On the Inhomogeneous Hall's Ray of Period-One Quadratics

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CONTENTS

For quadratics with period-one negative continued fraction expansions,

$$\theta = \cfrac{1}{a - \cfrac{1}{a - \cfrac{1}{a - \cdots}}},$$

we show that the inhomogeneous Lagrange spectrum,

$$\boldsymbol{L}(\theta) := \left\{ \liminf_{|n|\to\infty} |n|\,\|n\theta - \gamma\| : \gamma \in \mathbb{R},\ \gamma \notin \mathbb{Z}+\theta\mathbb{Z} \right\},$$

contains an inhomogeneous Hall's ray $[0, c(\theta)]$ with

$$c(\theta) = \tfrac{1}{4}\left(1 - O(a^{-1/2})\right).$$

We describe gaps in the spectrum showing that this is essentially best possible. Pictures of computed spectra are included. Investigating such pictures led us to these results.

1. INTRODUCTION AND STATEMENT OF RESULTS

For a fixed irrational real number θ and real γ one defines the *inhomogeneous approximation constant*

$$M(\theta,\gamma) := \liminf_{|n|\to\infty} |n|\,\|n\theta - \gamma\|.$$

By varying γ (not of the form $n+m\theta$) one obtains the *inhomogeneous Lagrange spectrum* of θ

$$\boldsymbol{L}(\theta) := \{M(\theta,\gamma) : \gamma \in \mathbb{R},\ \gamma \notin \mathbb{Z}+\theta\mathbb{Z}\} \subseteq \left[0, \tfrac{1}{4}\right]. \tag{1-1}$$

Arguably, the inhomogeneous analogue of the classical Lagrange spectrum $\boldsymbol{L} = \{M(\theta,0)^{-1} : \theta \in \mathbb{R}\}$ should be the set of $M(\theta,\gamma)^{-1}$ rather than the set of $M(\theta,\gamma)$, but in order to work with more easily illustrated bounded intervals we decided to avoid the unnecessary complication of taking reciprocals (it is of course trivial to translate our results should the reader prefer the convention of inverting everything). We are interested here in the largest interval $[0, c(\theta)]$ contained in this spectrum,

$$c(\theta) := \sup\{c : [0,c] \subseteq \boldsymbol{L}(\theta)\},$$

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usually referred to as the *inhomogeneous Hall's ray.* Using a Hall style Cantor dissection argument, Cusick, Moran and Pollington [Cusick et al. 1996] have shown that the larger spectrum of one-sided inhomogeneous constants, $\liminf_{n\to\infty} n\|n\theta-\gamma\|$, contains a non-degenerate Hall's ray when θ is the golden ratio. A similar argument can in fact be used to show that $c(\theta)>0$ for any quadratic θ [Crisp et al. $\geq$ 2001]. In the special case that the partial quotients of θ tend to infinity the spectrum will consist solely of the ray $[0,\frac{1}{4}]$—a result implicit in [Fukasawa 1926] and [Barnes 1956].

Here we examine the spectrum only in the simplest case, when θ has a period-one negative continued fraction expansion. We use a constructive approach to obtain very precise results (seemingly not obtainable using the Cantor dissection method). Since changing a finite number of partial quotients does not affect the spectrum we shall work with the purely periodic representative, and suppose from now on that

$$\theta=\tfrac{1}{2}\big(a-\sqrt{a^2-4}\,\big)=\cfrac{1}{a-\cfrac{1}{a-\cfrac{1}{a-\cdots}}}. \tag{1–2}$$

Theorem 1. *For θ of the form* (1–2) *the spectrum* (1–1) *contains the interval* $[0,c(\theta)]$ *with*

$$c(\theta)=\frac{1}{4}\left(1-O\left(\frac{1}{\sqrt{a}}\right)\right). \tag{1–3}$$

More precisely, as $a\to\infty$,

$$\frac{1}{4}\left(1-(1+o(1))\frac{10}{\sqrt{a}}\right) \leq c(\theta)\leq\frac{1}{4}\left(1-(1+o(1))\frac{2\sqrt{2}}{\sqrt{a}}\right). \tag{1–4}$$

Pinner [2000b] has shown how to use an appropriate θ-expansion of γ to evaluate $M(\theta,\gamma)$, which in the period-one case reduces to the classical β-expansion of Rényi [1957] and Parry [1960] (with $\beta=1/\theta$),

$$\gamma=\sum_{i=1}^{\infty}\tfrac{1}{2}(a-2+t_i)\theta^i.$$

The t_i will be a sequence of integers in $[-(a-2),a]$ with the same parity as a (and no blocks a, $a-2,\ldots,$ $a-2$, a or a, $a-2$, $a-2,\ldots$). If the sequence does not contain infinitely many *endpoint configurations* $t_i=a$, we have

$$M(\theta,\gamma)=\frac{1}{4}\frac{M^*(\theta,\gamma)}{1-\theta^2}$$

with

$$M^*(\theta,\gamma):=\liminf_{i\to\infty}s^*(i),$$
$$s^*(i):=\min\{s_1^*(i),s_2^*(i),s_3^*(i),s_4^*(i)\},$$

where

$$s_1^*(i):=(1-\theta+d_i^-)(1-\theta+d_i^+),$$
$$s_2^*(i):=(1+\theta+d_i^-)(1+\theta-d_i^+),$$
$$s_3^*(i):=(1-\theta-d_i^-)(1-\theta-d_i^+),$$
$$s_4^*(i):=(1+\theta-d_i^-)(1+\theta+d_i^+),$$

with

$$d_i^-=\sum_{j=1}^{i}t_{i+1-j}\theta^j,\quad d_i^+=\sum_{j=1}^{\infty}t_{i+j}\theta^j.$$

