frac{K}{m}\right)^2 + 2\pi^2 p_{\min} K \geq 2\pi^2 \frac{K^{1-\alpha}}{1-\alpha}. \tag{2.10}$$

The inequality (2.9) is equivalent to $K < p_{\max}^{-\frac{1}{\alpha}}$. And the inequality (2.10) can be written in the form

$$\omega(K) := \frac{K}{2\pi^2 m^2} + p_{\min} - \frac{K^{-\alpha}}{1-\alpha} \geq 0,$$

where $\omega : \mathbb{R} \to \mathbb{R}$ is continuous on $(0,\infty)$, $\lim_{K\to 0+}\omega(K) = -\infty$, $\lim_{K\to\infty}\omega(K) = \infty$ and $\omega'(K) > 0$ for each $K \in (0,\infty)$. Therefore there exists a unique $K_0 > 0$ such that $\omega(K_0) = 0$. Since ω is strictly increasing, K satisfies (2.9) and (2.10) iff $K \in [K_0, p_{\max}^{-\frac{1}{\alpha}})$. From (2.8) we get $\omega(p_{\max}^{-\frac{1}{\alpha}}) > 0$, which implies that the interval $[K_0, p_{\max}^{-\frac{1}{\alpha}})$ is nonempty. Let us choose some $K \in [K_0, p_{\max}^{-\frac{1}{\alpha}})$. According to Theorem 2.3 (i) there exists a classical solution x of Eq. (2.7) greater than K. If $\min_{t\in\mathbb{R}} x(t) < p_{\max}^{-\frac{1}{\alpha}}$, then there exists $t_0 \in \mathbb{R}$ such that $x(t_0) < p_{\max}^{-\frac{1}{\alpha}}$, $x'(t_0) = 0$ and $x''(t_0) \geq 0$. In view of (2.7) we get

$$x''(t_0) = -(x(t_0))^{-\alpha} + p(t_0) < -p(t_0) + p(t_0) = 0,$$

which is a contradiction. Therefore the classical solution is bounded from below by $p_{\max}^{-\frac{1}{\alpha}}$.

The assertions (ii) and (iii) follow directly from Theorem 2.3 (ii), (iii) and from the inequality $K < p_{\max}^{-\frac{1}{\alpha}}$. □

The feasibility of the obtained result is illustrated in the following example.

Example 2.5. We consider Lazer–Solimini equation (2.7), where $\alpha \in (0,1)$, $m \in \mathbb{N}$ and $p(t) = b\sin t + c$ with $b, c > 0$. Then $p_{\max} = b + c$ and $p_{\min} = c - b$, and so the condition (2.8) can be written as

$$m < \frac{1}{\pi}\sqrt{\frac{1-\alpha}{2(b+c)^{\frac{1}{\alpha}}(2b + \alpha(c-b))}}. \tag{2.11}$$

For instance, the condition (2.11) is valid for these values of the parameters:

- $\alpha = 0.5$, $b = 0.01$, $c = 0.11$, $m \le 3$, or
- $\alpha = 0.1$, $b = 0.01$, $c = 0.51$, $m \le 30$.

Remark 2.6. Let us note that the assumptions of Theorem 2.3 always fail to be satisfied for high m. Indeed, let the assumptions of Theorem 2.3 hold for each $m \in \mathbb{N}$ with $K = K_m$ in (2.5) and (2.6), i.e.

$$g(K_m) > p_{\max} \tag{2.12}$$

and

$$\left(\frac{K_m}{m}\right)^2 + 2\pi^2 p_{\min} K_m \ge 2\pi^2 \int_0^{K_m} g(x)\,\mathrm{d}x \tag{2.13}$$

for each $m \in \mathbb{N}$. From (2.1), (2.2) and (2.4) it follows that there exists $\overline{K} > 0$ such that $g(\overline{K}) = p_{\max}$ and according to (2.12) and (2.1) also $\overline{K} > K_m$ for every $m \in \mathbb{N}$, i.e. $\{K_m\}$ is bounded. On the other hand, from (2.1) and (2.12) we get

$$\int_0^{K_m} g(x)\,\mathrm{d}x \ge \int_0^{K_m} g(K_m)\,\mathrm{d}x = g(K_m)K_m > p_{\max}K_m.$$