When γ does contain infinitely many $t_i=a$ one needs to check the minimum of $s_1^*(i)$ and $s_2^*(i)$ for both γ and its *negative* $1-\theta-\gamma$ (in this case we have $M^*(\theta,\gamma)\leq\theta$, and for large a the value will be small and lie well within the inhomogeneous Hall's ray). When γ does not contain any $t_i=a$ the expansion for $1-\theta-\gamma$ simply replaces the t_i by $-t_i$, interchanging $s_3^*(i),s_4^*(i)$ and $s_1^*(i),s_2^*(i)$. Of course if the sequence t_i is eventually periodic with period r then we can replace the liminf by a min over cutting the purely periodic sequence (more precisely a doubly infinite sequence with that period) at the r places in its period. Since the $M(\theta,\gamma)$ obtained from the γ with periodic expansions (the $\gamma\in\mathbb{Q}(\theta)$) are dense in $\boldsymbol{L}(\theta)$ one would expect computing values for small periods to give a reasonable approximation to the spectrum. The figures on pages 490–491 show the spectra obtained by computing the approximation constants corresponding to all possible θ-expansion periods of length at most 7. An interface to produce similar pictures can be found at http://ctl.unbc.ca/CMS/LSC/. In [2000b] the spectrum $\boldsymbol{L}(\theta)$ was described down to the first limit point δ_∞, showing an infinite sequence from the largest point $\delta_0=\frac{1}{4}\left(1-(1+o(1))\frac{2}{a}\right)$ to $\delta_\infty=\frac{1}{4}\left(1-(1+o(1))\frac{4}{a}\right)$. The figures clearly suggest holes beyond the first limit point. Examining the configurations that seemed to correspond to the edges of these holes enabled us to identify $O(\sqrt{a})$ of these gaps extending down to

$\frac{1}{4}\left(1-(1+o(1))\sqrt{8}/\sqrt{a}\right)$, giving the upper bound in Theorem 1.

With

$$u := \begin{cases} 0 & \text{if } a \text{ is even,} \\ 1 & \text{if } a \text{ is odd,} \end{cases}$$

define

$$I_1(t) := \left(1-\theta-\frac{((2t-u)\theta+u\theta^2)}{1-\theta^2}\right) \times \left(1-\theta-\frac{(u\theta+(2t-u)\theta^2)}{1-\theta^2}\right),$$

$$I_2(t) := \left(1-(2t-u+3)\theta+\frac{(2+u)\theta^2+(2t-u-2)\theta^3}{1-\theta^2}\right) \times \left(1+(1-u)\theta+\frac{(2t+u-2)\theta^2+(2-u)\theta^3}{1-\theta^2}\right),$$

$$E_1(t) := \left(1-\theta+\frac{-(2t-u)\theta+(2-u)\theta^2}{1-\theta^2}\right) \times \left(1-\theta+\frac{(2-u)\theta-(2t-u)\theta^2}{1-\theta^2}\right);$$

moreover for $t \equiv a \pmod 2$ set

$$E_2(t) := \left(1-\theta-t\theta+\frac{(t+2)\theta^2+(t-4)\theta^3}{1-\theta^2}\right)^2,$$

and for $t \not\equiv a \pmod 2$

$$E_2(t) := \left(1-(t+1)\theta+(t+2)\theta^2+(t-3)\theta^3\right)^2 -\theta^2\left(1+\theta-2\theta^2+\frac{((t+3)\theta^3+(t-5)\theta^4)}{1-\theta^2}\right)^2.$$

Theorem 2. *Suppose that $a \geq 10$. With I_1, I_2, E_1, E_2 as above, there are always the following gaps in the spectrum $\boldsymbol{L}(\theta)$:*

1. $$I(t) := \tfrac{1}{4}(1-\theta^2)^{-1}\left(I_2(t), I_1(t)\right)$$

$$\text{for } t \in \begin{cases} \left[0,\ \sqrt{2a-8}-3\right] & \text{if } a \text{ is even,} \\ \left[\frac{1}{4}(\sqrt{4a-3}+1),\ \sqrt{2a-8}-2\right] & \text{if } a \text{ is odd.} \end{cases}$$

2. $$E(t) := \tfrac{1}{4}(1-\theta^2)^{-1}\left(E_2(t), E_1(t)\right),$$

$$\text{for } (2+u) \leq t \leq \sqrt{2a-6u+16}-(4-u).$$

The endpoints of these intervals are achieved with expansions t_i consisting of the following blocks, where $(a,b,)^k$ denotes k repetitions of the block a,b and where $k_i \to \infty$:

$I_1(t): -(2t-u),\ -u,$

$I_2(t): (2+u),\ -(2t+2-u),\ (2-u,\ 2t+u-2,)^{k_i}$
$(2-u),\ -(2t+2-u),\ (2+u,\ 2t-u-2,)^{k_i}$

$E_1(t): -(2t-u),\ (2-u),$

$E_2(t): (-t,\ -t,)(t+2,\ t-4,)^{k_i}(t+2)$ *if* $t \equiv a \pmod 2$,
$-(t+1),\ -(t-1),\ ((t+3),\ (t-5),)^{k_i}(t+3),$
$-(t-1),\ -(t+1),\ (t+1),\ (t-1),$
$(-(t+3),\ -(t-5),)^{k_i}-(t+3),\ (t-1),\ (t+1)$ *if* $t \not\equiv a \pmod 2$.

Finally it seems reasonable to ask two questions:

Question 1. Can (1–3) be improved to a precise asymptotic result? For example perhaps

$$c(\theta) = \tfrac{1}{4}\left(1-(1+o(1))\sqrt{8/a}\right).$$

Indeed can the top of the ray be determined exactly?

Question 2. Is there an absolute constant $c_0 > 0$ such that the spectrum of an arbitrary quadratic θ always contains a ray $[0, c(\theta)]$ with $c(\theta) > c_0$? Is it true that $c(\theta) \to \frac{1}{4}$ as the size of the smallest partial quotient in the period of the continued fraction expansion of θ tends to infinity? We can claim no numerical evidence to suggest that these are true (or even give an upper bound on the optimal c_0) but it is worth remarking that both do hold for the largest point in the spectrum; see [Pinner 2000a], for example.

2. THE EXISTENCE OF HALL'S RAY AND ITS ASYMPTOTICS

We now prove Theorem 1 (assuming Theorem 2, which is proved in Section 3). The upper bound in (1–4) follows from Theorem 2. The lower bound will follow from Lemma 2 below. The proof is constructive. Since we are only interested in an asymptotic result we shall assume that a is large (one can show the existence of a ray for much smaller values).

Lemma 1. *Suppose that $a \geq 4^8$ and that*

$$\xi = \left(1-\theta-(m_1+u_0)\theta\right)^2,$$

with $u_0 \in \left[\frac{1}{6}(\sqrt{10}-3), \frac{1}{6}(\sqrt{10}+3)\right]$ and $4\sqrt{a} \leq m_1 \leq a-10a^{1/4}$. Then for $i \geq 1$ we can successively write

$$\xi = \left(1-\theta-\sum_{j=1}^{i}(m_j+l_j)\theta^j-u_i\theta^i\right) \times \left(1-\theta-\sum_{j=1}^{i}(m_j-l_j)\theta^j-u_i\theta^i\right)$$

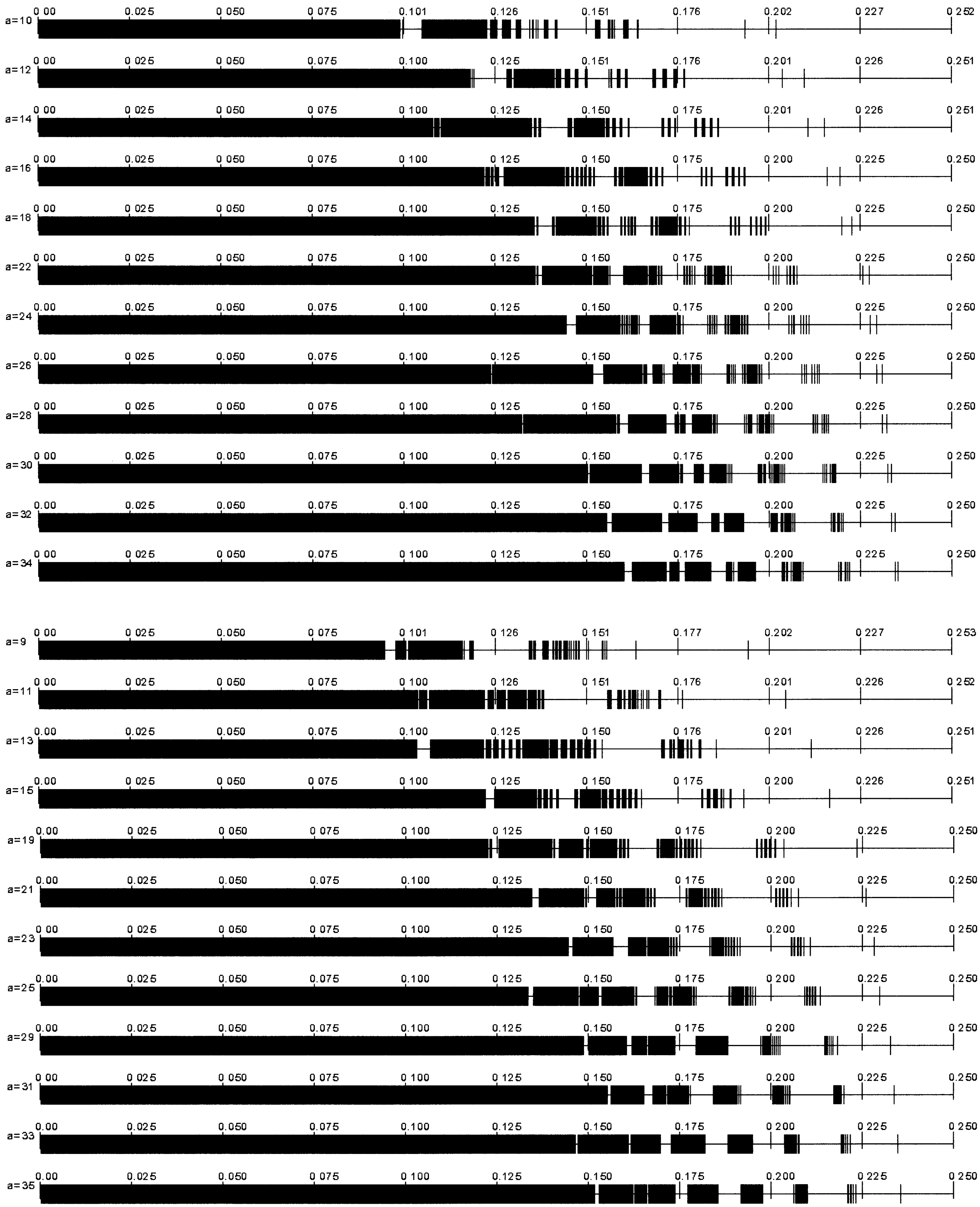

FIGURE 1. Spectra for θ of period a (values corresponding to γ with θ-expansions of period at most seven).