This estimate together with (2.13) yields an inequality

$$\left(\frac{K_m}{m}\right)^2 + 2\pi^2 p_{\min} K_m > 2\pi^2 p_{\max} K_m,$$

which gives

$$\left(\frac{K_m}{m}\right)^2 > 2\pi^2 (p_{\max} - p_{\min}) K_m$$

and finally

$$K_m > 2\pi^2 (p_{\max} - p_{\min}) m^2$$

for every $m \in \mathbb{N}$. The last inequality contradicts the boundedness of $\{K_m\}$. Therefore, the (non)existence of subharmonic solutions of *arbitrary period* is still an open problem.

3 Auxiliary equation

First, let us state the main result from [6], which will be used here as the main existence principle.

Theorem 3.1 (see [6, Theorem 1.2]). *Let us assume that*

$$\left.\begin{array}{l} g : (0,\infty) \to (0,\infty) \text{ is locally Lipschitz continuous function,} \\ \text{there exists } \varepsilon > 0 \text{ such that } g \text{ is strictly decreasing on } (0,\varepsilon), \end{array}\right\} \tag{3.1}$$

g satisfies (2.2), p fulfills (2.3) and

$$\frac{1}{2\pi}\int_0^{2\pi} p(s)\,\mathrm{d}s =: \bar{p} < 0 = g(\infty) := \lim_{x\to\infty} g(x). \tag{3.2}$$

Then, for any $m \in \mathbb{N}$, there exist at least two $2m\pi$-periodic bouncing solutions of Eq. (1.1) with one bounce in each period. Moreover, for any $n, m \in \mathbb{N}$, $n \ge 2$, there exists at least one $2m\pi$-periodic bouncing solution of Eq. (1.1) with n bounces in each period.

In the current paper we will use this result under slightly different assumptions. More precisely, we replace assumption (3.1) by (2.1) and assumption (3.2) by

$$\bar{p} < 0, \quad 0 \le g(\infty) := \lim_{x\to\infty} g(x). \tag{3.3}$$

Theorem 3.2. *Let us assume that (2.1), (2.2), (2.3) and (3.3) hold. Then the assertions of Theorem 3.1 remain valid.*

Proof. Decreasing character of g in (3.1) is used in the paper [6] only to prove uniqueness in the singular IVP (1.1), $x(t_0) = 0$, $x'(t_0) = y_0 > 0$, see [6, Remark 2.4]. Since this uniqueness was already proved in [9] under the assumptions (2.1)–(2.3), the replacement of (3.1) by (2.1) in Theorem 3.2 is correct.

In [6], only the positivity of the function g is used, not the fact $g(\infty) = 0$. Therefore the replacement of (3.2) by (3.3) is also correct. □

Now, we introduce the auxiliary equation

$$x'' + f(x) = p(t), \tag{3.4}$$

where $f : (0,\infty) \to (0,\infty)$ is defined by

$$f(x) = \begin{cases} g(x) & \text{if } x \in (0,K], \\ g(K) & \text{if } x > K, \end{cases} \tag{3.5}$$

with g, p and K satisfying (2.1), (2.2), (2.3) and (2.5). From (2.5) it follows that there exists $\varepsilon > 0$ such that

$$g(K) - p_{\max} > \varepsilon$$

and therefore

$$f(x) - p_{\max} > \varepsilon \tag{3.6}$$

for each $x > 0$.

Theorem 3.3. *Let us assume that (2.1), (2.2), (2.3) and (3.3) hold and let $f : \mathbb{R} \to \mathbb{R}$ be defined by (3.5). Then the assertions of Theorem 3.2 are valid for Eq. (3.4).*

Proof. Let us consider the differential equation

$$x'' + h(x) = r(t) \tag{3.7}$$

with

$$h(x) = f(x) - p_{\max} - \frac{\varepsilon}{2}, \quad x > 0 \tag{3.8}$$

and

$$r(t) = p(t) - p_{\max} - \frac{\varepsilon}{2}, \quad t \in \mathbb{R}. \tag{3.9}$$

By (2.3), (3.8) and (3.9), we see that r is a continuous 2π-periodic function, so r fulfills condition (2.3).