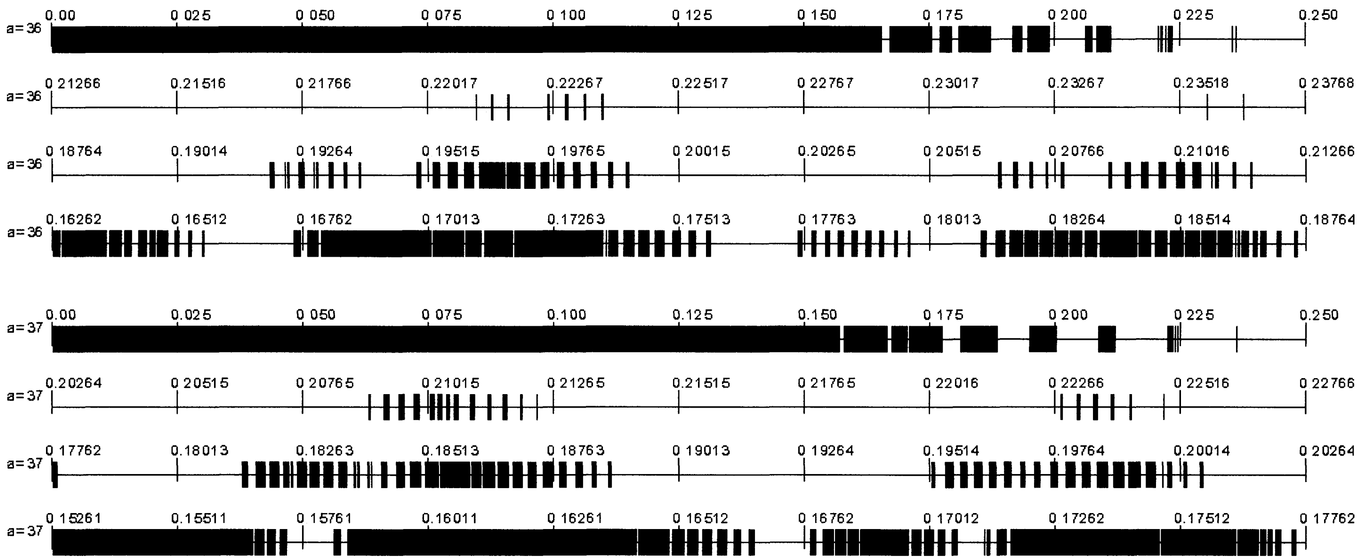

FIGURE 2. Spectra for $a = 36, 37$ (and γ of period at most seven) with zooms.

with integers m_i, l_i, *satisfying* $m_i+l_i \equiv a \pmod 2$,

$$\sqrt{2u_0}\sqrt{a-m_1}\left(1+29(a-m_1)^{-1/2}\right)^{-1} \leq l_1 \leq \sqrt{2u_0}\sqrt{a-m_1},$$

and for $i \geq 2$

$$0 \leq u_{i-1} \leq \frac{2\sqrt{2u_0}}{\sqrt{a-m_1}}\left(1+2a^{-1/8}\right)a\theta,$$

$$0 \leq m_i = \lfloor u_{i-1}/\theta \rfloor \leq \frac{2\sqrt{2u_0}}{\sqrt{a-m_1}}\left(1+2a^{-1/8}\right)a,$$

$$-1 \leq l_i \leq \frac{\sqrt{a-m_1}}{\sqrt{2u_0}}\left(1+\frac{29}{\sqrt{a-m_1}}\right) < \frac{1}{\sqrt{2u_0}}\frac{a}{\sqrt{a-m_1}}\left(1+a^{-1/8}\right).$$

Proof. The first step, $i = 1$, amounts to solving

$$l_1^2\theta = (2-2\theta-2m_1\theta-(u_1+u_0)\theta)(u_0-u_1).$$

Take l_1 to be the integer $l_1 \equiv m_1+a \pmod 2$ such that

$$\sqrt{\frac{(2-2\theta-2m_1\theta-u_0\theta)u_0}{\theta}} = l_1+\lambda_1$$

with $0 \leq \lambda_1 < 2$. We have $l_1 < \sqrt{2(a-m_1)u_0}$ and

$$l_1 > \sqrt{(2a-2\theta-2-2m_1-u_0)u_0}-2 > \sqrt{2(a-m_1)u_0}\left(1+29(a-m_1)^{-1/2}\right)^{-1}.$$

Hence

$$\theta\lambda_1^2-2\lambda_1\sqrt{(2-2\theta-2m_1\theta-u_0\theta)u_0\theta} = -u_1(2-2\theta-2m_1\theta-u_1\theta)$$

gives

$$0 \leq u_1 < \frac{2\lambda_1\sqrt{u_0\theta}}{\sqrt{2-2\theta-2m_1\theta-u_0\theta}} < \frac{4\sqrt{u_0}}{\sqrt{2a-2m_1-4}} < \frac{4\sqrt{u_0}}{\sqrt{2a-2m_1}}\left(1+a^{-1/4}\right)a\theta.$$

We proceed now by induction on i and, choosing $m_{i+1} = \lfloor u_i/\theta \rfloor$, write

$$\xi = \left(1-\theta-\sum_{j=1}^{i}(m_j+l_j)\theta^j-(m_{i+1}+\delta_{i+1})\theta^{i+1}\right) \times\left(1-\theta-\sum_{j=1}^{i}(m_j-l_j)\theta^j-(m_{i+1}+\delta_{i+1})\theta^{i+1}\right)$$

with $0 \leq \delta_{i+1} < 1$, and the claim amounts to

$$\delta_{i+1}\left(2-2\theta-2\sum_{j=1}^{i+1}m_j\theta^j-\delta_{i+1}\theta^{i+1}\right) = l_{i+1}\left(2\sum_{j=1}^{i}l_j\theta^j+l_{i+1}\theta^{i+1}\right) +u_{i+1}\left(2-2\theta-2\sum_{j=1}^{i+1}m_j\theta^j-u_{i+1}\theta^{i+1}\right).$$

Writing $B := \sqrt{a-m_1}(1+29(a-m_1)^{-1/2})/\sqrt{2u_0}$ and choosing $l_{i+1} \equiv m_{i+1}+a \pmod 2$ so that

$$\frac{\left(2-2\theta-2\sum_{j=1}^{i+1}m_j\theta^j-\delta_{i+1}\theta^{i+1}\right)\delta_{i+1}}{2\sum_{j=1}^{i}l_j\theta^j+B\theta^{i+1}} = l_{i+1}+\lambda_{i+1}$$

with $0 \le \lambda_{i+1} < 2$, we have

$$-1 \le l_{i+1} < \frac{2-2\theta-2m_1\theta}{2l_1\theta-2\theta^2/(1-\theta)} < \frac{a-m_1}{l_1} \le B,$$

and u_{i+1} is left to satisfy

$$u_{i+1}\left(2-2\theta-2\sum_{j=1}^{i+1} m_j\theta^j - u_{i+1}\theta^{i+1}\right) = \lambda_{i+1}\left(2\sum_{j=1}^{i} l_j\theta^j + B\theta^{i+1}\right) + l_{i+1}(B-l_{i+1})\theta^{i+1},$$

and, writing $D := 2\sqrt{2u_0}(1+2a^{-1/8})a/\sqrt{a-m_1}$, for $a \ge 4^8$ we have

$$\begin{aligned} 0 \le u_{i+1} &< \frac{\lambda_{i+1}\,(2l_1\theta+2B\theta^2/(1-\theta))+\frac14 B^2\theta^{i+1}}{2-2\theta-2m_1\theta-2D\theta^2/(1-\theta)} \\ &< \frac{4\sqrt{u_0}\sqrt{2a-2m_1}}{2a-2m_1-3} \\ &\quad \times\left(1+\frac{1+10a^{-1/8}}{2u_0}\frac{\theta}{1-\theta}+\frac{11}{32\sqrt{2}u_0^{3/2}\sqrt{a}}\right) \\ &< D\theta. \end{aligned}$$

□

Lemma 2. *For $a \ge 4^8$ the spectrum $\boldsymbol{L}(\theta)$ contains the interval*

$$\left[0,\ \tfrac14(1-c_0(1+3a^{-1/8})a^{-1/2})\right],$$

with

$$c_0 := 2\sqrt{3}\left(\sqrt{\sqrt{10}-3}+\sqrt{\sqrt{10}+3}\right) = 9.994\cdots.$$

Proof. Observe first that any ξ in $[U_1, U_2]$, where $U_1 = 10^2a^{1/2}\theta^2$ and $U_2 := 1-c_0(1+3a^{-1/8})a^{-1/2}$, can be written in the form $\xi = (1-\theta-(m_1+u_0)\theta)^2$ with $\frac12 c_0(1+\frac52 a^{-1/8})\sqrt{a} < m_1 < a-10a^{1/4}$ and $u_0 \in \left[\frac16(\sqrt{10}-3),\ \frac16(\sqrt{10}+3)\right]$. Hence we can write $\xi = vv'$ with

$$v = 1-\theta-\sum_{i=1}^{\infty} e_i\theta^i, \quad e_i = m_i+l_i,$$

$$v' = 1-\theta-\sum_{i=1}^{\infty} e_i'\theta^i, \quad e_i' = m_i-l_i,$$

where the integers m_i, l_i satisfy the conditions of Lemma 1. Clearly then by taking a γ_0 whose sequence t_i consists of increasingly long blocks of the form $\cdots -e_3', -e_2', -e_1', -e_1, -e_2, -e_3, \ldots$, we have $s^*(i) = s_1^*(i) \to \xi$ for the $(t_i, t_{i+1}) = (-e_1', -e_1)$. We show that $s^*(i) > \xi$ for the remaining $(t_i, t_{i+1}) \ne (-e_1', -e_1)$. It will be enough to show that for $i \ne 1$

$$m_i+l_i+1 < m_1-(l_1+1)-\frac{(l_1+1)^2\theta}{1-\theta-(m_1+l_1+1)\theta}. \tag{2–1}$$

Since $m_1+l_1 < m_1+\sqrt{2u_0}\sqrt{a-m_1} < a-130$ we certainly have $|d_i^-|$, $|d_{i-1}^+| \le (|t_i|+1)\theta$. Hence, under the assumption $(t_i, t_{i+1}) \ne (-e_1', -e_1)$,

$$s^*(i) \ge (1-\theta-(m_j+l_j+1)\theta)(1-\theta-(m_{j+\varepsilon}+l_{j+\varepsilon}+1)\theta)$$

for some $j \ne 1$ and $\varepsilon = \pm 1$, and (2–1) gives

$$\begin{aligned} s^*(i) &> (1-\theta-(m_j+l_j+1)\theta)(1-\theta-(m_1+l_1+1)\theta) \\ &> (1-\theta-m_1\theta)^2 > \xi. \end{aligned}$$

Now

$$\frac{(l_1+1)\theta}{1-\theta-(m_1+l_1+1)\theta} < \frac{\sqrt{2u_0}\sqrt{a-m_1}+1}{a-m_1-2-\theta-\sqrt{2u_0}\sqrt{a-m_1}} < \frac{3}{\sqrt{a-m_1}}$$

and $l_1(1+3(a-m_1)^{-1/2}) < \sqrt{2u_0}a/\sqrt{a-m_1}$, giving

$$\begin{aligned} (m_i+l_i+1)+(l_1+1)+\frac{(l_1+1)^2\theta}{(1-\theta-(m_1+l_1+1)\theta)} &< (1+2a^{-1/8})\left(3\sqrt{2u_0}+\frac{1}{\sqrt{2u_0}}\right)\frac{a}{\sqrt{a-m_1}} \\ &< (1+2a^{-1/8})\frac12 c_0\frac{a}{\sqrt{a-m_1}} =: E. \end{aligned}$$

Plainly $E < m_1$ for $m_1 = \frac12 c_0(1+\frac52 a^{-1/8})\sqrt{a}$ and $m_1 = a-10a^{1/4}$ (and all the m_1 in between by concavity). So (2–1) holds and $M^*(\theta, \gamma_0) = \xi$.