By (2.1) and (3.6) we see that h is locally Lipschitz continuous, positive and nonincreasing on $(0,\infty)$ which means that h fulfills conditions (2.1).

Using (2.2), (3.6), (3.8) and (3.9), we get

$$\bar{r} = \bar{p} - p_{\max} - \frac{\varepsilon}{2} < 0$$

and

$$\lim_{x\to\infty} h(x) = g(K) - p_{\max} - \frac{\varepsilon}{2} \geq \varepsilon - \frac{\varepsilon}{2} > 0.$$

Therefore also conditions (2.2) and (3.3) are satisfied. From Theorem 3.2 we get that assertions of Theorem 3.1 are valid for Eq. (3.7).

Note that Eq. (3.7) is equivalent to Eq. (3.4). Indeed Eq. (3.7) is obtained from Eq. (3.4) by subtracting the expression $p_{\max} + \varepsilon/2$ from both sides. Therefore the assertions of Theorem 3.1 remains valid also for Eq. (3.4). □

4 Bounds of bouncing solutions

By Theorem 3.3 there exist at least two $2m\pi$-periodic bouncing solutions of Eq. (3.4) with one bounce in each period and existence of at least one $2m\pi$-periodic bouncing solution of Eq. (3.4) with n ($n > 1$) bounces in each period. It remains to prove that all these solutions are bounded from above by the constant K and therefore they are also bouncing solutions of Eq. (1.1). This is the main purpose of this section.

To achieve this goal we use several auxiliary results from [9], namely Lemma 4.1, 4.2 and 4.3 from Section 4 of that paper. Here, we assume that (2.1)–(2.4) are satisfied – these are the same assumption as in [9, Section 4].

Let us consider an initial value problem (3.4),

$$x(t_0) = 0, \qquad x'(t_0+) = y_0, \tag{4.1}$$

where $t_0 \in \mathbb{R}$, $y_0 > 0$.

Lemma 4.1 (see [9, Lemma 8])**.**

(a) Let $t_0 \in \mathbb{R}$, $y_0 > 0$. Then there exists a finite $t_1 > t_0$ and a unique maximal solution x of IVP (3.4), (4.1) on (t_0, t_1) such that $x(t_1-) = 0$. Moreover there exists $a \in (t_0, t_1)$ such that

$$x'(a) = 0, \quad x' > 0 \ \text{ on } (t_0, a), \quad x' < 0 \ \text{ on } (a, t_1), \quad x'(t_1-) < 0.$$

(b) Let $t_1 \in \mathbb{R}$, $y_1 > 0$. Then there exists a finite $t_0 < t_1$ and a unique maximal solution x of TVP (3.4), $x(t_1) = 0, x'(t_1-) = -y_1$ on (t_0, t_1) such that $x(t_0+) = 0$. Moreover there exists $a \in (t_0, t_1)$ such that

$$x'(a) = 0, \quad x' > 0 \ \text{ on } (t_0, a), \quad x' < 0 \ \text{ on } (a, t_1), \quad x'(t_0+) > 0.$$

Further we need some estimates. First we define several useful functions

$$F(x) = \int_0^x f(s)\,\mathrm{d}s, \quad \alpha(x) = F(x) - p_{\max}x, \quad \beta(x) = F(x) - p_{\min}x, \quad x \in [0, \infty). \tag{4.2}$$

Finally, we will need the following assertions from [9].

Lemma 4.2 (see [9, Lemma 10])**.** *Let x be a maximal solution of Eq. (3.4) on the interval (t_0, t_1). Then*

$$\sqrt{2\alpha(x_{\max})} \leq x'(t_0+) \leq \sqrt{2\beta(x_{\max})}, \tag{4.3}$$

$$-\sqrt{2\beta(x_{\max})} \leq x'(t_1-) \leq -\sqrt{2\alpha(x_{\max})}, \tag{4.4}$$

$$\beta^{-1}\left(\frac{y_0^2}{2}\right) \le x_{\max} \le \alpha^{-1}\left(\frac{y_0^2}{2}\right), \tag{4.5}$$

$$t_1 - t_0 \le \frac{2y_0}{\varepsilon}, \tag{4.6}$$

$$\forall \eta \in (0, x_{\max}) \ : \ t_1 - t_0 > \sqrt{\frac{2(x_{\max} - \eta)}{f(\eta) - p_{\min}}}, \tag{4.7}$$

where α, β *are from* (4.2), $x(a) := x_{\max} := \max\{x(t) : t \in (t_0, t_1)\}$, *and* ε *is from* (3.6).