Likewise for any $k \ge 1$ a γ_k made from increasingly long blocks of

$$\ldots, -e_3', -e_2', -(e_1'+2), a, \underbrace{a-2, \ldots, a-2}_{k-1 \text{ times}}, e_1, e_2, \ldots$$

will have $s_2^*(i) \to \theta^k\xi$ when $(t_i, t_{i+1}) = (-(e_1'+2), a)$. The negative of this sequence is simply

$$\ldots, e_2', e_1', \underbrace{a-2, \ldots, a-2}_{k-1 \text{ times}}, a, -(e_1+2), -e_2, -e_3, \ldots$$

Trivially, since $e_j, e_j' \le m_1-l_1-2$ for $j \ne 1$, we have

$$s_1^*(i) \ge (1-\theta-(m_1+l_1+3)\theta)(1-\theta-(m_1-l_1-1)\theta) > \theta v' > \theta\xi,$$

while for $t_{i+1} \neq a$

$$s_2^*(i) \geq (1+\theta-(m_1+l_1+2)\theta+a\theta^2)(1+\theta-(a-2)\theta-a\theta^2) > v\theta > \theta\xi.$$

Finally when $(t_i, t_{i+1}) = (a-2 \text{ or } e_1', a)$ we have

$$s_2^*(i) \geq (1+\theta)(1+\theta-a\theta+(e_1'+1)\theta) > \theta(1+\theta) > \theta\xi.$$

Thus $M^*(\theta, \gamma_k) = \theta^k\xi$.

Hence for any $k \geq 0$ we can construct a γ with $M^*(\theta,\gamma)$ taking any value in $[\theta^k U_1, \theta^k U_2]$. Since $\theta U_2 > U_1$ we thereby obtain everything in $[0, U_2]$. □

3. EXISTENCE OF THE GAPS

We now prove the existence of the gaps given by Theorem 2.

Note that

$$I_1(t)-I_2(t) = \frac{2\theta^2(1-\theta)}{(1-\theta^2)^2}\kappa',$$
$$E_1(t)-E_2(t) = \frac{\theta^2}{(1-\theta^2)^2}\kappa$$

with

$$\kappa' = (1-u)(3+(2t-2)\theta-2\theta^2+\theta^3)+\theta(1+\theta)(4t^2-2t+1-a)+2\theta^2(1+t\theta),$$
$$\frac{\kappa}{1-\theta} = (2a+14-6u)-(t+4-u)^2+\theta t^2(5+8\theta+4\theta^2)+\theta(8-u)+2\theta^2(1+4\theta)-2\theta t(u+2+6\theta(1+\theta))$$

when $t \equiv a \pmod 2$, and

$$\kappa = (2a+13-6u)-(t+4-u)^2+t^2\theta\big(6+3\theta-2\theta^2(1-\theta^2)(3+2\theta)\big)+2\theta t\big(2-2u+(u-2)\theta-2\theta^2(1+\theta^2)+11\theta^3(1-\theta^2)\big)+\theta(10-2u+(u-7)\theta-8\theta^2+14\theta^3-32\theta^4+30\theta^5)$$

when $t \not\equiv a \pmod 2$. So $I_2(t) < I_1(t)$ for all $t \geq 0$ when a is even and for $(4t-1) \geq \sqrt{4a-3}$ when a is odd, and $E_2(t) < E_1(t)$ for all $t \leq \sqrt{2a-6u+16}-(4-u)$.

We suppose that γ has $M^*(\theta,\gamma)$ in $[I_2(t), I_1(t)]$ (Case I) or $[E_2(t), E_1(t)]$ (Case II), with t as in the statement of Theorem 2. Then from the rough lower bounds

$$E_2(t) \geq (1-(t+1)\theta)^2,$$
$$I_2(t) \geq 1-(2t+3-u)\theta$$

we can assume that all the $|t_i|$ are at most $2t+2v+u$ with $v=1$ in Case I and $v=0$ in Case II (it was shown in [Pinner 2000b] that if $t_i = a$ infinitely often then $M^*(\theta,\gamma) \leq \theta$ and if $|t_i| \geq k$ infinitely often then $M^*(\theta,\gamma) < 1-k\theta+\theta^2$, where a finite number of t_i can be changed without altering $M^*(\theta,\gamma)$). Now if we have $(t_i, t_{i+1}) = (-b, -c)$ (or $(-(b+2), (c+2))$) with $b+c = 2l+2t$, $l \geq 1$ then (since the remaining $|t_j| \leq 2t+u+2v$) we have $s_1^*(i)$ (or $s_2^*(i)$) bounded by

$$\left(1-\theta-b\theta+\frac{(2t+u+2v)\theta^2}{1-\theta}\right)\left(1-\theta-c\theta+\frac{(2t+u+2v)\theta^2}{1-\theta}\right)$$
$$= \left(1-\theta-(t+l)\theta+\frac{(2t+u+2v)\theta^2}{1-\theta}\right)^2-(t+l-c)^2\theta^2.$$

For $l \geq 1$ this is plainly less than $E_2(t)$ and for $l \geq 2$ is less than

$$B(t) := \left(1-(t+2)\theta+\frac{(t+3)\theta^2+(t-1)\theta^3}{1-\theta^2}\right)^2 = I_2(t)+\frac{\theta^2}{(1-\theta^2)^2}\Gamma,$$

where Γ, defined to be

$$(1+2\theta)((t+3-u)^2-(2a-8))-\theta(4t(5-u)+50-18u)-\theta^2(3t^2+2(5-u)t-u-9)-\theta^3(4t^2-20t-6(1-u))+\theta^4(4t^2+2u),$$

is certainly negative for $(t+3-u) \leq \sqrt{2a-8}$. So we can assume that each $|t_i+t_{i+1}|$ is at most $2t+2v$, and each $|t_i-t_{i+1}|$ is at most $2t+4+2v$. Thus if we have $(t_i, t_{i+1}) = (-b,-c)$ (or $(-(b+2),(c+2))$) with $b+c = 2t+2v$, then (since $t_{i-1} \leq 2t+2v+2-b+2\lambda$ with $\lambda = 1$ or 0, and $t_{i-1} = 2t+2v+2-b+2\lambda$ implies that $t_{i-2} \leq b-2-2\lambda$ and so on) $s_1^*(i)$ (or $s_2^*(i)$) is bounded by the quantity S defined as

$$\left(1-\theta-b\theta+\frac{(2t+2+2v+2\lambda-b)\theta^2+(b-2-2\lambda)\theta^3}{1-\theta^2}\right)\times\left(1-\theta-c\theta+\frac{(2t+2+2v+2\lambda-c)\theta^2+(c-2-2\lambda)\theta^3}{1-\theta^2}\right)$$
$$=\left(1-(t+1+v)\theta+\frac{(t+2+v+2\lambda)\theta^2+(t+v-2-2\lambda)\theta^3}{1-\theta^2}\right)^2-(b-t-v)^2\theta^2\left(\frac{1+2\theta}{1+\theta}\right)^2.$$

Since S is bounded by $B(t) < I_2(t)$ when $\lambda = 0$ and $v = 1$, in Case I we can successively rule out any $|t_i - t_{i+1}| = 2t+6$ and $|t_i+t_{i+1}| = 2t+2$. So

$|t_i - t_{i+1}| \le 2t+4$ and $|t_i + t_{i+1}| \le 2t$. Since S is less than

$$E_2(t)+4\theta^2-(b-t)^2\theta^2+\begin{cases} 0 & \text{if } t\equiv a \pmod 2,\\ \theta^2(1+\theta)^2 & \text{if } t\not\equiv a \pmod 2,\end{cases}$$

(and hence less that $E_2(t)$ for $b \ne t$ or $t\pm1$) in Case II we can assume that $|t_i+t_{i+1}| \le 2t-2$ and $|t_i-t_{i+1}| \le 2t+2$ except for the blocks $\pm\{t_i,t_{i+1}\} = \{-t,-t\}$, $\{-(t+2),(t+2)\}$, $\{-(t+1),-(t-1)\}$ or $\{-(t+3),(t+1)\}$. Now if $t \equiv a \pmod 2$ and $(t_i,t_{i+1}) = (-t,-t)$ then (since $t_{i-1} \le t+2$ and $t_{i-1} = t+2$ implies $t_{i-2} \le t-4$ and $t_{i-2} = t-4$ implies $t_{i-3} \le t+2$, etc., and the same for $t_{i+2},t_{i+3},t_{i+4},\ldots$) we obtain $s_1^*(i) \le E_2(t)$ (with equality if the $(-t,-t)$ is preceded and succeeded by perpetual blocks $(t+2)$, $(t-4)$). Likewise for $(t_i,t_{i+1}) = (-(t+2),(t+2))$ we obtain $s_2^*(i) \le E_2(t)$ with equality only if the preceding and succeeding blocks take the form $(t+2),(t-4)$ and $-(t+2),-(t-4)$, but then $s_4^*(i+1)$ will be smaller so these can be dismissed. Likewise if $t \not\equiv a \pmod 2$ and $(t_i,t_{i+1}) = (-(t+1),(t+3))$ then since

$$(1+\theta+d_i^-) \le \left(1+\theta-(t+1)\theta+\frac{(t+3)\theta^2+(t-5)\theta^3}{1-\theta^2}\right)$$

we must have $t_{i+2} = -(t+1)$ (else trivially $s_2^*(i) < E_2(t)$). Hence

$$\begin{aligned} s_2^*(i) &= \big(1+\theta-(t+1)\theta+\theta d_{i-1}^-\big)\\ &\quad\times\big(1+\theta-(t+3)\theta+(t+1)\theta^2-\theta^2 d_{i+2}^+\big),\\ s_4^*(i+1) &= \big(1+\theta-(t+1)\theta+\theta d_{i+2}^+\big)\\ &\quad\times\big(1+\theta-(t+3)\theta+(t+1)\theta^2-\theta^2 d_{i-1}^-\big).\end{aligned}$$

Thus, writing

$$d = \min\{d_{i-1}^-, d_{i+2}^+\} \le \frac{(t+3)\theta+(t-5)\theta^2}{1-\theta^2},$$

we have

$$\begin{aligned}\min\{s_2^*(i), s_4^*(i+1)\} &\le \big(1+\theta-(t+1)\theta+\theta d\big)\times\\ &\qquad \big(1+\theta-(t+3)\theta+(t+1)\theta^2-\theta^2 d\big)\\ &< E_2(t).\end{aligned}$$

Similarly if $t_i,t_{i+1} = -(t-1),-(t+1)$ then

$$\begin{aligned} s_1^*(i) &\le \left(1-\theta-(t-1)\theta+\frac{(t+3)\theta^2+(t-5)\theta^3}{1-\theta^2}\right)\\ &\qquad\times\big(1-\theta-(t+1)\theta+\theta d_{i+1}^+\big).\end{aligned}$$