Lemma 4.3 (see [9, Lemma 13]). *There exists a continuous* 2π*-periodic function* $\psi : \mathbb{R} \to \mathbb{R}$, $\psi(\mathbb{R}) \subset \left[\sqrt{2\alpha(K)}, \sqrt{2\beta(K)}\right]$ *such that the solution* x *of IVP* (3.4), (4.1) *with* $y_0 = \psi(t_0)$ *has its maximum value* $x_{\max}$ *equal to* K, *for each* $t_0 \in \mathbb{R}$.

The following lemma is a generalization of [9, Lemma 11].

Lemma 4.4. *Let* x, $\tilde{x}$ *be two different maximal classical solutions of Eq.* (3.4) *defined on the intervals* (t_0, t_1), $(\tilde{t}_0, \tilde{t}_1)$, *respectively. If* $(t_0, t_1) \subset (\tilde{t}_0, \tilde{t}_1)$, *then*

$$0 < x(t) < \tilde{x}(t), \quad t \in (t_0, t_1).$$

Proof. Let us prove the lemma by contradiction. Let the assumptions be satisfied and there exists $\tau \in (t_0, t_1)$ such that $x(\tau) \ge \tilde{x}(\tau)$. We put $v(t) = x(t) - \tilde{x}(t)$, $t \in (t_0, t_1)$. Then $v(t_0+) \le 0$, $v(t_1-) \le 0$ and $v(\tau) \ge 0$. From the continuity of v it follows that there exists an interval $(\tau_0, \tau_1) \subset (t_0, t_1)$ such that $v(\tau_0+) = v(\tau_1-) = 0$ and $v(t) \ge 0$ for $t \in (\tau_0, \tau_1)$. This implies $v'(\tau_0+) \ge 0$. There are two possibilities:

Case A. If $v'(\tau_0+) = 0$, then x and $\tilde{x}$ would be solutions of the same IVP and according to the uniqueness (see Lemma 4.1), we get $x = \tilde{x}$, which is a contradiction.

Case B. Let $v'(\tau_0+) > 0$. From the Mean Value Theorem we get that there exists $\xi \in (\tau_0, \tau_1)$ such that $v'(\xi) = v(\tau_1-) - v(\tau_0+) = 0$. Since x and $\tilde{x}$ are solutions of Eq. (3.4) on (τ_0, τ_1) and f is decreasing, we get

$$v''(t) = x''(t) - \tilde{x}''(t) = -f(x(t)) + f(\tilde{x}(t)) \ge 0$$

for $t \in (\tau_0, \tau_1)$. Integrating this inequality over the interval (τ_0, ξ), we get $v'(\xi) \ge v'(\tau_0+) > 0$, which is also a contradiction. □

The next lemma is a very slight generalization of [9, Lemma 14].

Lemma 4.5. *Let* x *be a maximal solution of IVP* (3.4), (4.1) *with* $y_0 = \psi(t_0)$ *defined on* (t_0, t_1). *If*