Hence $t_{i+2},t_{i+3} = t+1,t-1$ (else trivially $s_1^*(i) \le E_2(t)$) and

$$\begin{aligned} s_1^*(i) &= \big(1-\theta-(t-1)\theta+\theta d_{i-1}^-\big)\\ &\quad\times\big(1-\theta-(t+1)\theta+(t+1)\theta^2\\ &\qquad\qquad +(t-1)\theta^3-\theta^3(-d_{i+3}^+)\big),\\ s_3^*(i+2) &= \big(1-\theta-(t-1)\theta+\theta(-d_{i+3}^+)\big)\\ &\quad\times\big(1-\theta-(t+1)\theta+(t+1)\theta^2\\ &\qquad\qquad +(t-1)\theta^3-\theta^3 d_{i-1}^-\big),\end{aligned}$$

with

$$d = \min\{d_{i-1}^-, -d_{i+3}^+\} \le \frac{(t+3)\theta+(t-5)\theta^2}{1-\theta^2}$$

and $\min\{s_1^*(i), s_3^*(i+2)\} \le E_2(t)$ (with equality if the preceding and succeeding $t_{i-1},t_{i-2},\ldots$ and $-t_{i+4}$, $-t_{i+5},\ldots$ consist of repeated blocks $(t+3),(t-5)$).

Hence in Case II we can assume that all the $|t_i+t_{i+1}| \le 2t-2$ and $|t_i-t_{i+1}| \le 2t+2$ or that γ is of the form claimed to achieve $E_2(t)$. Now if each $|t_i-t_{i+1}|$ is at most $2t+2+2v$ and each $|t_i+t_{i+1}|$ is at most $2t$ then, writing $K = \min\{2t+2+2v-b, 2t+2v-u\}$, observe that if $t_i = -b$, $0 \le b \le 2t+2v-u$ then

$$\begin{aligned} s_2^*(i) &\ge \left(1+\theta-\frac{(b\theta+(2t-b)\theta^2)}{1-\theta^2}\right)\\ &\qquad\qquad \left(1+\theta-\frac{(K\theta+(2t-K)\theta^2)}{1-\theta^2}\right)\\ &\ge \left(1+\theta-\frac{((2t-u+2v)\theta+(u-2v)\theta^2)}{1-\theta^2}\right)\\ &\qquad\qquad \left(1+\theta-\frac{((2+u)\theta+(2t-u-2)\theta^2)}{1-\theta^2}\right),\end{aligned}$$

the minimum occurring when $b = (2+u)$ or $2t+2v-u$, with this greater than $I_1(t)$ or $E_1(t)$ as $v = 1$ or 0. Similarly for $s_4^*(i)$. Hence for the γ of interest we need only consider $s_1^*(i)$ and $s_3^*(i)$.

Now if $t_i = -b$ with $u \le b \le 2t-u$ and all the $|t_i+t_{i+1}| \le 2t-2+2v$ then

$$\begin{aligned} s_1^*(i) &\ge \left(1-\theta+\frac{-b\theta-(2t-2+2v-b)\theta^2}{1-\theta^2}\right)\\ &\qquad\times\left(1-\theta+\frac{-(2t-2+2v-b)\theta-b\theta^2}{1-\theta^2}\right).\end{aligned}$$

Clearly the minimum occurs when

$$b = 2t-u \quad\text{or } u-2+2v,$$

equalling $I_1(t)$ for $v = 1$ and $E_1(t)$ for $v = 0$ (with equality for period $(2t-u),(u-2+2v)$). Likewise for the γ claimed to achieve $E_2(t)$ we easily have $s_1^*(i) > E_1(t)$ for $\{t_i, t_{i+1}\} \neq \{t+u, t-u\}$. Hence $E_2(t)$, $E_1(t)$ and $I_1(t)$ are attained as claimed with $M^*(\theta,\gamma) > E_1(t)$ for the γ remaining in Case II. In Case I it remains only to check the $t_i = -(2t+2-u)$. If a is even then (since $t_{i\pm1} \leq 2$ and $t_{i\pm1} = 2$ implies $t_{i\pm2} \leq 2t-2$ and $t_{i\pm2} = 2t-2$ implies $t_{i\pm3} \leq 2$ etc.) we must have $s_1^*(i) \leq I_2(t)$ (with equality if the $-(2t+2)$ is contained inside blocks $(2t-2,2)^k$, $-(2t+2)$, $(2,2t-2)^k$ with $k \to \infty$). Similarly if a is odd and $t_{i+1} = 1$ we have (since $t_{i-1} \leq 3$ and $t_{i-1} = 3$ implies $t_{i-4} \leq 2t-3$ and $t_{i-4} = 2t-3$ implies $t_{i-4} \leq 3$, while $t_{i+2} \leq 2t-1$ and $t_{i+2} = 2t-1$ implies $t_{i+3} \leq 1$ and so on) $s_1^*(i) \leq I_2(t)$ (with asymptotic equality for blocks $(2t-1,1)^k$, $-(2t+1)$, $(3,2t-3)^k$). Likewise if $t_{i-1} = 1$ using $s_1^*(i-1)$. Hence $I_2(t)$ is achieved as claimed. Finally if the $t_i = -(2t+1)$ all have $t_{i-1} = t_{i+1} = 3$ then

$$\begin{aligned} s_1^*(i) &\geq \left(1-(2t+2)\theta+\frac{3\theta^2-(2t+1)\theta^3}{1-\theta^2}\right) \\ &\qquad \times \left(1+2\theta+\frac{-(2t+1)\theta^2+3\theta^3}{1-\theta^2}\right) \\ &= I_1(t)+\frac{2\theta^3(a-4t)(a-1)}{(1-\theta^2)^2} \end{aligned}$$

and $M^*(\theta,\gamma) > I_1(t)$. □

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