$$\left(\frac{K}{m}\right)^2 + 2\pi^2 p_{\min} K \ge 2\pi^2 F(K), \tag{4.8}$$

then

$$t_1 - t_0 > 2m\pi. \tag{4.9}$$

Proof. Let us consider linear functions

$$q(t) = (t - t_0)\sqrt{2\beta(K)}, \quad t \in \mathbb{R}$$

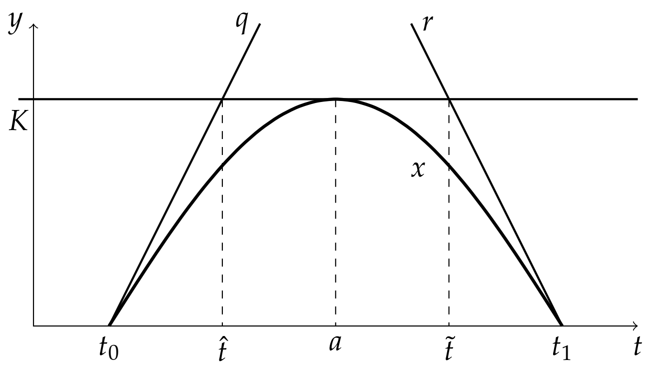

Figure 4.1: The solution x of IVP and auxiliary functions q and r from the proof of Lemma 4.5.

and

$$r(t) = (t_1 - t)\sqrt{2\beta(K)}, \quad t \in \mathbb{R}.$$

The graph of function q passes through some point $(\hat{t}, K)$, where $K = q(\hat{t}) = (\hat{t} - t_0)\sqrt{2\beta(K)}$. Consequently,

$$\hat{t} - t_0 = \frac{K}{\sqrt{2\beta(K)}}.$$

Similarly, the graph of function r passes through some point $(\tilde{t}, K)$, so

$$t_1 - \tilde{t} = \frac{K}{\sqrt{2\beta(K)}}.$$

The solution x is concave on (t_0, t_1) and from Lemma 4.3 we obtain that x has its maximum value equal to K. Denote $x(a) = K$, $a \in (t_0, t_1)$. Therefore $\hat{t} \in (t_0, a)$ and $\tilde{t} \in (a, t_1)$, see Figure 4.1. From (4.2) and assumption (4.8) we have

$$K \geq m\pi\sqrt{2\beta(K)}. \tag{4.10}$$

Finally we obtain

$$t_1 - t_0 = t_1 - a + a - t_0 > t_1 - \tilde{t} + \hat{t} - t_0 = \frac{2K}{\sqrt{2\beta(K)}} \geq \frac{2m\pi\sqrt{2\beta(K)}}{\sqrt{2\beta(K)}} = 2m\pi,$$

where the last inequality follows from (4.10). □

Finally, in the next lemma we get the upper bound of bouncing solutions.

Lemma 4.6. *Let x be $2\pi m$-periodic bouncing solution of Eq.* (3.4) *with n bounces in each period, $m, n \in \mathbb{N}$. Then $x(t) < K$ for each $t \in \mathbb{R}$.*

Proof. Let $t_0 \in \mathbb{R}$ be such that $x(t_0) = 0$. Then there exist bounces $t_1, \ldots, t_n \in \mathbb{R}$ such that $t_0 < t_1 < \cdots < t_n = t_0 + 2\pi m$. Let $\tilde{x}$ be a maximal classical solution of IVP (3.4), (4.1) with $y_0 = \psi(t_0)$ defined on the interval $(t_0, \tilde{t}_1)$. According to Lemma 4.5, $\tilde{t}_1 > t_0 + 2\pi m = t_n$. Then for $i = 0, \ldots, n-1$ we have $(t_i, t_{i+1}) \subset (t_0, \tilde{t}_1)$, which by Lemma 4.4 implies that $x(t) < \tilde{x}(t) \leq K$ for each $t \in (t_i, t_{i+1})$. This proves that x is lower than K on the interval $[t_0, t_0 + 2\pi m]$ and the rest follows from $2\pi m$-periodicity. □

Now, we are ready to prove the main theorem of this paper.

Proof of Theorem 2.3. Let (2.1)–(2.6) be satisfied. Case *(i)* is proved in [9]. Let us prove the cases *(ii)* and *(iii)*. Due to Theorem 3.3 for any $m \in \mathbb{N}$, there exists at least two $2m\pi$-periodic bouncing solutions of Eq. (3.4) with one bounce in each period and for any $n, m \in \mathbb{N}, n \geq 2$, there exists at least one $2m\pi$-periodic bouncing solution of Eq. (3.4) with n bounces in each period. By Lemma 4.6 every bouncing solution of Eq. (3.4) is lower than K. According to (3.5), these functions are also bouncing solutions of Eq. (1.1). □

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